the reduction of the feasible circulation problem and another combinatorial problem, the "problem of representatives", to the maximum flow problem and proceeds with some applications to matrix rounding and scheduling. After these motivations, the book returns to the "canonical material", viz. the Max-Flow Min-Cut Theorem and the Labeling Algorithm. Finally, the consequences of the Max-Flow Min-Cut Theorem for network connectivity, matchings, and flows with lower bounds are summarized. Chapter 7 is devoted to polynomial algorithms, namely, the shortest augmenting path algorithm, several versions of the pre-flow push algorithm, and scaling algorithms, and a (surprisingly brief) description of Dinitz' blocking flow approach. The authors do not follow the history, but give a more systematical, unified approach, which shows the inherent interrelations between the different concepts. Finally, Chap. 8 deals with additional topics, that is, the restrictions of the maximum flow problem to unit capacities, bipartite networks, and undirected, planar networks, respectively; applications of the maximum flow problem to network reliability, network connectivity, all pairs minimum cut value, and similar problems; and dynamic tree implementations of flow algorithms. Moreover, further applications are provided in the Introduction and in Chap. 19, which summarizes additional applications of network flow theory.

Several other important network flow problems are treated at some length: assignments, convex cost flows, generalized flows, and multicommodity flows. However, here the authors have selected only a few algorithms and additional topics and do not try to give a survey. As an example of these problems, let us consider Chap. 15, which is concerned with generalized flows. This chapter starts with a few instructive examples of occurrences of generalized flows in different application domains. Then, after some preliminaries, the generalized network simplex algorithm is developed in detail, again in a purely combinatorial manner in fact.

In summary, this book is comprehensive in the sense that network flow theory is discussed from many different points of view. Therefore, even people who are fairly familiar with the topic might find interesting new insights. On the other hand, people who are *not* familiar with it will find a good introduction, which reads very well and is easily understandable, although no "hard fact" at all is left out. Surely one reason for this is another difference to many other, similar books, namely, that the hard facts are developed along with the ideas behind them and with a broad practical context. I would like to read more books like this.

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ZOR – Mathematical Methods of Operations Research (1995) 41:255–275 ZOR

Routing in Grid Graphs by Cutting Planes

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Abstract: In this paper we study the following problem, which we call the weighted routing problem. Let be given a graph $G = (V, E)$ with non-negative edge weights $w_e \in \mathbb{R}_+$ and let $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, be a list of node sets. The weighted routing problem consists in finding mutually disjoint edge sets $S_1, \ldots, S_N$ such that, for each $k \in \{1, \ldots, N\}$, the subgraph $(V(S_k), S_k)$ contains an $[s, t]$-path for all $s, t \in T_k$ and the sum of the weights of the edge sets is minimal. Our motivation for studying this problem arises from the routing problem in VLSI-design, where given sets of points have to be connected by wires. We consider the weighted routing problem from a polyhedral point of view. We define an appropriate polyhedron and try to (partially) describe this polyhedron by means of inequalities. We describe our separation algorithms for some of the presented classes of inequalities. Based on these separation routines we have implemented a branch and cut algorithm. Our algorithm is applicable to an important subclass of routing problems arising in VLSI-design, namely to switchbox routing problems where the underlying graph is a grid graph and the list of node sets is located on the outer face of the grid. We report on our computational experience with this class of problem instances.

Keywords: Routing in VLSI-design, Switchbox Routing, Steiner Tree, Steiner Tree Packing, Cutting Plane Algorithm.

1 Introduction

One of the main topics in VLSI-design is the routing problem. The task is to connect so-called terminal sets via wires on a predefined area. In addition, certain design rules are to be taken into account and an objective function such as the wiring length must be minimized. The routing problem in general is too complex to be solved in one step. Depending on the user's choice of decomposing the chip design problem into a hierarchy of stages, on the underlying technology, and on the given design rules, various subproblems arise. Many of the routing problems that come up this way can be formulated in graphtheoretical terms as follows:

Problem 1.1: (The Weighted Routing Problem)

Instance:

A graph $G = (V, E)$ with non-negative edge weights $w_e \in \mathbb{R}_+$, $e \in E$.
A list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \geq 1$, with $T_k \subseteq V$ for all $k = 1, \ldots, N$.

