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ON THE ACYCLIC SUBGRAPH POLYTOPE

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The acyclic subgraph problem can be formulated as follows. Given a digraph with arc weights, find a set of arcs containing no directed cycle and having maximum total weight. We investigate this problem from a polyhedral point of view and determine several classes of facets for the associated acyclic subgraph polytope. We also show that the separation problem for the facet defining dicycle inequalities can be solved in polynomial time. This implies that the acyclic subgraph problem can be solved in polynomial time for weakly acyclic digraphs. This generalizes a result of Lucchesi for planar digraphs.

Key words: Acyclic Subgraph Problem, Feedback Arc Set Problem, Facets of Polyhedra, Polynomial Time Algorithms, Weakly Acyclic Digraphs.

1. Introduction and notation

The problem which we want to discuss in this paper comes up in various different sciences, has numerous applications, and is known under several quite unrelated names. It appears in slightly different but equivalent formulations some of which we shall briefly introduce.

A directed graph or *digraph* $D=(V,A)$ consists of a finite nonempty set V of *nodes* and a set A of *arcs* which are ordered pairs of different elements of V. (Since loops and parallel arcs are of no interest for our purposes, we do not consider them here.) The number of nodes of V is called the *order* of D. If $a=(u,v)\in A$ is an arc then a is said to go from u to v, or to be *incident from* u and *incident to* v; the nodes u, v are the *endnodes* of a; u is the *tail* and v is the *head* of a. Throughout the paper we assume that every digraph contains at least one arc.

A *graph* $G=[V,E]$ consists of a finite nonempty *node set* V and a set E of *edges* which are unordered pairs of different nodes called the *endnodes* of the edge. If $G=[V,E]$ is a graph then an *orientation* of G is a digraph which contains an arc (i,j) or (j,i) but not both whenever $ij\in E$.

If $D=(V,A)$ is a digraph then every digraph $D'=(V',A')$ with $V'\subseteq V$ and $A'\subseteq A$ is called a *subdigraph* of D, and D is called a *superdigraph* of D'. We also say that D contains D' and that D' is contained in D.

Let $D=(V,A)$ be a digraph, $(i,j)\in A$ and let k be a node not in V. Then the digraph $D'=(V\cup\{k\},(A\setminus\{(i,j)\})\cup\{(i,k),(k,j)\})$ is called the digraph obtained from D by *subdividing* (i,j). For a given digraph D the class of all digraphs obtainable from D by repeated subdivision is denoted by $S(D)$.

A nonempty set of arcs $P=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{k-2},v_{k-1}),(v_{k-1},v_k)\}$ in $D=(V,A)$ such that $v_i\neq v_j$ for $i\neq j$ is called a (v_1,v_k)-*dipath* of *length* $k-1$. If P is a (v_1,v_k)-dipath and $(v_k,v_1)\in A$ then $C=P\cup\{(v_k,v_1)\}$ is called a *dicycle* of *length* k or *k-dicycle*. A digraph $D=(V,A)$ or just an arc set A which contains no dicycle is called *acyclic*. (In our terminology it should be called adicyclic, but acyclic is standard.)

An instance of the (weighted) *acyclic subgraph problem* can be described as follows. We are given a digraph $D=(V,A)$ with arc weights $c_{ij}\in\mathbb{R}$ for every $(i,j)\in A$, and we look for an acyclic subdigraph $D'=(V,B)$ (resp. an acyclic arc set B) of D such that

$$c(B):=\sum_{(i,j)\in B} c_{ij}$$

is as large as possible.

A *feedback arc set* (or dicycle covering) in $D=(V,A)$ is an arc set $B\subseteq A$ such that every dicycle in D contains at least one arc of B. Given a digraph $D=(V,A)$ with arc wieghts c_{ij} then the problem to find a feedback arc set $B\subseteq A$ such that $c(B)$ is as small as possible is an instance of the (weighted) *feedback arc set problem.*

Clearly, for every feedback arc set $B\subseteq A$, the digraph $D'=(V,A')$ with $A'=A\setminus B$ is acyclic and vice versa. Thus a minimum weight feedback arc set determines a maximum weight acyclic subdigraph and vice versa.

A *tournament* is a digraph $D=(V,A)$ such that for every two nodes u and v, A contains exactly one arc with endnodes u and v. A tournament on n nodes is an orientation of the complete graph K_n. If we speak of a tournament T contained in a digraph D we assume that T is *spanning*, i.e. contains all nodes of D. An acyclic tournament obviously defines a linear ordering of the nodes of D. The *linear ordering problem* for a given arc weighted digraph D (D is usually assumed to be the complete digraph $D_n=(V,A_n)$) is to find a spanning acyclic tournament in D of maximum total arc weight.

The acyclic subgraph problem and the linear ordering problem are in an obvious way polynomially related. Suppose we have an algorithm to solve the acyclic subgraph problem and we want to find an optimal linear ordering on a complete digraph $D=(V,A)$ with arc weights c_{ij} for all $(i,j)\in A$. Set $M:=\max\{|c_{ij}|:(i,j)\in A\}+1$ and $c'_{ij}:=c_{ij}+M$. Then all new arc weights c'_{ij} are positive which implies that the optimum acyclic subdigraph is an acyclic tournament and therefore is an optimum solution of the linear ordering problem with respect to the original weights.

Suppose now we have an algorithm to solve the linear ordering problem, and a digraph $D=(V,A)$ with arc weights c_{ij} is given. Then we define new weights by setting $c'_{ij}:=\max\{0,c_{ij}\}$ for all $(i,j)\in A$ and add all arcs (i,j) which do not belong to D yet and assign the weight zero to them. It is easy to see that the optimum acyclic tournament on the complete digraph with the new weights determines an optimum acyclic subdigraph of D with respect to the original weights.

