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Routing in grid graphs by cutting planes

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In this paper we study the following problem, which we call the weighted routing problem. Let be given a graph $G = (V, E)$ with non-negative edge weights $w_e \in \mathbb{R}_+$ and integer edge capacities $c_e \in \mathbb{N}$ and let $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, be a list of node sets. The weighted routing problem consists in finding edge sets $S_1, \ldots, S_N$ such that, for each $k \in \{1, \ldots, N\}$, the subgraph $(V(S_k), S_k)$ contains an $[s,t]$-path for all $s, t \in T_k$, at most c_e of these edge sets use edge e for each $e \in E$, and such that the sum of the weights of the edge sets is minimal. Our motivation for studying this problem arises from the routing problem in VLSI-design, where given sets of points have to be connected by wires. We consider the weighted routing problem from a polyhedral point of view. We define an appropriate polyhedron and try to (partially) describe this polyhedron by means of inequalities. We briefly sketch our separation algorithms for some of the presented classes of inequalities. Based on these separation routines we have implemented a branch and cut algorithm. Our algorithm is applicable to an important subclass of routing problems arising in VLSI-design, namely to problems where the underlying graph is a grid graph and the list of node sets is located on the outer face of the grid. We report on our computational experience with this class of problem instances.

Key words: Routing in VLSI-design, Steiner tree, Steiner tree packing, cutting plane algorithm.

1. Introduction

One of the main topics in VLSI-design is the routing problem. Roughly described, the task is to connect so-called terminal sets via wires on a predefined area. In addition, certain design rules are to be taken into account and an objective function like the wiring length must be minimized. The routing problem in general is too complex to be solved in one step. Depending on the user's choice of decomposing the chip design problem into a hierarchy of stages, on the underlying technology, and on the given design rules, various subproblems arise. Many of the routing problems that come up this way can be formulated in graphtheoretical terms as follows:

Problem 1.1. (The Weighted Routing Problem)

Instance:

A graph $G = (V, E)$ with positive, integer edge capacities $c_e \in \mathbb{N}$ and non-negative edge weights $w_e \in \mathbb{R}_+$, $e \in E$.
A list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, with $T_k \subseteq V$ for all $k = 1, \ldots, N$.

Problem:

Find edge sets $S_1, \ldots, S_N \subseteq E$ such that

(i) $(V(S_k), S_k)$ *contains an* $[s,t]$*-path for all* $s,t \in T_k$ *for* $k = 1,\ldots,N$ *(where* $V(F)$ *is the set of nodes that are incident to an edge of* $F \subseteq E$*),*

(ii) $\sum_{k=1}^{N} |S_k \cap \{e\}| \leq c_e$ *for all* $e \in E$*,*

(iii) $\sum_{k=1}^{N} \sum_{e \in S_k} w_e$ *is minimal.*

We call the list of node sets $\mathcal{N}$ a *net list.* Any element $T_k \in \mathcal{N}$ is called a *set of terminals* or a *net* and the nodes $t \in T_k$ are called *terminals.* It is also customary to say *net k* instead of terminal set T_k. An edge set S that satisfies condition (i) for a terminal set T is called a *Steiner tree in G for T.* A N-tupel of edge sets $(S_1,\ldots,S_N)$ that satisfies (i) and (ii) is called a *routing* or a *Steiner tree packing.* If we are only interested in finding a feasible solution, i. e., we neglect condition (iii), we speak of the routing problem without the prefix "weighted".

Of particular interest in VLSI-Design are routing problems where the underlying graph is a grid graph. Among these are the channel routing and the switchbox routing problem. In these two cases, the graph is a complete rectangular grid, the edge capacities are equal to one and the terminal sets are located on the outer face of the grid. In the channel routing problem the terminal sets are restricted to lie on two opposite sides of the graph, whereas in the switchbox routing problem terminals may be located on all four sides.

It is not surprising that Problem 1.1 is $\mathcal{NP}$-complete or $\mathcal{NP}$-hard, respectively, even in many special cases. Among them are the minimal Steiner tree problem ([10], [4]) and the problem of packing N disjoint paths in a planar graph ([11]). Even the channel routing and the switchbox routing problem are $\mathcal{NP}$-complete ([16]).

We attack the (weighted) routing problem by using a polyhedral approach. In section 2, we define a polyhedron whose vertices are in one-to-one correspondence to the routings in the graph, and we try to describe this polyhedron by means of equations and inequalities. Section 3 deals with the separation problem for some classes of inequalities that are described in the previous section. Finally, in section 4 we report on some computational results we have obtained with our cutting plane algorithm. The test problems are switchbox routing problems discussed in the literature.

2. The routing polyhedron

In this section we define the routing polyhedron and describe some classes of valid and facet-defining inequalities. First, we introduce some notation.

We denote by $\mathbb{R}^E$ the vector space where the components of each vector are indexed by the elements of E, i. e., $x = (x_e)_{e\in E}$ for $x \in \mathbb{R}^E$. For an edge set $F \subseteq E$, we define the incidence vector $\chi^F \in \mathbb{R}^E$ by setting $\chi^F_e = 1$, if $e \in F$, and $\chi^F_e = 0$, otherwise. Furthermore, we abbreviate $\sum_{e\in F} x_e$ by $x(F)$ for an edge set F and a vector $x \in \mathbb{R}^E$. We denote by $\mathbb{R}^{\mathcal{N}\times E}$ the $N \cdot |E|$ – dimensional vector space $\mathbb{R}^E \times \ldots \times \mathbb{R}^E$. The components of a vector $x \in \mathbb{R}^{\mathcal{N}\times E}$ are indexed by x^k_e for $k \in \{1,\ldots,N\}$, $e \in E$. For a vector $x \in \mathbb{R}^{\mathcal{N}\times E}$ and $k \in \{1,\ldots,N\}$ we denote by $x^k \in \mathbb{R}^E$ the vector $(x^k_e)_{e\in E}$. If it is clear from the context we will abbreviate a vector $x = ((x^1)^T,\ldots,(x^N)^T)^T$ by $(x^1,\ldots,x^N)$. By the *incidence vector of a routing* $P = (S_1,\ldots,S_N)$ we mean the vector $(\chi^{S_1},\ldots,\chi^{S_N})$ or in short χ^P.