Problem:

Find edge sets $S_1, \ldots, S_N \subseteq E$ such that

(i) *$(V(S_k), S_k)$ contains an $[s, t]$-path for all $s, t \in T_k$ for $k = 1, \ldots, N$ (where $V(F)$ is the set of nodes that are incident to an edge of $F \subseteq E$),*

(ii) *no two edge sets of $S_1, \ldots, S_N$ have a common edge,*

(iii) *$\sum_{k=1}^{N} \sum_{e \in S_k} w_e$ is minimal.*

We call the list of node sets $\mathcal{N}$ a *net list*. Any element $T_k \in \mathcal{N}$ is called a *set of terminals* or a *net* and the nodes $t \in T_k$ are called *terminals*. It is also customary to say *net* k instead of terminal set T_k. An edge set S that satisfies condition (i) for a terminal set T is called a *Steiner tree in G for T*. A N-tupel of edge sets $(S_1, \ldots, S_N)$ that satisfies (i) and (ii) is called a *routing* or a *Steiner tree packing*. If we are only interested in finding a feasible solution, i.e., we neglect condition (iii), we speak of the routing problem without the prefix "weighted".

Of particular interest in VLSI-design are routing problems where the underlying graph is a grid graph. Among these are the channel routing and the switchbox routing problem. In these two cases, the graph is a complete rectangular grid and the terminal sets are located on the outer face of the grid. In the channel routing problem the terminal sets are restricted to lie on two opposite sides of the graph, whereas in the switchbox routing problem terminals may be located on all four sides.

It is not surprising that Problem 1.1 is $\mathcal{NP}$-hard, even in many special cases. Among them are the minimal Steiner tree problem ([K72], [GJ77]) and the problem of packing N disjoint paths into a planar graph ([KL84]). Even the channel routing and the switchbox routing problem are $\mathcal{NP}$-hard ([S87]).

We attack the (weighted) routing problem by using a polyhedral approach. In section 2, we define a polyhedron whose vertices are in one-to-one correspondence to the routings in the graph, and we try to describe this polyhedron by means of equations and inequalities. Section 3 deals with the separation problem for some classes of inequalities that are described in the previous section. Finally, in section 4 we report on some computational results we have obtained with our cutting plane algorithm. The test problems are switchbox routing problems discussed in the literature.

2 The Routing Polyhedron

In this section we define the routing polyhedron and describe some classes of valid and facet-defining inequalities. First, we introduce some notation.

We denote by $\mathbb{R}^E$ the Euclidean vector space of dimension $|E|$, where the components of each vector are indexed by the elements of E, i.e., $x = (x_e)_{e \in E}$ for $x \in \mathbb{R}^E$. For an edge set $F \subseteq E$, we define the incidence vector $\chi^F \in \mathbb{R}^E$ by setting $\chi^F_e = 1$, if $e \in F$, and $\chi^F_e = 0$, otherwise. Furthermore, we abbreviate $\sum_{e \in F} x_e$ by $x(F)$ for an edge set F and a vector $x \in \mathbb{R}^E$. We denote by $\mathbb{R}^{\mathcal{N} \times E}$ the $N \cdot |E|$-dimensional vector space $\mathbb{R}^E \times \cdots \times \mathbb{R}^E$. The components of a vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ are indexed by x^k_e for $k \in \{1, \ldots, N\}$, $e \in E$. For a vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ and $k \in \{1, \ldots, N\}$ we denote by $x^k \in \mathbb{R}^E$ the vector $(x^k_e)_{e \in E}$. If it is clear from the context we will abbreviate a vector $x = ((x^1)^T, \ldots, (x^N)^T)^T$ by $(x^1, \ldots, x^N)$. By the *incidence vector of a routing* $P = (S_1, \ldots, S_N)$ we mean the vector $(\chi^{S_1}, \ldots, \chi^{S_N})$ or in short χ^P.