The acyclic subgraph problem is known to be NP-complete for the class of all digraphs, cf. Garey and Johnson (1979). In fact it is even NP-complete for the class of digraphs with in- and outdegree at most three, as well as for line digraphs. The discussion above shows that the linear order problem is NP-complete for the same classes of digraphs.

The acyclic subgraph problem is known to be solvable in polynomial time for planar digraphs, cf. Lucchesi (1976). In section five we shall show the polynomial time solvability of this problem for a class of digraphs (called weakly acyclic) which contains the class of planar digraphs.

The main theoretical purpose of this paper is the study of polytopes associated with the acyclic subgraph problem. We shall briefly mention some concepts of polyhedral theory we need in the sequel.

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces in $\mathbb{R}^m$. A *polytope* is a bounded polyhedron or equivalently the convex hull of finitely many points. The dimension of a polyhedron P, denoted by $\dim P$, is the maximum number of affinely independent points in P minus one.

If $a \in \mathbb{R}^m \setminus \{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^T x \leq a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leq a_0\}$. We say that a valid inequality $a^T x \leq a_0$ defines a *face* of P if $\emptyset \neq P \cap \{x \mid a^T x = a_0\} \neq P$. A valid inequality $a^T x \leq a_0$ defines a *facet* of P if it defines a face of P and if there exist $\dim P$ affinely independent points in $P \cap \{x \mid a^T x = a_0\}$. Two face-defining inequalities $a^T x \leq a_0$, $b^T x \leq b_0$ are called *equivalent* if $P \cap \{x \mid a^T x = a_0\} = P \cap \{x \mid b^T x = b_0\}$.

A polyhedron $P \subseteq \mathbb{R}^m$ is called *full-dimensional* if $\dim P = m$. For every full-dimensional polyhedron there exists an inequality system $Ax \leq b$ with $P = \{x \mid Ax \leq b\}$ which is unique up to multiplication by a positive constant. If P is not full-dimensional then P is contained in the intersection of hyperplanes, i.e. P has a representation of the form $P = \{x \mid Ax \leq b,\ Dx = d\}$.

If $P = \{x \mid Ax \leq b,\ Dx = d\}$ then we say that the system $Ax \leq b$, $Dx = d$ is complete for P. If D has full rank and $\{x \mid Dx = d\}$ is the affine space spanned by P, then $Dx = d$ is called a *minimal equation system* for P. A complete system $Ax \leq b$, $Dx = d$ for P is called *nonredundant* with respect to P if $Dx = d$ is a minimal equation system and if the deletion of any inequality of $Ax \leq b$ results in a polyhedron larger than P. It is known that in such a case for every facet of P the system $Ax \leq b$ contains exactly one inequality defining it, i.e. every inequality of the system $Ax \leq b$ defines a facet of P and no two inequalities are equivalent.

We shall present a partial nonredundant system of inequalities for the polyhedron associated with the acyclic subgraph problem on a digraph D.

2. Valid inequalities and trivial facets of the acyclic subgraph polytope $P_{AC}(D)$

There is a natural way to associate a polytope with every instance of the acyclic subgraph problem such that every vertex of the polytope corresponds to an acyclic arc set and vice versa. More precisely, suppose $D = (V, A)$ is a digraph and let

$$\mathscr{A}(D) := \{B \subseteq A \mid B \text{ is acyclic in } D\} \tag{2.1}$$

be the set of acyclic arc sets in D. These are exactly the feasible solutions of the acyclic subgraph problem on D.

Let $\mathbb{R}^A$ denote the real vector space where every component of a vector $x \in \mathbb{R}^A$ is indexed by an arc $(i,j) \in A$. For convenience we write x_{ij} instead of $x_{(i,j)}$. For every arc set $B \subseteq A$ the incidence vector $x^B \in \mathbb{R}^A$ of B is defined by: $x^B_{ij} = 1$, if $(i,j) \in B$, and $x^B_{ij} = 0$ if $(i,j) \notin B$. The *acyclic subgraph polytope* $P_{AC}(D)$ for D is the convex hull of the incidence vectors of all acyclic arc sets in D, i.e.

$$P_{AC}(D) = \operatorname{conv}\{x^B \in \mathbb{R}^A \mid B \in \mathscr{A}(D)\}. \tag{2.2}$$

Thus every vertex of $P_{AC}(D)$ corresponds to an acyclic subdigraph (induced by an acyclic arc set) of D and vice versa. This implies that—in principle—the acyclic subgraph problem for D can be solved via the linear program

$$\max c^T x, \quad x \in P_{AC}(D). \tag{2.3}$$

In order to apply linear programming techniques, however, the definition of $P_{AC}(D)$ is not appropriate. What is needed is a description of $P_{AC}(D)$ by means of a system of inequalities. The number of inequalities necessary to describe $P_{AC}(D)$ is in general exponential in the order of D. In the sequel we shall explicitly describe classes of facets of $P_{AC}(D)$ with an exponential number of members. Since the acyclic subgraph problem is NP-complete we shall probably never be able to obtain an explicit linear description of $P_{AC}(D)$. Nevertheless, we shall show in a later section and in a subsequent paper (Grötschel, Jünger and Reinelt (1984a)) that the partial linear description of $P_{AC}(D)$ obtained here is quite useful from an algorithmic point of view both from the theoretical as well as the practical side.