We define now the *routing polyhedron* (also called the *Steiner tree packing polyhedron*) by

$$\begin{aligned}
\mathrm{STP}(G,\mathcal{N},c) := \mathrm{conv}\{\,(x^1,\ldots,x^N) \in \mathbb{R}^{\mathcal{N}\times E}\ |& \\
\text{(i)}\quad & \sum_{e\in\delta(W)} x_e^k \geq 1, && \text{for all } W \subset V,\ W\cap T_k \neq \emptyset, \\
& && (V\setminus W)\cap T_k \neq \emptyset,\ k=1,\ldots,N; \\
\text{(ii)}\quad & \sum_{k=1}^{N} x_e^k \leq c_e, && \text{for all } e\in E; \\
\text{(iii)}\quad & 0 \leq x_e^k \leq 1, && \text{for all } e\in E,\ k=1,\ldots,N; \\
\text{(iv)}\quad & x_e^k \in \{0,1\}, && \text{for all } e\in E,\ k=1,\ldots,N\},
\end{aligned} \tag{2.1}$$

where $\delta(W)$ in (2.1) (i) denotes the set of all edges with exactly one endnode in W. The inequalities (2.1) (ii) are called the *capacity inequalities* and the ones in (2.1) (iii) the *trivial inequalities.* If $N = 1$ we refer to $\mathrm{STP}(G,\mathcal{N},c)$ as the *Steiner tree polyhedron.* Obviously, each incidence vector of a routing satisfies (2.1) (i) – (iv), and vice versa, it is easy to see that each vector $x \in \mathbb{R}^{\mathcal{N}\times E}$ satisfying (2.1) (i) – (iv) is the incidence vector of a routing. Thus, the weighted routing problem reduces to the linear program $\min\{\sum_{k=1}^{N} w^T x^k \mid x \in \mathrm{STP}(G,\mathcal{N},c)\}$.

In order to apply linear programming techniques, a "good" description of the routing polyhedron by means of equations and inequalities is indespensible. To this end we must determine the dimension of the routing polyhedron. Unfortunately, this problem is $\mathcal{NP}$-complete, even for switchbox routing problems. This follows from the fact that the decision problem, "Does there exist a routing for a given instance $(G,\mathcal{N},c)$?", is $\mathcal{NP}$-complete (see [11], [16]).

Thus, we have decided to study the routing polyhedron for special problem instances for which the dimension can easily be determined and to look for facet-defining inequalities for these special instances. Clearly, such an approach is only sensible if the results can be carried over (at least partially) to practically interesting instances like switchbox routing problems.

For example, an instance $(G,\mathcal{N},c)$, where the graph G is complete, the net list $\mathcal{N} = \{T_1,\ldots,T_N\}$ is *disjoint* (that is $T_i \cap T_j = \emptyset$ for $i \neq j$) and the capacities are equal to one ($c = \mathbb{1}$), is an appropriate case. It can easily be verified that the corresponding routing polytope $\mathrm{STP}(G,\mathcal{N},\mathbb{1})$ is fulldimensional in this case. By applying the subsequent two lemmas we can transform any given valid (resp. facet-defining) inequality for this polytope to a valid inequality of the routing polytope corresponding to, for example, the switchbox routing problem.

Lemma 2.2. (Deletion of an edge)

Let $(G,\mathcal{N},c)$ be an instance of the routing problem. Let $a^Tx \geq \alpha$ be a valid inequality for $STP(G,\mathcal{N},c)$ and let us delete $f \in E$ from G. Then $\hat{a}^Tx \geq \alpha$ is a valid inequality for $STP(G\setminus f,\mathcal{N},c)$ where $\hat{a}_e^k = a_e^k$ for all $e \in E\setminus\{f\}$, $k \in \{1,\ldots,N\}$ (where $G\setminus f$ denotes the graph that is obtained by deleting edge f).

Lemma 2.3. (Splitting a node)

Let $(G,\mathcal{N},c)$ be an instance of the routing problem. Let $f \in E$ with $c_f = 1$ and let $\hat{a}^Tx \geq \alpha$ be a valid inequality for $STP(G\,/\,f,\mathcal{N},c)$ (where $G\,/\,f$ denotes the graph that is obtained by shrinking edge f). Then, $a^Tx \geq \alpha$ defines a valid inequality for $STP(G,\mathcal{N},c)$ with $a_e^k = \hat{a}_e^k$ for all $e \in E\setminus\{f\}$, $k \in \{1,\ldots,N\}$ and $a_f^k = 0$ for all $k = 1,\ldots,N$.

Lemma 2.2 follows from the fact that every routing of $(G\setminus f,\mathcal{N},c)$ is also a routing of $(G,\mathcal{N},c)$. A similar argument proves Lemma 2.3.

Let us now describe some facet-defining inequalities for $\mathrm{STP}(G,\mathcal{N},c)$. The first two theorems concern instances, where the graph is complete and the net list is disjoint. Afterwards we describe a class of facet-defining inequalities for the routing polytope of an instance that is strongly related to grid graph routing problems. The inequality stated in the last theorem is based on a condition that is necessary for the existence of a routing.

The first question that arises is: "Can the facet-defining inequalities for the Steiner tree polyhedron be extended to facet-defining inequalities of the routing polyhedron?" The following theorem gives an answer to this question.

Theorem 2.4.
Let $G=(V,E)$ be the complete graph with node set V and let $\mathcal{N}=\{T_1,\ldots,T_N\}$ be a disjoint net list. Let $\bar{a}^T x \geq \alpha$, $\bar{a} \in \mathbb{R}^E$, be a non-trivial facet-defining inequality for $STP(G,\{T_1\},\mathbb{1})$. Then, $a^T x \geq \alpha$ defines a facet for $STP(G,\mathcal{N},\mathbb{1})$, where $a \in \mathbb{R}^{\mathcal{N}\times E}$ is defined by $a_e^1 = \bar{a}_e$, $a_e^k = 0$ for all $k=2,\ldots,N$, $e \in E$.

For a proof of this theorem we refer to [7].

Next, we consider inequalities that combine two or more nets. We call such inequalities *joint.*

First of all, let us fix some notation. For two node sets $U, W \subseteq V$, we denote by $[U:W]$ all edges with one endpoint in U and one in W. Furthermore, $E(W)$ denotes all edges with both endpoints in W. For a cycle F, an edge uv is called a *diagonal* if $u,v \in V(F)$ and $uv \notin F$.

Let $G=(V,E)$ be a graph and $\mathcal{N}=\{T_1,T_2\}$ a net list. We call a cycle F an *alternating cycle with respect to* T_1,T_2 if $F \subseteq [T_1:T_2]$ and $V(F)\cap T_1 \cap T_2 = \emptyset$, see Figure 2.1. Moreover, let $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ be two sets of diagonals of the alternating cycle F with respect to T_1,T_2. The inequality

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq \tfrac{1}{2}|F| - 1$$

is called an *alternating cycle inequality*.