We define now the *routing polyhedron* (also called the *Steiner tree packing polyhedron*) by

$$\mathrm{STP}(G, \mathcal{N}) := \mathrm{conv}\{(x^1, \ldots, x^N) \in \mathbb{R}^{\mathcal{N} \times E} |$$

$$\begin{aligned}
&\text{(i)} \quad \sum_{e \in \delta(W)} x^k_e \geq 1, && \text{for all } W \subset V,\ W \cap T_k \neq \emptyset\ , \\
& && (V \setminus W) \cap T_k \neq \emptyset,\ k = 1, \ldots, N\ ; \\
&\text{(ii)} \quad \sum_{k=1}^{N} x^k_e \leq 1, && \text{for all } e \in E\ ; \qquad\qquad (2.1) \\
&\text{(iii)} \quad x^k_e \geq 0, && \text{for all } e \in E,\ k = 1, \ldots, N\ ; \\
&\text{(iv)} \quad x^k_e \in \{0, 1\}, && \text{for all } e \in E,\ k = 1, \ldots, N\}\ ,
\end{aligned}$$

where $\delta(W)$ in (2.1) (i) denotes the set of all edges with exactly one endnode in W. The inequalities (2.1) (ii) and (2.1) (iii) are called the *capacity inequalities* and *trivial inequalities*, respectively. If $N = 1$ we refer to $\mathrm{STP}(G, \mathcal{N})$ as the *Steiner tree polyhedron*. Obviously, each incidence vector of a routing satisfies (2.1) (i)–(iv), and vice versa, it is easy to see that each vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ satisfying (2.1) (i)–(iv) is the incidence vector of a routing. Thus, the weighted routing problem reduces to the linear program $\min\{\sum_{k=1}^{N} w^T x^k | x \in \mathrm{STP}(G, \mathcal{N})\}$.

In order to apply linear programming techniques, a "good" description of the routing polyhedron by means of equations and inequalities is indispensable. To this end, we must determine the dimension of the routing polyhedron. Unfortunately, this problem is $\mathcal{NP}$-hard, even for switchbox routing problems. This follows from the fact that the decision problem, "Does there exist a routing for a given instance $(G, \mathcal{N})$?", is $\mathcal{NP}$-complete (see [KL84], [S87]).

Thus, we have decided to study the routing polyhedron for special problem instances for which the dimension can easily be determined and to look for facet-defining inequalities for these special instances. Clearly, such an approach is only sensible if the results can be carried over (at least partially) to practically interesting instances like switchbox routing problems.

For example, an instance $(G, \mathcal{N})$, where the graph G is complete and the net list $\mathcal{N} = \{T_1, \ldots, T_N\}$ is *disjoint* (that is $T_i \cap T_j = \emptyset$ for $i \neq j$) is an appropriate case. It can easily be verified that the corresponding routing polytope $STP(G, \mathcal{N})$ is fulldimensional in this case. By applying the subsequent two lemmas we can transform any given valid (resp. facet-defining) inequality for this polytope to a valid inequality of the routing polytope corresponding to, for example, the switchbox routing problem.

Lemma 2.2: (Deletion of an edge). *Let $(G, \mathcal{N})$ be an instance of the routing problem. Let $a^T x \geq \alpha$ be a valid inequality for $STP(G, \mathcal{N})$ and let us delete $f \in E$ from G. Then $\bar{a}^T x \geq \alpha$ is a valid inequality for $STP(G\backslash f, \mathcal{N})$ where $\bar{a}^k_e = a^k_e$ for all $e \in E\backslash\{f\}$, $k \in \{1, \ldots, N\}$ (where $G\backslash f$ denotes the graph that is obtained by deleting edge f).*

Lemma 2.3: (Splitting a node). *Let $(G, \mathcal{N})$ be an instance of the routing problem. Let $f \in E$ and $\bar{a}^T x \geq \alpha$ be a valid inequality for $STP(G/f, \mathcal{N})$ (where G/f denotes the graph that is obtained by shrinking edge f). Then, $a^T x \geq \alpha$ defines a valid inequality for $STP(G, \mathcal{N})$ with $a^k_e = \bar{a}^k_e$ for all $e \in E\backslash\{f\}$, $k \in \{1, \ldots, N\}$ and $a^k_f = 0$ for all $k = 1, \ldots, N$.*

Lemma 2.2 follows from the fact that every routing of $(G\backslash f, \mathcal{N})$ is also a routing of $(G, \mathcal{N})$. A similar argument proves Lemma 2.3.

Let us now describe some facet-defining inequalities for $STP(G, \mathcal{N})$. The first two statements concern instances, where the graph is complete and the net list is disjoint. Afterwards we describe in more detail a class of facet-defining inequalities for the routing polytope of an instance that is strongly related to grid graph routing problems. The inequality stated in the last theorem is based on a cut condition that is necessary for the existence of a routing.