Since $P_{AC}(D)$ contains the zero vector and all unit vectors of $\mathbb{R}^A$, it follows that

$$\dim P_{AC}(D) = |A|, \tag{2.4}$$

i.e. $P_{AC}(D)$ is full-dimensional, which implies that the facet defining inequalities are unique up to multiplication by a constant. Clearly, $P_{AC}(D)$ is contained in the unit hypercube and is *monotone*, i.e. $0 \leq x \leq y \in P_{AC}(D)$ implies $x \in P_{AC}(D)$. Monotonicity implies that all facet defining inequalities $a^T x \leq a_0$ (except for non-negativity constraints) have nonnegative coefficients. It is easy to see which of the *trivial inequalities* (i.e. the hypercube constraints $0 \leq x_{ij} \leq 1$) define facets of $P_{AC}(D)$.

(2.5) Proposition. *Let $D = (V, A)$ be a digraph and $P_{AC}(D)$ the acyclic subgraph polytope for D.*

(a) *$x_{ij} \geq 0$ defines a facet of $P_{AC}(D)$ for all $(i,j) \in A$.*

(b) *For all $(i,j) \in A$, $x_{ij} \leq 1$ defines a facet of $P_{AC}(D)$ if and only if $(j,i) \notin A$.*

Proof. Trivial. □

By definition, an acyclic arc set contains no dicycle of D. This implies that the intersection of the arc set of every dicycle C with every acyclic arc set contains at most $|C|-1$ arcs. This immediately implies that the inequalities

$$x(C) := \sum_{(i,j)\in C} x_{ij} \leq |C|-1, \quad C \text{ a dicycle in } D, \tag{2.6}$$

are valid with respect to $P_{AC}(D)$. If C is a k-dicycle we call $x(C) \leq k-1$ a *k-dicycle inequality*. Validity of the k-dicycle inequalities implies that $P_{AC}(D)$ is contained in the polytope

$$P_C(D) := \{x \in \mathbb{R}^A \mid 0 \leq x_{ij} \leq 1 \quad \forall (i,j) \in A,$$
$$x(C) \leq |C|-1 \quad \forall \text{ dicycles } C \text{ in } D\}. \tag{2.7}$$

More importantly, but trivial to prove, we have the following

$$P_{AC}(D) = \operatorname{conv}\{x \in P_C(D) \mid x \text{ integral}\} \tag{2.8}$$

which shows that all integral solutions of the linear program $\max c^T x$, $x \in P_C(D)$ correspond to acyclic arc sets in D. More precisely, every vertex of $P_{AC}(D)$ (which is zero-one by definition) is a vertex of $P_C(D)$ and every integral vertex of $P_C(D)$ is a vertex of $P_{AC}(D)$. We shall study the relation between $P_C(D)$ and $P_{AC}(D)$ in more detail in the sequel.

For every integer $k \geq 3$, a digraph $D = (V, A)$ of order $2k$ is called a *simple k-fence* if V consists of two disjoint node sets $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$ such that

$$A = \bigcup_{i=1}^{k} (\{(u_i, w_i)\} \cup \{(w_i, v) \mid v \in U \setminus \{u_i\}\}).$$

The nodes in U are called the *upper nodes*, those in W the *lower nodes*. The arcs (u_i, w_i) going 'down' are called *pales*, the arcs (w_i, u_j), $i \neq j$, going 'up' are called *pickets*, see Fig. 2.1 for a 4-fence.

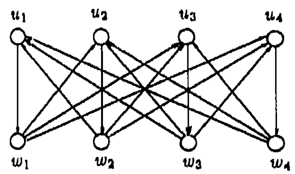

Fig. 2.1.

A simple k-fence is a particular orientation of the complete bipartite graph $K_{k,k}$. (A simple 2-fence would be a 4-dicycle.)

For two disjoint node sets $U = \{u_1, u_2, \ldots, u_k\}$ and $W = \{w_1, w_2, \ldots, w_k\}$, $F_k(U, W)$ denotes the arc set of the simple k-fence induced by U and W, i.e. the simple k-fence where U is the set of upper nodes and W the set of lower nodes. For every simple k-fence $D = (V, A)$ we call

$$x(A) \leq k^2 - k + 1 = |A| - k + 1 \tag{2.9}$$

the *simple k-fence inequality* of D. Sometimes it is convenient to specify the upper nodes U and the lower nodes W explicitly. Then we say that the simple k-fence inequality

$$x(F_k(U, W)) \leq k^2 - k + 1 \tag{2.10}$$

is the *simple k-fence inequality induced by U and W*.

(2.11) Proposition. *Let $D' = (V', A')$ be a digraph and let $D = (V, A)$ be a simple k-fence contained in D'. Then the simple k-fence inequality $x(A) \leq k^2 - k + 1$ is a face-defining inequality with respect to $P_{AC}(D')$. Moreover, if B is an acyclic arc set in D' with $x(B \cap A) = k^2 - k + 1$, then B contains one or two pales of D.*

Proof. Observe first that every set formed by all pickets and one pale of D is an acyclic arc set of cardinality $k^2 - k + 1$. Suppose now B is an acyclic arc set of D containing $l \geq 2$ pales. If the pales (u_i, w_i) and (u_j, w_j), $i \neq j$, are contained in B, then one of the pickets (w_i, u_j) or (w_j, u_i) cannot be in B, otherwise B would contain a dicycle. So, if B contains l pales, B contains at most $2\binom{k}{2} - \binom{l}{2}$ pickets. This implies

$$|B \cap A| \leq k^2 - k - (l^2 - l)/2 + l \leq k^2 - k + 1$$

which proves that the simple k-fence inequality is valid and face-defining. Moreover, it is obvious that in the inequality above, equality can only hold if $l = 2$ (or $l = 1$), and the construction above shows how to obtain an acyclic arc set B with $|B \cap A| = k^2 - k + 1$ containing exactly two pales. □

A *k-fence* is a digraph $D = (V, A)$ which can be obtained from a simple k-fence $D' = (V', A')$ by repeated subdivision of arcs, i.e. $D \in S(D')$. If U' resp. W' are the upper resp. lower nodes of D' then clearly these node sets correspond in a unique way to node sets U resp. W in D. We call the nodes in U resp. W the upper resp. lower nodes of D. For k-fences we can show

(2.12) Proposition. *Let $D' = (V', A')$ be a digraph and let $D = (V, A)$ be a k-fence contained in D'. Then the k-fence inequality*

$$x(A) \leq |A| - k + 1 \tag{2.13}$$

is a face defining inequality for $P_{AC}(D')$.