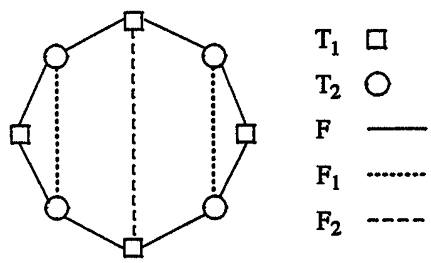

Fig. 1.

It is not difficult to see that the basic form of an alternating cycle inequality, i. e., $F_1 = F_2 = \emptyset$, is valid for STP $(G,\mathcal{N},\mathbb{1})$. The following theorem states conditions under which the alternating cycle inequality is facet-defining.

Here, we need some additional notation. Let F be a cycle. We say two diagonals uv and $u'v'$ *cross with respect to cycle* F if the endnodes appear in the sequence u,u',v,v' or u,v',v,u' by walking around the cycle. Two sets of diagonals F_1 and F_2 are *cross free* if for all $e_1 \in F_1$ and $e_2 \in F_2$ the edges e_1 and e_2 do not cross. Let $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ be two sets of diagonals of an alternating cycle F with respect to T_1,T_2. F_1 and F_2 are called *maximal cross free with respect to* F, if F_1 and F_2 are cross free and each diagonal $e_1 \in E(T_1)\setminus F_2$ crosses F_1 and each diagonal $e_2 \in E(T_2)\setminus F_1$ crosses F_2. Figure 2.1 shows an alternating cycle F with two maximal cross free sets of diagonals F_1 and F_2. Now, we can state the theorem.

Theorem 2.5.

Let $G = (V, E)$ be the complete graph with node set V and let $\mathcal{N} = \{T_1, T_2\}$ be a disjoint net list with $T_1 \cup T_2 = V$ and $|T_1| = |T_2| = l$, $l \geq 2$. Furthermore, let F be an alternating cycle with respect to T_1, T_2 such that $V(F) = V$ and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. Then the alternating cycle inequality

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq l - 1$$

defines a facet for $STP(G, \mathcal{N}, \mathbb{I})$ if and only if F_1 and F_2 are maximal cross free.

Again, a proof of Theorem 2.5 can be found in [7].

Next, we introduce the so-called grid inequalities. Let $G = (V, E)$ be a graph and $\mathcal{N} = \{T_1, T_2\}$ be a net list. Furthermore, let $\hat{G} = (\hat{V}, \hat{E})$ be a subgraph of G such that $\hat{G}$ is a complete $h \times 2$ grid graph with $h \geq 3$ (where a complete $h \times b$ grid graph is a grid graph with h rows and b columns). Assume that the nodes of V are numbered such that $\hat{V} = \{(i, j) \mid i = 1, \ldots, h, j = 1, 2\}$. Moreover, let $(1, 1), (h, 2) \in T_1$ and $(1, 2), (h, 1) \in T_2$. We call the inequality

$$(\chi^{E\setminus\hat{E}}, \chi^{E\setminus\hat{E}})^T x \geq 1$$

a $h \times 2$ *grid inequality.*

Let $G = (V, E)$ be a graph, $F \subset E$ and $u, v \in V$. We call a path $Q_F(u, v)$ from u to v in G a *quasi path from u to v with respect to F*, if there exists an edge $e \in Q_F(u, v)$ such that $e \in E \setminus F$ and $Q_F(u, v) \setminus \{e\} \subset F$. Then, the following theorem holds.

Theorem 2.6.

Let $\hat{G} = (\hat{V}, \hat{E})$ be a complete $h \times 2$ grid graph with $h \geq 3$. Let $\mathcal{N} = \{T_1, T_2\}$ be a net list where $T_1 = \{(1, 1), (h, 2)\}$ and $T_2 = \{(1, 2), (h, 1)\}$. Furthermore, let $G = (V, E)$ be a graph with $\hat{V} \subset V$, $\hat{E} \subset E$ such that the set of horizontal edges in $\hat{G}$, i. e., $\{[\{(i, 1)\} : \{(i, 2)\}] \mid i = 1, \ldots, h\}$, is a cut in G. Set $F = \hat{E}$ and let $F_1, F_2 \subset E \setminus F$, then the following holds. $STP(G, \mathcal{N}, \mathbb{I})$ is fulldimensional and the inequality

$$(\chi^{E\setminus(F\cup F_1)}, \chi^{E\setminus(F\cup F_2)})^T x \geq 1$$

defines a facet for $STP(G, \mathcal{N}, \mathbb{I})$ if and only if F_1 and F_2 satisfy the following properties (see Figure 2.2):

(i) *For all $i \in \{1, \ldots, h\}$ (at least) one of the following conditions is fullfilled in $(V, E \setminus F)$:*
  - (a) *There exist an index $k \in \{1, 2\}$, nodes $(r, l), (s, l) \in V$ with $r, s \in \{1, \ldots, h\}, l \in \{1, 2\}$ and a quasi path $Q_{F_k}((r, l), (s, l))$ such that $r \leq i - |k - l|$ and $s \geq i + 2 - |k - l|$ holds.*
  - (b) *The subsequent three requirements are satisfied:*
    - *There exist an index $k_1 \in \{1, 2\}$, nodes $(r_1, 1), (s_1, 1) \in V$ with $r_1, s_1 \in \{1, \ldots, h\}$ and a quasi path $Q_{F_{k_1}}((r_1, 1), (s_1, 1))$ such that $r_1 \leq i < i + 1 \leq s_1$ holds.*
    - *There exist an index $k_2 \in \{1, 2\}$, nodes $(r_2, 2), (s_2, 2) \in V$ with $r_2, s_2 \in \{1, \ldots, h\}$ and a quasi path $Q_{F_{k_2}}((r_2, 2), (s_2, 2))$ such that $r_2 \leq i < i + 1 \leq s_2$ holds.*
    - *There exist an index $k \in \{1, 2\}$, nodes $(r, l), (s, l) \in V$ where $r, s \in \{1, \ldots, h\}, r < s$, $l \in \{1, 2\}$ and a quasi path $Q_{F_k}((r, l), (s, l))$ with the additional properties:*

      $s - r \geq 2$, *if* $r \in \{i - 1, i\}$,

$r \neq i$, *if* $k \neq l$,
$s \neq i$, *if* $k = l$.

(ii) $\bigcap_{Q \in \mathcal{Q}} Q = \emptyset$, *where* $\mathcal{Q} = \{Q \subset E \mid$ *there exist an index* $k \in \{1,2\}$ *and nodes* $u, v \in V$ *such that* Q *is a quasi path from* u *to* v *in* $(V, E \setminus F)$ *with respect to* $F_k\}$.