The first question that arises is: "Can the facet-defining inequalities for the Steiner tree polyhedron be extended to facet-defining inequalities of the routing polyhedron?" This can be answered positively. More precisely, let $G = (V, E)$ be the complete graph with node set V and let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a disjoint net list. Let $\bar{a}^T x \geq \alpha$, $\bar{a} \in \mathbb{R}^E$, be a non-trivial facet-defining inequality for $STP(G, \{T_1\})$. Then, $a^T x \geq \alpha$ defines a facet for $STP(G, \mathcal{N})$, where $a \in \mathbb{R}^{1 \times E}$ is defined by $a^1_e = \bar{a}_e$, $a^k_e = 0$ for all $k = 2, \ldots, N$, $e \in E$. This has been shown in [GMW92a].

Next, we consider inequalities that combine two or more nets. First of all, we fix some notation. For two node sets $U, W \subseteq V$, we denote by $[U : W]$ all edges with one endpoint in U and one in W. Furthermore, $E(W)$ denotes all edges with both endpoints in W. For a cycle F, an edge uv is called a *chord* if $u, v \in V(F)$ and $uv \notin F$.

Let $G = (V, E)$ be a graph and $\mathcal{N} = \{T_1, T_2\}$ a net list. We call a cycle F an *alternating cycle with respect to* T_1, T_2 if $F \subseteq [T_1 : T_2]$ and $V(F) \cap T_1 \cap T_2 = \emptyset$,

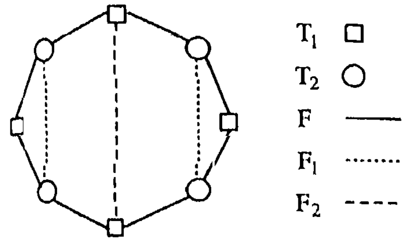


Fig. 2.1.

see Figure 2.1. Moreover, let $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ be two sets of chords of the alternating cycle F with respect to T_1, T_2. The inequality

$$(\chi^{E\backslash(F \cup F_1)}, \chi^{E\backslash(F \cup F_2)})^T x \geq \tfrac{1}{2}|F| - 1$$

is called an *alternating cycle inequality*.

It is not difficult to see that the basic form of an alternating cycle inequality, i.e., $F_1 = F_2 = \emptyset$, is valid for $STP(G, \mathcal{N})$, yet not facet-defining. In fact, one can derive necessary and sufficient conditions on the choice of the sets F_1 and F_2 such that the corresponding alternating cycle inequality defines a facet of $STP(G, \mathcal{N})$. The details are reported in [GMW92a].

Let us now introduce the so-called grid inequalities. Suppose, $G = (V, E)$ is a graph and $\mathcal{N} = \{T_1, T_2\}$ is a net list. Furthermore, let $H = (X, F)$ be a subgraph of G such that H is a complete $h \times 2$ grid graph with $h \geq 3$ (where a complete $h \times b$ grid graph is a grid graph with h rows and b columns). Assume that the nodes of X are numbered such that $X = \{(i, j) | i = 1, \ldots, h, j = 1, 2\}$. Moreover, let $(1, 1), (h, 2) \in T_1$ and $(1, 2), (h, 1) \in T_2$. We call the inequality

$$(\chi^{E \backslash F}, \chi^{E \backslash F})^T x \geq 1$$

an $h \times 2$ *grid inequality*.

Let $G = (V, E)$ be a graph, $F \subseteq E$ and $u, v \in V$. We call a path $Q_F(u, v)$ from u to v in G a *quasi path from u to v with respect to F*, if there exists an edge $e \in Q_F(u, v)$ such that $e \in E\backslash F$ and $Q_F(u, v)\backslash\{e\} \subseteq F$. The following theorem presents necessary and sufficient conditions for the grid inequality to be facet-defining. These conditions, in particular (i), are very technical and hard to verify. However, from these conditions we can derive some relaxations that are easy to check and that apply, for example, to switchbox routing instances. It turns out that such conditions are only sufficient for the corresponding grid inequality to be facet-defining.