Proof. Analogous to (2.11). □

We shall now construct a class of digraphs, called Möbius-ladders, which can be obtained by linking an odd number of dicycles in a particular way.

Let $C_1, C_2, \ldots, C_k$ be a sequence of different dicycles in a digraph $D=(V,A)$ such that the following holds:

(2.14) $k \geq 3$ and k is odd.

(2.15) C_i and C_{i+1} $(i=1,\ldots,k-1)$ have a directed path P_i in common, C_1 and C_k have a dipath P_k in common.

(2.16) Given any dicycle C_j, $j \in \{1,\ldots,k\}$, set

$$J=\{1,\ldots,k\} \cap (\{j-2, j-4, j-6, \ldots\} \cup \{j+1, j+3, j+5, \ldots\}).$$

Then every set $(\bigcup_{i=1}^k C_i)\setminus\{e_i \mid i \in J\}$ contains exactly one dicycle (namely C_j), where e_i, $i \in J$, is any arc contained in the dipath P_i.

(2.17) The cardinality of a smallest feedback arc set in $\bigcup_{i=1}^k C_i$ is $(k+1)/2$ (or equivalently the largest acyclic arc set has cardinality $|\bigcup_{i=1}^k C_i| - (k+1)/2$.)

Then we call the arc set $M=\bigcup_{i=1}^k C_i$ a *Möbius-ladder*. For convenience we say that the dicycles C_i, C_{i+1}, $i=1,\ldots,k-1$ and C_1, C_k are *adjacent* (with respect to M). Assumption (2.17) implies immediately that for any Möbius-ladder M contained in a digraph D the inequality

$$x(M) \leq |M| - \frac{k+1}{2} \tag{2.18}$$

is valid with respect to $P_{AC}(D)$.

The requirements (2.14), ..., (2.17) are of course not easy to check for a given arc set M. (In fact, the problem of checking (2.17) is NP-complete.) They are however precisely those assumptions which we need to make a certain proof method work, cf. Theorem (3.5). (2.17) implies validity of (2.18); and (2.16) implies that the sets $M\setminus\{e_i \mid i \in J\}$ minus any arc in C_j are maximum cardinality acyclic arc sets and that there are enough acyclic arc sets of this kind to find $|M|$ whose incidence vectors are linearly independent. For even k, the construction does not give anything interesting. We might in fact also consider single cycles as Möbius ladders for $k=1$.

A Möbius ladder (for $k=9$) is depicted in Fig. 2.2.

If $C_1, \ldots, C_k$ is a sequence of directed cycles satisfying (2.14) and (2.15) and if no two different nonadjacent cycles C_i, C_j have a node in common, then the union of these cycles clearly forms a Möbius-ladder. Such a situation is depicted in Fig. 2.2. It may however well be that different nonadjacent cycles have a node or even a path in common, cf. Figs 2.3 and 2.4.

It should be clear how to generate large classes of Möbius-ladders from the examples shown in Figs 2.2—2.4.

It would be interesting to find equivalent characterizations of Möbius-ladders which are 'nicer' than those given in (2.14)—(2.17). It is easy to see that the axioms (2.14)—(2.17) imply that no two different paths P_i, P_j have a common arc (they may however have a common node, see Fig. 2.3). Moreover, all dicycles C_i have length at least three. In case $k=3$, all dicyclces C_i have to have length at least four; in fact, it is easy to see that the class of Möbius-ladders that can be obtained from 3 dicycles coincides with the class of 3-fences. In other words, for $k=3$, the classes

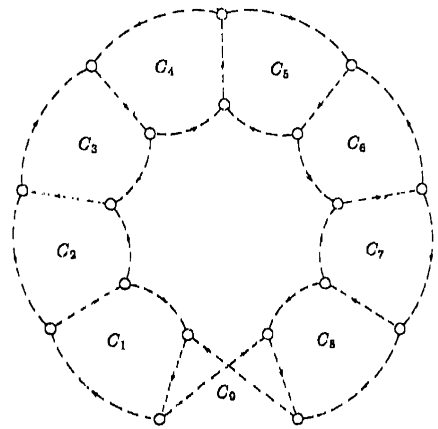

Fig. 2.2.

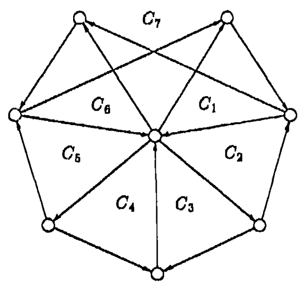

Fig. 2.3.

of inequalities (2.18) and (2.13) are identical. (This however is the only overlap among the classes of inequalities introduced in this section.)

3. Facets of $P_{AC}(D)$

We shall now show that the inequalities (2.6), (2.13) and (2.18) introduced in Section 2 define facets of the acyclic subgraph polytope. We start with the dicycle inequalities.

(3.1) Theorem. *Let C be a dicycle in a digraph $D=(V,A)$. Then the dicycle inequality $x(C) \leq |C|-1$ defines a facet of $P_{AC}(D)$.*

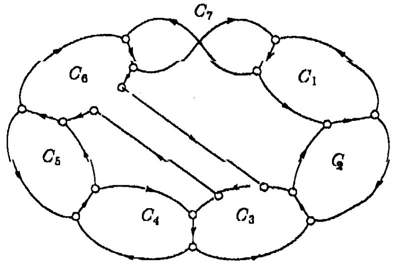

Fig. 2.4.