(iii) *For all* $u, v \in V(F)$, $u \neq v$, *and* $k \in \{1,2\}$ *there does not exist a path from* u *to* v *in* (V, F_k).

(iv) F_1 *and* F_2 *are maximal with respect to the properties (i) – (iii).*

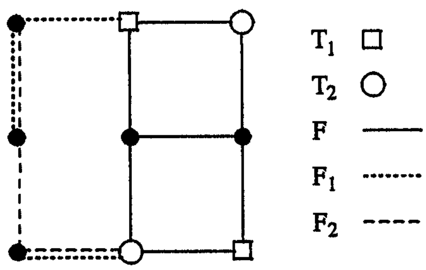

Fig. 2.

Proof.

For ease of exposition we introduce the following notation. For an edge $uv \in E$, we also use the symbol $[u,v]$. Let C denote the set of horizontal edges in $\hat{G}$, i. e., $C := \{[(i,1),(i,2)] \mid i = 1,\ldots,h\}$. For indices $k \in \{1,2\}$, $r, s \in \{1,\ldots,h\}$, we set $Q_k^+(r,s) = Q_{F_k}((r,2),(s,2))$ and $Q_k^-(r,s) = Q_{F_k}((r,1),(s,1))$. Let be given indices $s \in \{1,2\}$, $i_0, i_1 \in \{1,\ldots,h\}, i_0 \leq i_1$. $J_s^{i_0,i_1}$ denotes all vertical edges in G between nodes (i_0,s) and (i_1,s), i. e., $J_s^{i_0,i_1} := \{[(i,s),(i+1,s)] \mid i = i_0,\ldots,i_1-1\}$. Moreover, set $S_1^i = J_1^{1,i} \cup \{[(i,1),(i,2)]\} \cup J_2^{i,h}$ and $S_2^i = J_2^{1,i} \cup \{[(i,2),(i,1)]\} \cup J_1^{i,h}$ for $i = 1,\ldots,h$. Finally, for a routing $P = (S_1, S_2)$ and an edge $e \in E$, we designate $(S_1 \cup \{e\}, S_2)$ by $P \cup_1 e$ and $(S_1, S_2 \cup \{e\})$ by $P \cup_2 e$. If $e \in S_1 \cup S_2$, we simply write $e \in P$.

Let us start by proving that properties (i) to (iv) imply that the $h \times 2$ grid inequality defines a facet for STP $(G, \mathcal{N}, \mathbb{1})$ and that the routing polyhedron is fulldimensional.

The validity of $a^T x \geq 1$ with $a = (\chi^{E \setminus (F \cup F_1)}, \chi^{E \setminus (F \cup F_2)})$ is easy to see. Obviously, there does not exist a routing in $(V(F), F)$, since all nodes of $V(F)$ have at most degree three with respect to F and all terminals have degree two with respect to F. This together with property (iii) implies that the inequality is valid.

Now let $b^T x \geq \beta$ be a facet-defining inequality of STP $(G, \mathcal{N}, \mathbb{1})$ with $F_a := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^T x = 1\} \subseteq F_b := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^T x = \beta\}$. In the following we show that b is a multiple of a.

(1) $b_e^k = 0$ *for all* $e \in F_k$, $k = 1, 2$.

Due to (ii) there exists a quasi path $\bar{Q}$ in $(V, E \setminus F)$ with $e \notin \bar{Q}$. W. l. o. g. let $\bar{Q}$ be a quasi path with respect to F_1. Since C is a cut in G, we know that $\bar{Q} = Q_1^+(r,s)$ or $\bar{Q} = Q_1^-(r,s)$ with $r, s \in \{1,\ldots,h\}, r < s$. We consider the case $\bar{Q} = Q_1^+(r,s)$ (the other case can be shown analogously). Set $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1, S_2)$ and $P' = P \cup_k e$ are routings with $\chi^{P'}, \chi^P \in F_a$, and we obtain that $0 = b^T \chi^{P'} - b^T \chi^P = b_e^k$.

(2) $b_e^k = 0$ *for* $e \in F$, $k = 1, 2$.

First, let us note that, for a given $i \in \{1,\ldots,h\}$, property (a) in (i) is obviously equivalent to the property:

- There exist indices $r,s \in \{1,\ldots,h\}$ and a quasi path $Q_2^+(r,s)$ with $r \le i < i+1 < s$ or
- there exist indices $r,s \in \{1,\ldots,h\}$ and a quasi path $Q_2^-(r,s)$ with $r < i < i+1 \le s$ or
- there exist indices $r,s \in \{1,\ldots,h\}$ and a quasi path $Q_1^-(r,s)$ with $r \le i < i+1 < s$ or
- there exist indices $r,s \in \{1,\ldots,h\}$ and a quasi path $Q_1^+(r,s)$ with $r < i < i+1 \le s$.

Depending on edge e we distinguish the following cases.

(α) $e = [(i,1),(i+1,1)]$ *with* $i \in \{1,\ldots,h-1\}$. *Property (i) guarantees that one of the following quasi paths exists.*

- $Q_2^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^i$ and $S_2 = S_2^s \cup Q_2^+(r,s) \setminus J_2^{r,s}$.
- $Q_2^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^i$ and $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$.
- $Q_1^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^s \cup Q_1^-(r,s) \setminus J_1^{r,s}$ and $S_2 = S_2^{i+1}$.
- $Q_1^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^{i+1}$.
- $Q_k^+(r,s)$ with $k \in \{1,2\}$, $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 \le s$.
  If $k=1$, choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^s$. Otherwise, choose $S_1 = S_1^r$ and $S_2 = S_2^s \cup Q_2^+(r,s) \setminus J_2^{r,s}$.

(β) $e = [(i,2),(i+1,2)]$ *with* $i \in \{1,\ldots,h-1\}$. *Property (i) implies that one of the following quasi paths exists.*

- $Q_2^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^{i+1}$ and $S_2 = S_2^s \cup Q_2^+(r,s) \setminus J_2^{r,s}$.
- $Q_2^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^{i+1}$ and $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$.
- $Q_1^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^s \cup Q_1^-(r,s) \setminus J_1^{r,s}$ and $S_2 = S_2^i$.
- $Q_1^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^i$.
- $Q_k^-(r,s)$ with $k \in \{1,2\}$, $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 \le s$.
  If $k=1$, choose $S_1 = S_1^s \cup Q_1^-(r,s) \setminus J_1^{r,s}$ and $S_2 = S_2^r$. Otherwise, choose $S_1 = S_1^s$ and $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$.