Theorem 2.4: Let $H = (X, F)$ be a complete $h \times 2$ grid graph with $h \geq 3$. Let $\mathcal{N} = \{T_1, T_2\}$ be a net list where $T_1 = \{(1, 1), (h, 2)\}$ and $T_2 = \{(1, 2), (h, 1)\}$. Furthermore, let $G = (V, E)$ be a graph with $X \subset V$, $F \subset E$ such that the set of horizontal edges in H, i.e., $\{[\{(i, 1)\} : \{(i, 2)\}] | i = 1, \ldots, h\}$, is a cut in G. Let F_1, $F_2 \subset E \backslash F$, then the following holds. $STP(G, \mathcal{N})$ is fulldimensional and the inequality

$$(\chi^{E\backslash(F\cup F_1)}, \chi^{E\backslash(F\cup F_2)})^T x \geq 1$$

defines a facet for $STP(G, \mathcal{N})$ if and only if F_1 and F_2 satisfy the following properties (see Figure 2.2.):

(i) *For all $i \in \{1, \ldots, h\}$ (at least) one of the following conditions is fullfilled in $(V, E\backslash F)$:*
  (a) *There exist an index $k \in \{1, 2\}$, nodes $(r, l), (s, l) \in V$ with $r, s \in \{1, \ldots, h\}$, $l \in \{1, 2\}$ and a quasi path $Q_{F_k}((r, l), (s, l))$ such that $r \leq i - |k - l|$ and $s \geq i + 2 - |k - l|$ holds.*
  (b) *The subsequent three requirements are satisfied:*
    (b1) *There exist an index $k_1 \in \{1, 2\}$, nodes $(r_1, 1), (s_1, 1) \in V$ with $r_1, s_1 \in \{1, \ldots, h\}$ and a quasi path $Q_{F_{k_1}}((r_1, 1), (s_1, 1))$ such that $r_1 \leq i < i + 1 \leq s_1$ holds.*
    (b2) *There exist an index $k_2 \in \{1, 2\}$, nodes $(r_2, 2), (s_2, 2) \in V$ with $r_2, s_2 \in \{1, \ldots, h\}$ and a quasi path $Q_{F_{k_2}}((r_2, 2), (s_2, 2))$ such that $r_2 \leq i < i + 1 \leq s_2$ holds.*
    (b3) *There exist an index $k \in \{1, 2\}$, nodes $(r, l), (s, l) \in V$ where $r, s \in \{1, \ldots, h\}$, $r < s$, $l \in \{1, 2\}$ and a quasi path $Q_{F_k}((r, l), (s, l))$ with the additional properties:*

$$s - r \geq 2, \text{ if } r \in \{i - 1, i\} ,$$

$$r \neq i, \text{ if } k \neq l ,$$

$$s \neq i, \text{ if } k = l .$$

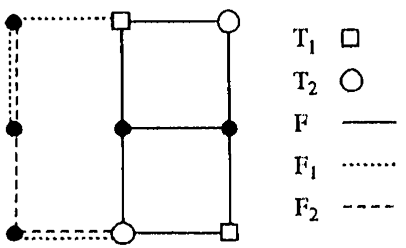


Fig. 2.2.

(ii) $\bigcap_{Q \in \mathcal{Q}} Q = \emptyset$, *where $\mathcal{Q} = \{Q \subset E |$ there exist an index $k \in \{1, 2\}$ and nodes $u, v \in V$ such that Q is a quasi path from u to v in $(V, E\backslash F)$ with respect to $F_k\}$.*
(iii) *For all $u, v \in V(F)$, $u \neq v$, and $k \in \{1, 2\}$ there does not exist a path from u to v in (V, F_k).*
(iv) *F_1 and F_2 are maximal with respect to the properties* (i)–(iii).