Proof. Suppose C is a k-dicycle in D. It is trivial to see that the k dipaths obtained from C by removing one arc from C form a set of acyclic arc sets whose incidence vectors in $\mathbb{R}^n$ are linearly independent and satisfy $x(C) \leq k-1$ with equality.

Now let $(i,j) \in A$ be an arc not in C. If both nodes i, j are in C (i.e. (i,j) is a chord of C) then remove the arc from C whose [illegible] to obtain a dipath P. If one of the endnodes of (i,j) is not in C let P_{ij} be any dipath of length $k-1$ contained in C. It is obvious that each of the arc sets $P_{ij} \cup \{(i,j)\}$ is acyclic and satisfies $x(C) \leq k-1$ with equality.

Moreover, the incidence vectors of all arc sets constructed above (k dipaths and $|A|-k$ dipaths plus an arc) are clearly linearly independent. This proves the theorem. □

The following subdivision lemma will help proving that the k-fence inequalities define facets.

(3.2) Lemma. *Let $D=(V,A)$ be a digraph and $a^{\mathrm{T}}x \leq a_0$ be a nontrivial inequality defining a facet of $P_{\mathrm{AC}}(D)$. Let $D'=(V',A')$ be the digraph obtained from D by subdividing the arc $(i,k) \in A$ into the arcs (i,j), $(j,k) \in A'$. Set*

$$a'_{uv} := a_{uv} \quad \text{for all } (u,v) \in A \cap A',$$

$$a'_{ij} := a'_{jk} := a_{ik},$$

$$a'_0 := a_0 + a_{ik}.$$

Then the inequality $a'^{\mathrm{T}}x \leq a'_0$ defines a facet of $P_{\mathrm{AC}}(D')$.

Proof. The validity of $a'^{\mathrm{T}}x \leq a'_0$ for $P_{\mathrm{AC}}(D')$ is obvious.

Since $a^{\mathrm{T}}x \leq a_0$ is nontrivial, there exist $m=|A|$ acyclic arc sets $B_1, \ldots, B_m$ in D whose incidence vectors satisfy the inequality with equality and which are linearly independent. Let M denote the (m,m)-matrix whose rows are the incidence vectors of $B_1, \ldots, B_m$, and assume that the column corresponding to arc (i,k) is the last column of M.

Now extend $B_1, \ldots, B_m$ to acyclic arc sets $B'_1, \ldots, B'_m$ of D' as follows. Keep all the arcs in $A' \cap B_s$, $s \in \{1, \ldots, m\}$. If $(i,k) \in B_s$ replace this arc by path (i,j), (j,k). If $(i,k) \notin B_s$ then add arc (j,k) to B_s. In addition we construct an arc set B'_{m+1} by taking any B_s, $s \in \{1, \ldots, n\}$, such that $(i,k) \notin B_s$ (since $a^{\mathrm{T}}x \leq a_0$ is nontrivial such a B_s exists) and adding the arc (i,j). It is obvious from the construction that the arc sets $B'_1, \ldots, B'_{m+1}$ are acyclic and that their incidence vectors satisfy $a'^{\mathrm{T}}x \leq a_0$ with equality.

Let M' denote the $(m+1, m+1)$-matrix whose rows are the incidence vectors of $B'_1, B'_2, \ldots, B'_{m+1}$ and assume that the last two columns of M' correspond to the arcs (i,j) and (j,k). It is clear from the construction that the submatrix of M' consisting of the first m rows and m columns is the matrix M. Moreover, the first m entries of the last column are equal to one, while the $(m+1)$-st entry equals zero.

Since M has rank m, the first m rows of M' are linearly independent. All row vectors of M' satisfy the equation $a'^{\mathrm{T}}x = a_0$ and the first m row vectors satisfy the equation $x_{jk}=1$. This equation is not satisfied by the last row of M'. This implies that the rows of M' are linearly independent which proves that $a'^{\mathrm{T}}x \leq a_0$ is a facet of D'. □

Actually, lemma (3.2) can also be used to prove theorem (3.1) in a less direct way.

(3.3.) Theorem. *Let $D'=(V',A')$ be a digraph and $D=(V,A)$ be a simple k-fence contained in D'. Then the simple k-fence inequality $x(A) \leq |A|-k+1$ defines a facet of $P_{\mathrm{AC}}(D')$.*

Proof. Let us denote the simple k-fence inequality for D by $a^{\mathrm{T}}x \leq \alpha$. Assume that $b^{\mathrm{T}}x \leq \beta$ is a facet defining inequality for $P_{\mathrm{AC}}(D')$ which has the following property. If a vertex $x \in P_{\mathrm{AC}}(D')$ satisfies $a^{\mathrm{T}}x = \alpha$ then x also satisfies $b^{\mathrm{T}}x = \beta$. If we can prove that $a = \gamma b$ for some $\gamma > 0$ then we know that $a^{\mathrm{T}}x \leq \alpha$ defines a facet for $P_{\mathrm{AC}}(D')$ and are done.

For ease of notation we may assume that the simple k-fence D is the simple k-fence induced by $U=\{1,2,\ldots,k\}$, $W=\{k+1,\ldots,2k\}$. In the proof of proposition (2.11) we have shown that the acyclic arc sets contained in A which have exactly $|A|-k+1$ arcs are those which contain either one pale and all pickets, or two pales, say $(i,k+i)$ and $(j,k+j)$, and all pickets except for one of the two pickets $(k+i,j)$, $(k+j,i)$.