(γ) $e = [(i,1),(i,2)]$ *with* $i \in \{1,\ldots,h-1\}$. *From property (i) we know that one of the following quasi paths exists.*

- $Q_2^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^{i+1}$ and $S_2 = S_2^s \cup Q_2^+(r,s) \setminus J_2^{r,s}$.
- $Q_2^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^{i+1}$ and $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$.
- $Q_1^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r \le i < i+1 < s$.
  Choose $S_1 = S_1^s \cup Q_1^-(r,s) \setminus J_1^{r,s}$ and $S_2 = S_2^{i+1}$.

- $Q_1^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^{i+1}$.
- $Q_{F_k}((r,l),(s,l))$ with $k,l \in \{1,2\}, r,s \in \{1,\ldots,h\}, r < s$ such that $s - r \ge 2$, if $r \in \{i-1,i\}$, and $r \ne i$, if $k \ne l$, and $s \ne i$, if $k = l$.
  First, we consider the case $r \notin \{i-1,i\}$. If $s \ne i$, choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^s$, if $k = 1$ and $l = 2$. In the other cases ($k = l = 1, k = l = 2$ and $k = 2, l = 1$) Steiner trees S_1 and S_2 can be chosen similarly. If $s = i$, we have $k \ne l$. Choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^{s-1}$, if $k = 1, l = 2$, otherwise set $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$ and $S_1 = S_1^{s-1}$. If $r = i-1$, we know that $s \ge i+1$. In accordance to $r \notin \{i-1,i\}$ and $s \ne i$ we can choose appropriate Steiner trees S_1 and S_2 in this case as well. If $r = i$, we have $s \ge i+2$ and $k = l$. If $k = 1$, choose $S_1 = S_1^s \cup Q_1^-(r,s) \setminus J_1^{r,s}$ and $S_2 = S_2^{s-1}$. Otherwise, set $S_2 = S_2^s \cup Q_2^+(r,s) \setminus J_2^{r,s}$ and $S_1 = S_1^{s-1}$.

(δ) $e = [(h,1),(h,2)]$. *From property (i) it follows that one of the following quasi paths exists.*

- $Q_2^-(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < h-1 < h \le s$.
  Choose $S_1 = S_1^{h-1}$ and $S_2 = S_2^r \cup Q_2^-(r,s) \setminus J_1^{r,s}$.
- $Q_1^+(r,s)$ with $r,s \in \{1,\ldots,h\}$ such that $r < h-1 < h \le s$.
  Choose $S_1 = S_1^r \cup Q_1^+(r,s) \setminus J_2^{r,s}$ and $S_2 = S_2^{h-1}$.
- $Q_{F_k}((r,l),(h,l))$ with $k,l \in \{1,2\}, r,s \in \{1,\ldots,h\}, r < s$ such that $r \le h-2$ and $s \ne h$, if $k = l$.
  This is a special case of the corresponding case in (γ).

We conclude that in all cases $P = (S_1,S_2)$ and $P' = P \cup_k e$ are routings with $\chi^{P'}, \chi^P \in F_a$, and we obtain that $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(3) $b_e^k = \beta$ *for all* $e \in E \setminus (F \cup F_k)$, $k = 1,2$.

Let $e \in E \setminus (F \cup F_1)$. From property (iv) we know that there exist nodes $u, v \in V$ and a quasi path $Q_{F_1}(u,v)$ from u to v with $e \in Q_{F_1}(u,v)$. Suppose $u = (r,2)$ and $v = (s,2)$ for some $r,s \in \{1,\ldots,h\}, r < s$. The case $u = (r,1)$ and $v = (s,1)$ can be shown accordingly (note that these are the only possible cases, since C is a cut in G). We choose $S_1 = S_1^r \cup Q_{F_1}(u,v) \setminus J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1,S_2)$ is a routing with $\chi^P \in F_a$, and, by taking (1) and (2) into account we have that $\beta = b^T\chi^P = b_e^1$. Similarly, we obtain $b_e^2 = \beta$.

It remains to be shown that STP$(G,\mathcal{N},\mathbb{I})$ is fulldimensional. Due to (1)–(3) it suffices to construct a routing P with $a^T\chi^P \ge 2$. Let $e \in E \setminus (F \cup F_1)$. Property (i) guarantees that such an edge exists. Moreover, property (iv) implies that there exist nodes $u,v \in V$ and a quasi path $Q_{F_1}(u,v)$ from u to v with $e \in Q_{F_1}(u,v)$. W. l. o. g. let $u = (r,2)$ and $v = (s,2)$ for some $r,s \in \{1,\ldots,h\}, r < s$. From (ii) it follows that there exist an index $k \in \{1,2\}$ and a quasi path Q' with respect to F_k such that $e \notin Q'$. Let $\{e'\} = Q' \cap \{E \setminus (F \cup F_k)\}$. Obviously, $e \ne e'$. We choose $S_1 = S_1^r \cup Q_{F_1}(u,v) \setminus J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1,S_2) \cup_k e'$ is a routing with $a^T\chi^P = 2$.

In the remainder of the proof we show that properties (i) to (iv) are also necessary. We start by proving property (iii).

(iii) Suppose there exists a path W from (r,l) to (s,l) in $V(F_k)$ with $r < s$ and $k,l \in \{1,2\}$. We consider the case $k = 1$ and $l = 2$ (the other cases can be shown similarly). We choose $S_1 = S_1^r \cup W \setminus J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1,S_2)$ is a routing with $a^T\chi^P = 0$, a contradiction.

Since property (iii) holds and since there does not exist a routing in $(V(F), F)$, we know that, for each edge-minimal routing P (an *edge-minimal* routing $P = (S_1, \ldots, S_N)$ is a routing where each Steiner tree S_k is edge-minimal, that is, S_k is a tree whose leaves are terminals) with $a^T\chi^P = 1$, there exists exactly one quasi path Q with $Q \subset P$. In the following we denote this unique quasi path by Q^P. Let us now show the remaining properties.