Proof: For the case of exposition we introduce the following notation. For an edge $uv \in E$, we also use the symbol $[u, v]$. Let C denote the set of horizontal edges in G, i.e., $C := \{[(i, 1), (i, 2)] | i = 1, \ldots, h\}$. For indices $k \in \{1, 2\}$, $r, s \in \{1, \ldots, h\}$, we set $Q_k^l(r, s) = Q_{F_k}((r, 2), (s, 2))$ and $Q_k^r(r, s) = Q_{F_k}((r, 1), (s, 1))$. Let be given indices $s \in \{1, 2\}$, $i_0, i_1 \in \{1, \ldots, h\}$, $i_0 \leq i_1$. $J_s^{i_0, i_1}$ denotes all vertical edges in G between nodes (i_0, s) and (i_1, s), i.e., $J_s^{i_0, i_1} := \{[(i, s), (i + 1, s)] | i = i_0, \ldots, i_1 - 1\}$. Moreover, set $S_1^i = J_1^{1, i} \cup \{[(i, 1), (i, 2)]\} \cup J_2^{i, h}$ and $S_2^i = J_2^{1, i} \cup \{[(i, 2), (i, 1)]\} \cup J_1^{i, h}$ for $i = 1, \ldots, h$. Finally, for a routing $P = (S_1, S_2)$ and an edge $e \in E$, we designate $(S_1 \cup \{e\}, S_2)$ by $P \cup_1 e$ and $(S_1, S_2 \cup \{e\})$ by $P \cup_2 e$. If $e \in S_1 \cup S_2$, we simply write $e \in P$.

Let us start by proving that properties (i) to (iv) imply that the $h \times 2$ grid inequality defines a facet for $STP(G, \mathcal{N})$ and that the routing polyhedron is fulldimensional.

The validity of $a^T x \geq 1$ with $a = (\chi^{E\backslash(F\cup F_1)}, \chi^{E\backslash(F\cup F_2)})$ is easy to see. Obviously, there does not exist a routing in $(V(F), F)$, since all nodes of $V(F)$ have at most degree three with respect to F and all terminals have degree two with respect to F. This together with property (iii) implies that the inequality is valid.

Now let $b^T x \geq \beta$ be a facet-defining inequality of $STP(G, \mathcal{N})$ with $I_a := \{x \in STP(G, \mathcal{N}) | a^T x = 1\} \subseteq I_b := \{x \in STP(G, \mathcal{N}) | b^T x = \beta\}$. In the following we show that b is a multiple of a.

(1) $b_e^k = 0$ for all $e \in F_k$, $k = 1, 2$.

From (i) we know that $\mathcal{Q} \neq \emptyset$. Due to (ii) there exists a quasi path $\bar{Q}$ in $(V, E\backslash F)$ with $e \notin \bar{Q}$. W.l.o.g. let $\bar{Q}$ be a quasi path with respect to F_1. Since C is a cut in G, we know that $\bar{Q} = Q_1^l(r, s)$ or $\bar{Q} = Q_1^r(r, s)$ with $r, s \in \{1, \ldots, h\}$, $r < s$. We consider the case $\bar{Q} = Q_1^l(r, s)$ (the other case can be shown analogously). Set $S_1 = S_1^r \cup Q_1^l(r, s) \backslash J_2^{r, s}$ and $S_2 = S_2^s$. Then, $P = (S_1, S_2)$ and $P' = P \cup_k e$ are routings with $\chi^{P'}, \chi^P \in I_a$, and we obtain that $0 = b^T \chi^{P'} - b^T \chi^P = b_e^k$.

(2) $b_e^k = 0$ for $e \in F$, $k = 1, 2$.

First, let us note that, for a given $i \in \{1, \ldots, h\}$, property (a) in (i) guarantees the existence of one of the following four quasi paths:

- $Q_2(r, s)$ with $r, s \in \{1, \ldots, h\}$ such that $r < i < i + 1 \leq s$.
- $Q_2^l(r, s)$ with $r, s \in \{1, \ldots, h\}$ such that $r \leq i < i + 1 < s$.
- $Q_1(r, s)$ with $r, s \in \{1, \ldots, h\}$ such that $r \leq i < i + 1 < s$.
- $Q_1^l(r, s)$ with $r, s \in \{1, \ldots, h\}$ such that $r < i < i + 1 \leq s$.

Depending on edge e we need to distinguish several cases. It turns out that in the forthcoming analysis, the above four cases can be treated quite similarly. To present our proof in a condensed and more convenient form, we refrain from dealing with all the above cases separately and give the construction only for the first case, i.e., we assume that, if property (a) in (i) holds, there exists a quasi path $Q_2^-(r, s)$ with $r, s \in \{1, \dots, h\}$ such that $r \le i < i+1 < s$.

(α) $e = [(i, 1), (i+1, 1)]$ with $i \in \{1, \dots, h-1\}$. Property (i) guarantees that one of the following quasi paths exists.

