The Graphs for which All Strong Orientations Are Hamiltonian

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ABSTRACT

We show that the only nontrivial strongly orientable graphs for which every strong orientation is Hamiltonian are complete graphs and cycles.

A graph $G$ is strongly orientable if it has a strongly connected orientation. Robbins [5] has shown that $G$ is such a graph if and only if $G$ is connected and has no bridges. Our object is to determine the graphs such that all strong orientations are Hamiltonian. We define the standard orientation of $H$ shown in Figure 1 as the digraph $E$ with arcs $(i, i+1)$ for $i=1, \ldots, k-1$; $(i, i+1)$ for $i=k+1, \ldots, p$ and two more arcs $(1, p), (k, 1)$.

We use the convention of [2] that $p$ is the number of points of $G$. By definition, a tournament is an orientation of the complete graph $K_n$. It is well known [1; 3; 4, p. 306] that a nontrivial tournament is strong if and only if it is Hamiltonian. Obviously both strong orientations of the cycle $C_n$ are Hamiltonian. We now show that $K_3$ and $C_3$ are the only such graphs. As usual, $V(G)$ and $E(G)$ denote the point set and edge set of $G$.

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Theorem. If $G$ is a nontrivial strongly orientable graph for which every strong orientation is Hamiltonian, then $G$ is either the cycle $C_n$ or the complete graph $K_n$.

Proof. Let $G$ be a graph satisfying the hypothesis of the theorem and let $C_n: 1, 2, \ldots, p, 1$ be a Hamiltonian cycle of $G$. Assume that the theorem is false, i.e., that there is such a graph $G \neq C_n, K_n$. Then $C_n$ has at least one chord, say $(1, k)$, in $G$ but not all possible chords of $C_n$ are in $G$. Let $H$ be the subgraph of $G$ consisting just of $C_n$ and this chord. The standard orientation $E$ of $H$ is clearly strong and therefore every orientation of $G$ that contains $E$ is also strong. In the following, the point $p + 1$ is identified with point 1. We prove the theorem in two steps.

Step 1. If edge $(r, s) \in E(G)$, then edge $(r + j, s + j) \in E(G)$ for all $j = 1, \ldots, p$. It suffices to show that if $(1, s) \in E(G)$, then $(1 + j, s + j) \in E(G)$ for $j = 1, 2, \ldots, p$ for the argument is similar when $r > 1$.

The assertion is true by definition for $r = 1, s = 1 + 1$ as these are precisely the edges of $C_n$.

We already know that $(1, k)$, $2 \not\subset k \not\subset p$, is an edge of $G$. We now orient the edges $(i, j) \in E(G)$ as the arc $(i, j)$ under the following five conditions, which are exhaustive:

(a) $2 \leq i < j \leq k$;
(b) $p \leq i \leq j \leq k$;
(c) $i \in \{k + 1, \ldots, p\}$ and $j \in \{2, \ldots, k\}$;
(d) $i = 1$ and $j \in \{2, 3, \ldots, k - 1, \ldots, p\}$, and
(e) $i = k$ and $j = 1$.

Call this digraph $D$. The orientation of $H$ is standard by construction, therefore $D$ is strong. Let $C$ be any Hamiltonian cycle in $D$. Since $h_D(p - 1, C)$ has to contain $(1, p)$, the indegree of $p - 1$ in $D - 1$ (with point 1 deleted) is one, hence $C$ must contain $(p, p - 1)$. Continuing this reasoning we conclude that $C$ also contains the arcs $(p - 1, p), (p - 2, \ldots, k + 2, k + 1)$. Now point 2 can be reached in $D$ only if $(2, k + 1) \in E(G)$. By repeating this argument Step 1 is proved.

Step 2. If edge $(r, s)$ is a chord of $C_n$, then edge $(r, s + 1)$ is in $G$. It suffices to show that if $(2, k + 1) \in E(G)$, then $(2, k + 2) \in E(G)$.

We may assume for convenience that $k$ is chosen so that $(2, k + 1) \in E(G)$. Step 1 implies that $(1, k) \in E(G)$. Let $(2, k + 1)$ be the orientation of $(2, k + 1)$ and orient all other edges as in Step 1. Since the orientation on $H$ is standard, the digraph $D'$ obtained in this way is strong. The same arguments as in the proof of Step 1 show that every Hamiltonian cycle in $D'$ has to contain the arcs $(1, p), (p, p - 1), \ldots, (k + 3, k + 2)$. The point 2 can be included in $C$ only if $(k + 2, 2)$ is an arc of $D'$, i.e., if $(2, k + 2) \in E(G)$. This proves Step 2.

We can now complete the proof of the theorem. By assumption $G$ contains some chords of $C_n$ but not all. Thus because of Step 1 there exist points $r, s$ such that $(r, s) \in E(G)$ and $(r, s + 1) \notin E(G)$. This contradicts Step 2 and the theorem is proved.

References