(i) We claim that there exists an edge $e \in F$ such that $e \in P$ for all edge-minimal routings P with $\chi^P \in F_a$, if property (i) does not hold. This implies that $F_a \subsetneq \{x \in \text{STP}(G, \mathcal{N}, \mathbb{I}) \mid x_e^1 + x_e^2 = 1\}$, a contradiction. Suppose now, property (i) does not hold. Then, there exists an $i \in \{1, \ldots, h\}$ for which the required quasi paths in (i) do not exist. By negation of condition (a) and (b) we distinguish the following cases:

(b1) For every index $k_1 \in \{1,2\}$ and every pair of nodes $(r_1, 1), (s_1, 1) \in V$ with $r_1, s_1 \in \{1, \ldots, h\}$, there does not exist a quasi path $Q_{F_{k_1}}((r_1, 1), (s_1, 1))$ such that $r_1 \leq i < i + 1 \leq s_1$ holds.
Let $P = (S_1, S_2)$ be any edge-minimal routing with $\chi^P \in F_a$ and let $e = [(i, 1), (i + 1, 1)]$. Suppose $e \notin P$. Since condition (a) of (i) does not hold, the unique quasi path $Q^P_{F_k}((r, l)(s, l))$, where $k, l \in \{1,2\}$ and $r, s \in \{1, \ldots, h\}$, satisfies $r > i - |k - l|$ or $s < i + 2 - |k - l|$. Moreover, $r \leq i$ and $s \geq i + 1$, since $e \notin P$. If $r > i - |k - l|$, we know that $k \neq l$ and $r = i$. Thus, according to assumption (b1), we obtain that $l = 1$ and $k = 2$. Hence, $e \in S_1$, a contradiction. On the other hand, if $s < i + 2 - |k - l|$, we have that $k = l$ and $s = i + 1$. Thus, according to assumption (b1), we obtain in this case $k = l = 1$. Hence, $e \in S_2$, a contradiction.

(b2) For every index $k_2 \in \{1,2\}$ and every pair of nodes $(r_2, 2), (s_2, 2) \in V$ with $r_2, s_2 \in \{1, \ldots, h\}$, there does not exist a quasi path $Q_{F_{k_2}}((r_2, 2), (s_2, 2))$ such that $r_2 \leq i < i + 1 \leq s_2$ holds.
Analogously, it can be shown that $e = [(i, 2), (i + 1, 2)]$ is an element of every edge-minimal routing P with $\chi^P \in F_a$.

(b3) For every set of indices $k, l \in \{1,2\}$ and $r, s \in \{1, \ldots, h\}, r < s$ that satisfy $s - r \geq 2$, if $r \in \{i - 1, i\}$, and $r \neq i$, if $k \neq l$, and $s \neq i$, if $k = l$, there does not exist a quasi path $Q_{F_k}((r, l), (s, l))$.
Let $P = (S_1, S_2)$ be any edge-minimal routing with $\chi^P \in F_a$ and let $e = [(i, 1), (i, 2)]$. Suppose $e \notin P$. Due to assumption (b3) we know that for the unique quasi path $Q^P_{F_k}((r, l)(s, l))$ one of the following cases holds. Either $r = i - 1$ and $s = i$, or $r = i$ and $s = i + 1$, or $r = i$ and $k \neq l$, or $s = i$ and $k = l$. It is easy to see that in all four cases edge e must be in P, a contradiction.

Hence, property (i) is necessary, indeed.

(ii) Now, suppose there exists an edge $e \in \bigcap_{Q \in \mathcal{Q}} Q$. Since for each edge-minimal routing P with $a^T\chi^P = 1$, there exists a unique quasi path, we conclude that e is contained in every such routing. Thus, $F_a \subsetneq \{x \in \text{STP}(G, \mathcal{N}, \mathbb{I}) \mid x_e^1 + x_e^2 = 1\}$, a contradiction.

(iv) Suppose F_1 and F_2 are not maximal with respect to properties (i) to (iii). Then, choose $F_1' \subseteq E \setminus F$ and $F_2' \subseteq E \setminus F$ such that $F_1 \cup F_2 \subset F_1' \cup F_2'$, and F_1' and F_2' are maximal with respect to properties (i) – (iii). Due to part 1 of the proof, we know that $(\chi^{E \setminus (F \cup F_1')}, \chi^{E \setminus (F \cup F_2')})^T x \geq 1$ defines a facet of $\text{STP}(G, \mathcal{N}, \mathbb{I})$. By summing up this facet-defining inequality together with the valid inequalities $x_e^1 \geq 0$ for all $e \in F_1' \setminus F_1$ and $x_e^2 \geq 0$ for all $e \in F_2' \setminus F_2$ we obtain $a^T x \geq 1$. Hence, $a^T x \geq 1$ does not define a facet of $\text{STP}(G, \mathcal{N}, \mathbb{I})$, a contradiction.

This completes the proof. □

Now we turn to the last class of inequalities we intend to describe in this paper. For a node set $W \subseteq V$ we define $S(W) = \{k \in \{1,\dots,N\} \mid T_k \cap W \neq \emptyset,\ T_k \cap (V \setminus W) \neq \emptyset\}$. We call a cut induced by a node set W *critical for* $(G,\mathcal{N},c)$, if $s(W) := c(\delta(W)) - |S(W)| \leq 1$. If V_1, V_2, V_3 is a partition of V (that is V_1, V_2, V_3 are pairwise disjoint node sets with $V_1 \cup V_2 \cup V_3 = V$) such that $\delta(V_1)$ is a critical cut and if $T_1 \cap V_1 = \emptyset$ and $T_1 \cap V_i \neq \emptyset$ for $i = 2,3$, we call the inequality

$$x^1([V_2 : V_3]) \geq 1$$

a *critical cut inequality (with respect to T_1)*. Then, the following theorem holds.

Theorem 2.7.
Let $G = (V,E)$ be a graph with $V = \{u,v,w\}$. Moreover, let $\mathcal{N} = \{T_1,\dots,T_N\}$ be a net list such that all terminal sets are of cardinality two and $T_1 = \{u,v\}$. Set $E_{ij} := \{e \in E \mid e$ is incident to i and $j\}$ for $i,j \in V$ and $N_i = \{k \in \{1,\dots,N\} \mid i \in T_k\}$ for $i \in V$. Assume that $|E_{uv}| \geq 2$, $N_w = \{2,\dots,N\}$, $|E_{uw}| \geq |N_u| - 1$, $|E_{vw}| \geq |N_v| - 1$ and $|E_{uw}| + |E_{vw}| = N$; see Figure 2.3. Then, the inequalilty

$$x^1(E_{uv}) \geq 1$$

defines a facet for $STP(G,\mathcal{N},\mathbb{1})$.

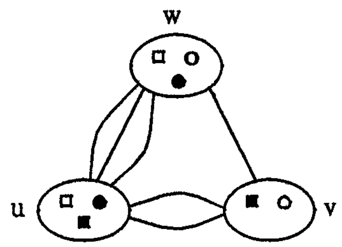

Fig. 3.

Proof. Let $a = (\chi^{E_{uv}}, 0, \dots, 0) \in \mathbb{R}^{\mathcal{N} \times E}$. First, we show that the inequality is valid. Let S_1 be a Steiner tree for T_1 with $S_1 \cap E_{uv} = \emptyset$. Then, we know that $S_1 \cap E_{uw} \neq \emptyset$ and $S_1 \cap E_{vw} \neq \emptyset$. Thus, $|\delta(w) \setminus S_1| \leq |E_{uw}| + |E_{vw}| - 2 = N - 2$. Since $|N_w| = N - 1$ and $1 \notin N_w$, there can not exist Steiner trees $S_2,\dots,S_N$ such that $P = (S_1,\dots,S_N)$ is a routing. So, we conclude that $a^T x \geq 1$ is valid.