- $Q_2^-(r, s)$ with $r, s \in \{1, \dots, h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^i$ and $S_2 = S_2^s \cup Q_2^-(r, s)\backslash J_1^{r,s}$.
- $Q_k^+(r, s)$ with $k \in \{1, 2\}$, $r, s \in \{1, \dots, h\}$ such that $r \le i < i+1 \le s$.
  If $k = 1$, choose $S_1 = S_1^r \cup Q_1^+(r, s)\backslash J_2^{r,s}$ and $S_2 = S_2^s$. Otherwise, choose $S_1 = S_1^r$ and $S_2 = S_2^s \cup Q_2^+(r, s)\backslash J_2^{r,s}$.

(β) $e = [(i, 2), (i+1, 2)]$ with $i \in \{1, \dots, h-1\}$. This case is symmetric to (α).
(γ) $e = [(i, 1), (i, 2)]$ with $i \in \{1, \dots, h\}$. From property (i) we know that one of the following quasi paths exists.

- $Q_2^-(r, s)$ with $r, s \in \{1, \dots, h\}$ such that $r < i < i+1 \le s$.
  Choose $S_1 = S_1^{i+1}$, if $i \le h-1$, and $S_1 = S_1^h$, otherwise. Set $S_2 = S_2^s \cup Q_2^-(r, s)\backslash J_1^{r,s}$.
- $Q_{F_k}((r, l), (s, l))$ with $k, l \in \{1, 2\}$, $r, s \in \{1, \dots, h\}$, $r < s$ such that $s - r \ge 2$, if $r \in \{i-1, i\}$, and $r \ne i$, if $k \ne l$, and $s \ne l$, if $k = l$.
  First, we consider the case $r \notin \{i-1, i\}$. If $s \ne i$, choose $S_1 = S_1^r \cup Q_1^+(r, s)\backslash J_2^{r,s}$ and $S_2 = S_2^s$, if $k = 1$ and $l = 2$. In the other cases ($k = l = 1$, $k = l = 2$ and $k = 2$, $l = 1$) Steiner trees S_1 and S_2 can be chosen similarly. If $s = i$, we have $k \ne l$. Choose $S_1 = S_1^r \cup Q_1^+(r, s)\backslash J_2^{r,s}$ and $S_2 = S_2^{s-1}$, if $k = 1$, $l = 2$, otherwise set $S_2 = S_2^r \cup Q_2(r, s)\backslash J_1^{r,s}$ and $S_1 = S_1^{s-1}$. If $r = i-1$, we know that $s \ge i+1$. In accordance to $r \notin \{i-1, i\}$ and $s \ne i$ we can choose appropriate Steiner trees S_1 and S_2 in this case as well. If $r = i$, we have $s \ge i+2$ and $k = l$. If $k = 1$, choose $S_1 = S_1^s \cup Q_1^-(r, s)\backslash J_1^{r,s}$ and $S_2 = S_2^{s-1}$. Otherwise, set $S_2 = S_2^s \cup Q_2^+(r, s)\backslash J_2^{r,s}$ and $S_1 = S_1^{s-1}$.

We conclude that in all cases $P = (S_1, S_2)$ and $P' = P \cup_k e$ are routings with $\chi^{P'}$, $\chi^P \in I_\alpha$, and we obtain that $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(3) $b_e^k = \beta$ for all $e \in E\backslash(F \cup F_k)$, $k = 1, 2$.

Let $e \in E\backslash(F \cup F_1)$. From property (iv) we know that there exist nodes $u, v \in V$ and a quasi path $Q_{F_1}(u, v)$ from u to v with $e \in Q_{F_1}(u, v)$. Suppose $u = (r, 2)$ and $v = (s, 2)$ for some $r, s \in \{1, \dots, h\}$, $r < s$. The case $u = (r, 1)$ and $v = (s, 1)$ can be shown accordingly (note that these are the only possible cases, since C is a cut in G). We choose $S_1 = S_1^r \cup Q_{F_1}(u, v)\backslash J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1, S_2)$ is a routing with $\chi^P \in I_\alpha$, and, by taking (1) and (2) into account we have that $\beta = b^T\chi^P = b_e^1$. Similarly, we obtain $b_e^2 = \beta$.