Since $b^{\mathrm{T}}x \leq \beta$ defines a facet of $P_{\mathrm{AC}}(D')$ we have $b \geq 0$. Moreover, from the characterization of the acyclic arc sets in A containing $|A|-k+1$ arcs of A it is easy to deduce that $b_{i,k+i} > 0$ for all pales $(i,k+i)$. By multiplying $b^{\mathrm{T}}x \leq \beta$ with an appropriate constant $\gamma > 0$ we may therefore assume that

$$b_{1,k+1} = 1.$$

Let B_1, resp. B_2, be the acyclic arc sets containing pale $(1,k+1)$, resp. pale $(i,k+i)$, $2 \leq i \leq k$, and all pickets of D. Then the incidence vectors of B_1 and B_2

satisfy $a^Tx \leq \alpha$ and hence $b^Tx \leq \beta$ with equality. This implies $0 = \beta - \beta = b^Tx^{B_1} - b^Tx^{B_2} = b_{1,k+1} - b_{i,k+i}$, i.e.

$$b_{i,k+i} = 1 \quad \text{for } i = 1, \ldots, k.$$

Now let $(k+i, j)$, $1 \leq i, j \leq k$, $i \neq j$, be any picket. Let B_1 be the acyclic arc set containing the pales $(i, k+i)$, $(j, k+j)$ and all pickets except for $(k+i, j)$. Then the incidence vectors of B_1 and B_2 satisfy $a^Tx \leq \alpha$ with equality, and as above we have $0 = b^Tx^{B_1} - b^Tx^{B_2} = b_{j,k+j} - b_{k+i,j}$ which implies

$$b_{k+i,j} = 1 \quad \text{for all } 1 \leq i, j \leq k,\ i \neq j.$$

It is trivial to see that $b_{ij} = 0$ for all $(i,j) \in A'$ with $(i,j) \notin A$, and hence we have shown that $a = b$ holds which proves our theorem. □

We remark at this point that A. Schrijver proved nonintegrality of $P_{C}(D)$ by introducing a class of valid inequalities for $P_{AC}(D)$ whose minimal member is the simple 3-fence-inequality (cf. Kaas (1981)). However, this inequality is the only one in that class defining a facet of $P_{AC}(D)$.

(3.4) Theorem. *Let $D = (V, A)$ be a digraph and $D' = (V', A')$ be a k-fence contained in D. Then the k-fence inequality $x(A') \leq |A'| - k + 1$ defines a facet of $P_{AC}(D)$.*

Proof By definition D' can be obtained from a simple k-fence D'' by repeated subdivision of arcs. Since the simple k-fence inequality for D'' defines a facet of $P_{AC}(D'')$ repeated application of lemma (3.2) yields that the k-fence inequality for D' defines a facet of $P_{AC}(D')$.

We now have to show that for every arc $(i,j) \in A \setminus A'$ there exists an acyclic arc set $B_{ij} \subseteq A'$ containing $|A'| - k + 1$ arcs such that $B_{ij} \cup \{(i,j)\}$ is acyclic. This can be done by enumeration of several trivial cases.

Since $x(A') \leq |A'| - k + 1$ defines a facet of $P_{AC}(D')$ there are $|A'|$ acyclic arc sets in A' whose incidence vectors (in $\mathbb{R}^{A}$) are linearly independent and satisfy the k-fence inequality with equality. Similarly, each of the incidence vectors of the $|A| - |A'|$ acyclic arc sets $B_{ij} \cup \{(i,j)\}$ satisfies $x(A') \leq |A'| - k + 1$ with equality and—by construction—is linearly independent from all the other incidence vectors. This proves that the k-fence inequality defines a facet of $P_{AC}(D)$. □

(3.5) Theorem. *Let M be a Möbius-ladder defined by the dicycles $C_1, \ldots, C_k$ of $D = (V, A)$. Then the Möbius-ladder inequality $x(M) \leq |M| - (k+1)/2$ defines a facet of $P_{AC}(D)$.*

Proof. For convenience we set $a^Tx := x(M) \leq |M| - (k+1)/2 := \alpha$, and we want to show that $F_a := \{x \in P_{AC}(D) \mid a^Tx = \alpha\}$ is a facet of $P_{AC}(D)$.

Assume now that $b^Tx \leq \beta$ is valid with respect to $P_{AC}(D)$ and that $F_b := \{x \in P_{AC}(D) \mid b^Tx = \beta\}$ is a facet of $P_{AC}(D)$ with $F_a \subseteq F_b$. If we can show that $b^T = \gamma a^T$ for some $\gamma > 0$ we are done.

Since F_b is a facet we know that $b_{uv} \geq 0$ for all $(u, v) \in A$. Moreover, it is trivial to see that for at least one arc $(u, v) \in M$, $b_{uv} > 0$. Assume dicycle C_j, $1 \leq j \leq k$, consists of the arcs $f_1, f_2, \ldots, f_s$ and that $b_{f_1} = \gamma > 0$.

Let J be the index set (with respect to C) of the id . (2.16) of M, for every $i \in \{1, \ldots, k\}$ we choose any arc, say e_i, contained in P_i. By (2.16) $B = (\bigcup_{i=1}^{k} C_i) \setminus \{e_i \mid i \in J\}$ contains the dicycle C_j but no other dicycles. Therefore, the arc sets $B_r := B \setminus \{f_r\}$, $r = 1, \ldots, s$, are acyclic and their incidence vectors are contained in F_a. Hence they are also contained in F_b which for every $r \in \{2, \ldots, s\}$ implies

$$0 = \beta - \beta = b^Tx^{B_1} - b^Tx^{B_r} = b_{f_r} - b_{f_1} = \gamma - b_{f_r}.$$

From this we obtain $b_{uv} = \gamma$ for all arcs $(u, v) \in C_j$.