Suppose, $b^T x \geq \beta$ is a facet-defining inequality of $\text{STP}(G,\mathcal{N},c)$ such that $F_a := \{x \in \text{STP}(G,\mathcal{N},c) \mid a^T x = 1\} \subseteq F_b := \{x \in \text{STP}(G,\mathcal{N},c) \mid b^T x = \beta\}$. In the following we show that b is a multiple of a. We prove this statement for the case $|E_{uw}| = |N_u|$ and $|E_{vw}| = |N_v| - 1$. The other case, $|E_{uw}| = |N_u| - 1$ and $|E_{vw}| = |N_v|$, can be shown accordingly (note that $|N_u| + |N_v| = N + 1$).

For a routing $P = (S_1,\dots,S_N)$ and an edge $e \in E$, we abbreviate $(S_1,\dots,S_k \cup \{e\},\dots,S_N)$ by $P \cup_k e$.

(1) $b_e^k = 0$ *for* $e \in E_{uw}$ *and* $k = 1,\dots,N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv}$. Furthermore, for every $k \in N_u \setminus \{1\}$, we set $S_k := \{e_k\}$ with $e_k \in E_{uw} \setminus \{e\}$ such that the edge sets S_k, $k \in N_u \setminus \{1\}$, are pairwise disjoint. Similarly, for every $k \in N_v \setminus \{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v \setminus \{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw} \setminus \{e\}| = |N_u| - 1$. Thus, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in F_a$. This yields $0 = b^T\chi^{P'} - b^T\chi^P = b^k_e$.

(2) $b^k_e = 0$ *for* $e \in E_{uv}$ *and* $k = 2, \ldots, N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv} \setminus \{e\}$. This is possible, since $|E_{uv}| \geq 2$. Furthermore, for $k \in N_u \setminus \{1\}$, we choose $S_k := \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u \setminus \{1\}$, are pairwise disjoint. Analogously, for $k \in N_v \setminus \{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v \setminus \{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw}| = |N_u|$. Hence, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in F_a$, and we obtain that $0 = b^T\chi^{P'} - b^T\chi^P = b^k_e$.

(3) $b^k_e = 0$ *for* $e \in E_{vw}$ *and* $k = 1, \ldots, N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv}$. Furthermore, for $k \in N_u \setminus \{1\}$, we set $S_k := \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u \setminus \{1\}$, are mutually disjoint. This is possible, since $|E_{uw}| = |N_u|$. Let $k_0 \in N_v$ and $\{f_1\} = E_{uw} \setminus \bigcup_{k \in N_u \setminus \{1\}} S_k$. This edge f_1 exists, since $|E_{uw}| = |N_u|$. Moreover, let $f_2 \in E_{uv} \setminus S_1$. This edge also exists, since $|E_{uv}| \geq 2$. Set $S_{k_0} := \{f_1, f_2\}$. For $k \in N_v \setminus \{1, k_0\}$, we choose $S_k := \{e_k\}$ with $e_k \in E_{vw} \setminus \{e\}$ such that the edge sets S_k, $k \in N_v \setminus \{1, k_0\}$, are pairwise disjoint. Again, this is possible, since $|E_{vw} \setminus \{e\}| = |N_v| - 2$. Thus, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in F_a$, and we conclude that $0 = b^T\chi^{P'} - b^T\chi^P = b^k_e$.

(4) $b^1_e = b^1_f$ *for* $e, f \in E_{uv}$.

Set $S_1 = \{f\}$ and $S'_1 = \{e\}$. Furthermore, for $k \in N_u \setminus \{1\}$, we choose $S_k := \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u \setminus \{1\}$, are pairwise disjoint. Similarly, for $k \in N_v \setminus \{1\}$, we set $S_k := \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v \setminus \{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw}| = |N_u|$. Thus, $P = (S_1, S_2, \ldots, S_N)$ and $P' = (S'_1, S_2, \ldots, S_N)$ are routings with $\chi^P, \chi^{P'} \in F_a$. This yields $0 = b^T\chi^{P'} - b^T\chi^P = b^1_e - b^1_f$.

Hence, we know that b is a multiple of a. To complete the proof we show that STP $(G, \mathcal{N}, \mathbb{1})$ is fulldimensional. Otherwise, $a^Tx \geq 1$ defines an equality of STP $(G, \mathcal{N}, \mathbb{1})$. Due to (1)–(4) it suffices to construct a routing P with $a^T\chi^P \geq 2$. Choose $S_1 = E_{uv}$. Moreover, for $k \in N_u \setminus \{1\}$, we set $S_k := \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u \setminus \{1\}$, are mutually disjoint. Similarly, for $k \in N_v \setminus \{1\}$, we choose $S_k := \{e_k\}$ with $e_k \in E_{vw}$ such that S_k, $k \in N_v \setminus \{1\}$, are pairwise disjoint. Then, $P = (S_1, \ldots, S_N)$ is a routing, and we have that $a^T\chi^P = |E_{uv}| \geq 2$. This completes the proof. □

We have introduced before four classes of inequalities for STP $(G, \mathcal{N}, c)$ which we proved to be facet-defining for special routing instances. Though in the first two theorems a complete graph and a disjoint net list is assumed, we can transform the results to grid graph routing problems by applying Lemma 2.2 and Lemma 2.3. Theorem 2.6 is directly applicable to grid graph instances, because a complete rectangular grid graph occurs as a subgraph of G. Theorem 2.7 can be interpreted as a polyhedral formulation of the so-called *cut condition* which says that $s(W)$ has to be non-negative for all $W \subset V$. This condition has been intensively studied in the literature especially for grid graphs (see for example [3], [12], [15]).

3. Separation

In this section we briefly discuss the separation problem for some classes of inequalities introduced in the last section. The *separation problem* for a class of inequalities can be stated as follows. "Given a vector $y \in \mathbb{R}^{\mathcal{N}\times E}$, decide whether y satisfies all inequalities of the given class. If not, find an inequality of this class that is violated by y."

Among the known facet-defining inequalities for the Steiner tree polyhedron let us consider the so-called *Steiner partition inequalities.* These inequalities are of the form

$$\sum_{i=1}^{p} x(\delta(V_i)) \geq p-1,$$

where $x \in \mathbb{R}^E$ and $V_1, \ldots, V_p$, $p \geq 2$ is a partition of V such that each V_i contains at least one terminal. Observe that the Steiner partition inequalities are a generalization of the inequalities in (2.1) (i).

