The Graphs for which All Strong Orientations Are Hamiltonian

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ABSTRACT

We show that the only nontrivial strongly orientable graphs for which every strong orientation is Hamiltonian are complete graphs and cycles.

A graph G is *strongly orientable* if it has a strongly connected orientation. Robbins [5] has shown that G is such a graph if and only if G is connected and has no bridges. Our object is to determine the graphs such that all strong orientations are Hamiltonian. We define the *standard orientation* of H shown in Figure 1 as the digraph E with arcs $(i, i+1)$ for $i=1,\cdots,k-1$; $(i, i-1)$ for $i=k+1,\cdots,p$ and two more arcs $(1, p)$, $(k, 1)$.

We use the convention of [2] that p is the number of points of G. By definition, a *tournament* is an orientation of the complete graph K_p. It is well known [1; 3; 4, p. 306] that a nontrivial tournament is strong if and only if it is Hamiltonian. Obviously both strong orientations of the cycle C_p are Hamiltonian. We now show that K_p and C_p are the only such graphs. As usual, $V(G)$ and $E(G)$ denote the point set and edge set of G.

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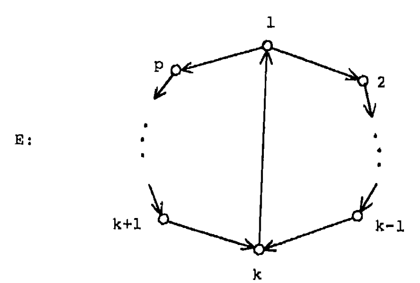

FIGURE 1. The standard orientation of the labeled graph H of order p and size $p+1$.

Theorem. If G is a nontrivial strongly orientable graph for which every strong orientation is Hamiltonian, then G is either the cycle C_p or the complete graph K_p.

Proof. Let G be a graph satisfying the hypothesis of the theorem and let C_p: $1, 2, \cdots, p, 1$ be a Hamiltonian cycle of G. Assume that the theorem is false, i.e., that there is such a graph $G \neq C_p, K_p$. Then C_p has at least one chord, say $\{1, k\}$, in G but not all possible chords of C_p are in G. Let H be the subgraph of G consisting just of C_p and this chord. The standard orientation E of H is clearly strong and therefore every orientation of G that contains E is also strong. In the following, the point $p+j$ is identified with point j. We prove the theorem in two steps.

Step 1. If edge $\{r, s\} \in E(G)$, then edge $\{r+j, s+j\} \in E(G)$ for all $j = 1, \cdots, p$. It suffices to show that if $\{1, s\} \in E(G)$, then $\{1+j, s+j\} \in E(G)$ for $j = 1, 2, \cdots, p$ for the argument is similar when $r > 1$.

The assertion is true by definition for $r=1$, $s=1+1$ as these are precisely the edges of C_p.

We already know that $\{1, k\}$, $2 \neq k \neq p$, is an edge of G. We now orient the edges $\{i, j\} \in E(G)$ as the arc (i, j) under the following five conditions, which are exhaustive:

(a) $2 \leq i < j \leq k$;
(b) $p \geq i > j \geq k$;
(c) $i \in \{k+1, \cdots, p\}$ and $j \in \{2, \cdots, k\}$;
(d) $i = 1$ and $j \in \{2, 3, \cdots, k-1, \cdots, p\}$, and
(e) $i = k$ and $j = 1$.

Call this digraph D. The orientation of H is standard by construction, therefore D is strong. Let C be any Hamiltonian dicycle in D. Since $id_D(p) = 1$, C has to contain $(1, p)$. The indegree of $p-1$ in $D-1$ (D with point 1 deleted) is one, hence C must contain $(p, p-1)$. Continuing this reasoning we conclude that C also contains the arcs $(p-1, p-2), \cdots, (k+2, k+1)$. Now point 2 can be reached in D only if $\{2, k+1\} \in E(G)$. By repeating this argument Step 1 is proved.

Step 2. If edge $\{r, s\}$ is a chord of C_p, then edge $\{r, s+1\}$ is in G. It suffices to show that if $\{2, k+1\} \in E(G)$, then $\{2, k+2\} \in E(G)$.

We may assume for convenience that k is chosen so that $\{2, k+1\} \in E(G)$. Step 1 implies that $\{1, k\} \in E(G)$. Let $(2, k+1)$ be the orientation of $\{2, k+1\}$ and orient all other edges as in Step 1. Since the orientation on H is standard, the digraph D' obtained in this way is strong. The same arguments as in the proof of Step 1 show that every Hamiltonian dicycle C in D' has to contain the arcs $(1, p), (p, p-1), \cdots, (k+3, k+2)$. The point 2 can be included in C only if $(k+2, 2)$ is an arc of D', i.e., if $\{2, k+2\} \in E(G)$. This proves Step 2.

We can now complete the proof of the theorem. By assumption G contains some chords of C_p but not all. Thus because of Step 1 there exist points r, s such that $\{r, s\} \in E(G)$ and $\{r, s+1\} \notin E(G)$. This contradicts Step 2 and the theorem is proved. ∎

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