By our choice, the arc e_j is contained in C_j and C_{j+1}. Using the same argument as before we can show that $b_{uv} = \gamma$ for all (u, v) in C_{j+1}. And by iterating this procedure we get that $b_{uv} = \gamma$ for all $(u, v) \in M$.

We still have to show that for all arcs $(u, v) \in A \setminus M$, $b_{uv} = 0$ holds. This can be easily derived by using the arc sets B_r constructed above. Altogether we obtain $b = \gamma a$ which proves our claim. □

The results of this section show that for every digraph $D = (V, A)$ we have the following inclusion:

$$P_{AC}(D) \subseteq P := \Big\{ x \in \mathbb{R}^A \mid 0 \leq x_{ij} \leq 1,$$
$$x(C) \leq |C| - 1 \quad \text{for all dicycles } C \text{ in } D,$$
$$x(A') \leq |A'| - k + 1 \quad \text{for all } k\text{-fences } (V', A') \text{ in } D,$$
$$x(M) \leq |M| - \frac{k+1}{2} \quad \text{for all Möbius-ladders } M \text{ in } D \Big\}. \tag{3.6}$$

Moreover, we know that all facets of P are also facets of $P_{AC}(D)$.

From an algorithmic point of view it is important to know whether the facet separation problem for the classes of facets defined above can be solved in polynomial time, i.e. whether the problem, given $y \in \mathbb{Q}^A$, is there a facet in the class which is violated by y, is polynomially solvable. We do not know how to handle the class of simple k-fence inequalities, nor what to do with Möbius-ladders inequalities, but we shall show in Section 5 how violated dicycle inequalities can be recognized in polynomial time.

4. Weakly acyclic digraphs

We now introduce a class of digraphs $D = (V, A)$ for which we shall show that the acyclic subgraph problem for these graphs can be solved in polynomial time.

The definition of these digraphs is based on the polytope $P_C(D)$, cf. (2.7), which the intersection of the half-spaces given by

$$0 \leq x_{ij} \leq 1 \quad \text{for all } (i,j) \in A, \tag{4.1}$$

$$x(C) \leq |C| - 1 \quad \text{for all dicycles } C \text{ in } D. \tag{4.2}$$

(4.3) Definition. A digraph $D=(V,A)$ is called *weakly acyclic* if the acyclic subgraph polytope $P_{AC}(D)$ equals $P_C(D)$. Digraphs which are not weakly acyclic are called *strongly cyclic.* □

It is clear that acylic digraphs are weakly acyclic, since obviously a digraph D is acyclic if and only if $P_{AC}(D)$ is the unit hypercube. Since the k-fence inequality $x(A) \leq |A| - k + 1$ defines a facet of $P_{AC}(D)$ for $D=(V,A)$, no k-fence can be weakly acyclic. Moreover, theorem (3.4) implies that every digraph containing a k-fence is strongly cyclic.

On the other hand, if D is weakly acyclic then any digraph obtained from D by adding a source or a sink is weakly acyclic, more generally, if D' and D'' are two node-disjoint weakly acyclic digraphs and we create a new digraph D from D' and D'' by adding some arcs going from a node in D' to a node in D'', then D is weakly acyclic. Similarly, if D' and D'' are node disjoint and weakly acyclic then the digraph obtained by identifying a node in D' and a node in D'' is weakly acyclic.

A particular interesting class of weakly acyclic digraphs is given by the following observation.

(4.4) Remark. Planar digraphs are weakly acyclic.

Proof. In view of (2.8) we have to show that all vertices of the polyhedron $P_C(D)$ for a planar digraph D are integral.

Consider the polyhedron

$$\bar{P}_C(D) := \{y \in \mathbb{R}^A \mid 0 \leq y_{ij} \leq 1,\ y(C) \geq 1 \text{ for all dicycles } C \text{ in } D\}.$$

$\bar{P}_C(D)$ can be obtained from $P_C(D)$ by making the variable substitution $y_{ij} := 1 - x_{ij}$ for all $(i,j) \in A$. This implies that every vertex of $P_C(D)$ corresponds to a vertex of $\bar{P}_C(D)$ and vice versa. The planar version of the theorem of Lucchesi and Younger (1978) states that for planar digraphs $\bar{P}_C(D)$ is integral. This implies that $P_C(D)$ has integral vertices only. □

A direct consequence of (4.4) is the following observation.

(4.5) Corollary. *Let $a^{\mathrm{T}}x \leq a_0$ define a nontrivial facet of $P_{AC}(D)$ for some digraph $D=(V,A)$. Let $D'=(V',A')$ be the subdigraph of D defined by $A' := \{(i,j) \in A \mid a_{ij} > 0\}$. Then D' is nonplanar or a dicycle.* □

The study of weakly dicyclic digraphs is still in its infancy. We know some further weakly acyclic digraphs, but no 'nice' classes. Also, there is not much known about minimally strongly cyclic digraphs, i.e. digraphs which are strongly cyclic but where removal of any arc results in a weakly acyclic digraph. One example of a minimally strongly cyclic digraph is the simple 3-fence. It is also easy to see that every digraph that can be obtained from a simple 3-fence by subdivision of arcs is minimally strongly cyclic. i.e. all 3-fences are digraphs of this kind. Thus, there are minimally strongly cyclic digraphs of order n for all $n \geq 6$.