Grötschel and Monma have characterized the conditions under which the Steiner partition inequalities are facet-defining for the Steiner tree polyhedron ([5]). Unfortunately, the separation problem for this class of inequalities is $\mathcal{NP}$-hard in general ([6]). But, if we restrict G to be planar and the terminals to be on the outer face we have developed an exact separation algorithm. We do not want to describe the rather complicated algorithm in this paper. The main idea is to construct a graph G_D and a set of terminals D such that each Steiner partition inequality corresponds exactly to a Steiner tree in G_D for a subset of D. In order to find the minimal Steiner tree in G_D among ali subsets of D, dynamic programming techniques are applied. For details we refer the interested reader to [9].

About the separation problem of the alternating cycle inequalities much less is known. For general instances, we even do not know the complexity of the problem. In the case that G is planar and all terminals lie on the outer face we have tried to apply the same ideas as sketched above. This dynamic programming based method works efficiently and successful in practice, but due to the more complex structure of the inequalities this method no longer yields an exact separation algorithm. There are examples where the returned value of the algorithm indicates a violated constraint which is indeed not violated. However, it can be shown that the returned value is a lower bound for the most violated constraint, and thus the algorithm may prove that no violated alternating cycle inequality exists (cf. [9]).

In order to find violated grid inequalities we proceed as follows. First of all, we concentrate on valid (not necessarily facet-defining) inequalities, i. e., we neglect properties (i) and (ii) of Theorem 2.6. Nevertheless, the properties that $\hat{G}$ has to be a complete rectangular $h\times 2$ grid graph and that $T_1 = \{(1,1),(h,2)\}$ and $T_2 = \{(1,2),(h,1)\}$ are still quite restrictive. They are usually not satisfied by practical problem instances, even not by switchbox routing problems. Our idea was to relax these conditions such that the corresponding inequality $(\chi^{E\setminus(\hat{E}\cup F_1)}, \chi^{E\setminus(\hat{E}\cup F_2)})^T x \geq 1$ (where $F_1, F_2 \subset E \setminus \hat{E}$ are chosen apropriately) remains valid. For finding violated inequalities of this (new) class we proceed in a greedy like fashion. We refrain here from explaining the details and refer the interested reader to [14].

Finally, we have implemented an algorithm for finding critical cuts if the given instance is a switchbox routing problem. The algorithm makes use of the following lemma.

Lemma 3.1.

Let G be a complete rectangular grid graph and $\mathcal{N}$ a net list such that all terminal sets lie on the outer face. Let $W \subset V$, $\emptyset \neq W \neq V$, be a set of nodes and let the cut induced by W be critical with respect to the given instance $(G, \mathcal{N}, \mathbb{I})$. Then, one of the following statements is true.

(i) *There exists a node $w \in V$ such that $\delta(w)$ is a critical cut with respect to $(G, \mathcal{N}, \mathbb{I})$.*

(ii) *There exists a horizontal or vertical cut which is critical with respect to $(G, \mathcal{N}, \mathbb{I})$. (A cut F is called* horizontal *if there exists some $i \in \{1, \ldots, h-1\}$ such that $F = \{uv \in E \mid u = (i,j)$ and $v = (i+1, j)$ for some $j \in \{1, \ldots, b\}\}$; a vertical cut is defined accordingly).*

Our algorithm checks all cuts in 3.1 (i) and (ii). Thus, we are sure that the algorithm finds a critical cut if there exists one. Suppose $\delta(W)$ for $W \subset V$ is a critical cut. If there exits a terminal set T_k with $T_k \subset W$ we know that a Steiner tree S_k for T_k of a edge-minimal routing cannot use edges of $E(V \setminus W)$. Thus, we can fix all corresponding variables to zero. After this step is performed, the critical cut inequalities are automatically separated by the separation algorithm for the Steiner partition inequalities (cf. [8]).

4. Computational results

In this section we report on our experimental results we have obtained with a branch and cut algorithm. We have tested our algorithm on switchbox routing problems that are discussed in literature. Table 4.1 summarizes the data.

Table 4.1

name	height	width	nets	variables	ref
difficult switchbox	15	23	24	15648	[1]
more difficult switchbox	15	22	24	14952	[2]
terminal intensive switchbox	16	23	24	16728	[13]
dense switchbox	17	15	19	9082	[13]
augmented dense switchbox	18	16	19	10298	[13]
modified dense switchbox	17	16	19	9709	[2]
pedagogical switchbox	16	15	22	9878	[2]

Column 1 presents the name used in literature. In column 2 and 3 the height and width of the underlying grid graph is given. Column 4 contains the number of nets. Columns 5 shows the resulting number of 0/1 variables. Finally, the last column states the reference to the paper the example is taken from. In all examples the edge weights as well as the edge capacities are identical to one.

In Table 4.2 the results of our branch and cut algorithm are summarized. Column 2 gives the best feasible solution we have obtained with a primal heuristic. The entries in column 3 are the objective function values of the linear program (rounded up to the next integer) when no further violated constraints are found, i. e., when branching is performed for the first time. This values are lower bounds for the whole problem. In column 4 the percental deviation of the best solution from the lower bound is given. Column 5 (resp. 6) gives the number of cutting plane iterations (resp. the number of nodes in the branching tree). Finally, the last column reports on the running times. The values are stated in minutes obtained on a SPARC-Workstation.

Table 4.2

example	best sol.	LP value	gap	iter.	B&C	CPU-time
difficult switchbox	464	464	0.0%	69	3	1564:15
more difficult switchbox	452	452	0.0%	53	1	983:23
terminal intensive switchbox	537	536	0.2%	163	13	3755:44
dense switchbox*	441	438	0.7%	119	4	1017:43
augmented dense switchbox*	469	467	0.4%	105	1	4561:41
modified dense switchbox	452	452	0.0%	51	1	387:03
pedagogical switchbox	331	331	0.0%	77	5	251:58

For the two examples "dense switchbox" and "augmented dense switchbox" marked with an asterisk, the execution of the branch and cut algorithm was stopped after the time given in the last column, because no further progress could be achieved. We believe that the values given in column 2 are optimal, but we are not yet able to prove this with the cutting plane algorithm. All other problem instances are solved to optimality. The running times in the last column are surely quite high. This is due to the fact that we were interested in finding an optimal solution. On the other hand, a provable quality guarantee of 5% can be given after at most 5 minutes for all these problem instances, which shows that our methodology is approaching practical usability. In fact, standard routing algorithms are rarely able to provide any quality guarantee at all.

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