5. Polynomial solvability of the acyclic subgraph problem for weakly acyclic digraphs

We shall now show that for any digraph $D=(V,A)$ the linear program

$$\begin{aligned} &\max c^{\mathrm{T}}y, \\ &x(C) \leq |C| - 1 \quad \text{for all dicycles } C \text{ in } D, \\ &0 \leq x_{ij} \leq 1 \quad \text{for all } (i,j) \in A \end{aligned} \tag{5.1}$$

can be solved in polynomial time, in spite of the fact that (5.1) may have a number of inequalities exponential in the order of D. We shall make use of the ellipsoid method as described in Grötschel, Lovász and Schrijver (1981).

In order to show the polynomial solvability of (5.1) it is sufficient to solve the separation problem for $P_C(D)$ in polynomial time. This can be stated as follows

(5.2) Separation problem for $P_C(D)$. *Given a point $y \in \mathbb{Q}^A$. Determine whether $y \in P_C(D)$ and if not find a vector $d \in \mathbb{Q}^A$ such that $d^{\mathrm{T}}y > d^{\mathrm{T}}x$ for all $x \in P_C(D)$ (a separating hyperplane).*

Clearly, if $y \in \mathbb{Q}^A$ is given one can check by substitition whether y satisfies the trivial inequalities $0 \leq x_{ij} \leq 1$. Hence, if one of these is violated we have found a separating hyperplane.

For our further discussion we may therefore assume that the given $y \in \mathbb{Q}^A$ satisfies $0 \leq y_{ij} \leq 1$ for all $(i,j) \in A$.

For every arc in A we define a 'weight' $w_{ij} := 1 - y_{ij}$. If C is any dicycle in D, then clearly $y(C) \leq |C| - 1$ if and only if $w(C) \geq 1$. This implies that we can check whether y violates a dicycle inequality by finding a dicycle C^* whose weight $w(C^*)$ is minimum. Namely, if the minimum weight $w(C^*)$ satisfies $w(C^*) \geq 1$ then all dicycle inequalities $x(C) \leq |C| - 1$ are satisfied by y, if $w(C^*) < 1$, then $y(C^*) > |C^*| - 1$ and hence a separating hyperplane is found.

What remains to be shown is that, given a digraph $D=(V,A)$ with arc weights w_{ij} for all $(i,j) \in A$, a shortest dicycle under w can be found in polynomial time. But this is easy by making appropriate modifications of any polynomial time shortest dipath algorithm (like the Dijkstra or Floyd-Warshall method). Hence we have proved:

(5.3) Theorem. *There is an algorithm which for any digraph $D = (V, A)$ and any objective function $c \in \mathbb{Q}^A$ solves the linear program max $c^T x$, $x \in P_{C}(D)$ in polynomial time.* □

The version of the ellipsoid method described in Grötschel, Lovász and Schrijver (1981) in fact finds an optimum vertex solution of (5.1). So in case $P_{AC}(D) = P_C(D)$ the incidence vector of an optimum acyclic arc set is found and we get:

(5.4) Theorem. *The acyclic subgraph problem for weakly acyclic diagraphs can be solved in polynomial time.* □

Lucchesi (1976) has designed a polynomial time algorithm to solve the acyclic subgraph problem for planar digraphs; since by (4.4) planar digraphs are weakly acyclic, theorem (5.4) generalizes this result.

In fact, the linear program max $c^T x$, $x \in P_C(D)$ can be used as a linear relaxation of the acyclic subgraph problem for general digraphs in the framework of a branch and bound algorithm. We have investigated this approach with respect to the triangulation of input-output matrices. This method turns out to be quite successful, and our results in this regard is the subject of (Grötschel, Jünger, Reinelt (1984a, b, 1985)).

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FACETS OF THE LINEAR ORDERING POLYTOPE

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Let D_n be the complete digraph on n nodes, and let P^n_{LO} denote the convex hull of all incidence vectors of arc sets of linear orderings of the nodes of D_n (i.e. these are exactly the acyclic tournaments of D_n). We show that various classes of inequalities define facets of P^n_{LO}, e.g. the 3-dicycle inequalities, the simple k-fence inequalities and various Möbius ladder inequalities, and we discuss the use of these inequalities in cutting plane approaches to the triangulation problem of input-output matrices, i.e. the solution of permutation resp. linear ordering problems.

Key words: Facets of Polyhedra, Linear Ordering Problem, Triangulation Problem, Permutation Problem.

1. Introduction and notation

This paper is a continuation of our paper Grötschel, Jünger and Reinelt (1985) on the acyclic subgraph polytope. The polytope associated with linear orderings is a face of the acyclic subgraph polytope. Our main objective is to investigate which of the inequalities shown to define facets of the acyclic subgraph polytope in our former paper also define facets of the linear ordering polytope. We adopt the notations in graph theory and polyhedral theory of that paper.

A *linear ordering* (or *permutation*) of a finite set V with $|V| = n$ is a bijective mapping $\sigma: \{1, 2, \ldots, n\} \to V$. For $u, v \in V$ we say that u is 'better than' or 'before' v if $\sigma^{-1}(u) < \sigma^{-1}(v)$. Among all possible linear orderings of V we want to find a linear ordering which is the best according to some criterion. In many applications a 'value' or a 'cost' can be associated with a linear ordering in the following way. For every two elements $u, v \in V$ a value c_{uv} and a value c_{vu} are given which can be interpreted as the profit we obtain from having u 'before' v resp. v 'before' u in a linear ordering. Then the total value of a linear ordering clearly is given by

$$\sum_{\sigma^{-1}(u) < \sigma^{-1}(v)} c_{uv}.$$

Given a linear ordering of the nodes V of a digraph then the arc set $\{(u, v) | \sigma^{-1}(u) < \sigma^{-1}(v)\}$ forms an acyclic tournament on V, and similarly, if (V, T) is an acyclic tournament then this induces a linear ordering of V. Using this graph theoretical interpretation we can state the ***linear ordering problem*** as follows.