It remains to be shown that STP$(G, \mathcal{N})$ is fulldimensional. Due to (1)–(3) it suffices to construct a routing P with $a^T\chi^P \ge 2$. Let $e \in E\backslash(F \cup F_1)$. Property (i) guarantees that such an edge exists. Moreover, property (iv) implies that there exist nodes $u, v \in V$ and a quasi path $Q_{F_1}(u, v)$ from u to v with $e \in Q_{F_1}(u, v)$. W.l.o.g. let $u = (r, 2)$ and $v = (s, 2)$ for some $r, s \in \{1, \dots, h\}$, $r < s$. From (ii) it follows that there exist an index $k \in \{1, 2\}$ and a quasi path Q' with respect to F_k such that $e \notin Q'$. Let $\{e'\} = Q' \cap \{E\backslash(F \cup F_k)\}$. Obviously, $e \ne e'$. We choose $S_1 = S_1^r \cup Q_{F_1}(u, v)\backslash J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1, S_2) \cup_k e'$ is a routing with $a^T\chi^P = 2$.

In the remainder of the proof we show that properties (i) to (iv) are also necessary. We start by proving property (iii).

(iii) Suppose there exists a path W from (r, l) to (s, l) in $V(F_k)$ with $r < s$ and k, $l \in \{1, 2\}$. We consider the case $k = 1$ and $l = 2$ (the other cases can be shown similarly). We choose $S_1 = S_1^r \cup W\backslash J_2^{r,s}$ and $S_2 = S_2^s$. Then, $P = (S_1, S_2)$ is a routing with $a^T\chi^P = 0$, a contradiction.

Since property (iii) holds and since there does not exist a routing in $(V(F), F)$, we know that, for each edge-minimal routing P (an *edge-minimal* routing $P = (S_1, \dots, S_N)$ is a routing where each Steiner tree S_k is edge-minimal, that is, S_k is a tree whose leaves are terminals) with $a^T\chi^P = 1$, there exists exactly one quasi path Q with $Q \subseteq P$. In the following we denote this unique quasi path by Q^P. Let us now show the remaining properties.

(i) We claim that there exists an edge $e \in F$ such that $e \in P$ for all edge-minimal routings P with $\chi^P \in I_\alpha$, if property (i) does not hold. This implies that $I_\alpha \subseteq \{x \in \text{STP}(G, \mathcal{N}) \mid x_e^1 + x_e^2 = 1\}$, a contradiction. Suppose now, property (i) does not hold. Then, there exists an $i \in \{1, \dots, h\}$ for which the required quasi paths in (i) do not exist. By negation of condition (a) and (b) we distinguish the following cases:

(b1) does not hold. Let $P = (S_1, S_2)$ be any edge-minimal routing with $\chi^P \in I_\alpha$ and let $e = [(i, 1), (i+1, 1)]$. Suppose $e \notin P$. Since condition (a) of (i) does not hold, the unique quasi path $Q_{F_k}^P((r, l)(s, l))$, where k, $l \in \{1, 2\}$ and $r, s \in \{1, \dots, h\}$, satisfies $r > i - |k - l|$ or $s < i + 2 - |k - l|$. Moreover, $r \le i$ and $s \ge i+1$, since $e \notin P$. If $r > i - |k - l|$, we know that $k \ne l$ and $r = i$. Thus, since (b1) does not hold, we obtain that $l = 1$ and $k = 2$. Hence, $e \in S_1$, a contradiction. On the other hand, if $s < i + 2 - |k - l|$, we have that $k = l$ and $s = i + 1$. Since (b1) is not true, we obtain $k = l = 1$ and $e \in S_2$, a contradiction.

(b2) does not hold. Analogously, it can be shown that $e = [(i, 2), (i+1, 2)]$ is an element of every edge-minimal routing P with $\chi^P \in I_\alpha$.

(b3) does not hold. Let $P = (S_1, S_2)$ be any edge-minimal routing with $\chi^P \in I_a$ and let $e = [(i, 1), (i, 2)]$. Suppose $e \notin P$. Then, one of the following cases holds for the unique quasi path $Q^P_{F_k}((r, l)(s, l))$. Either $r = i - 1$ and $s = i$, or $r = i$ and $s = i + 1$, or $r = i$ and $k \neq l$, or $s = i$ and $k = l$. It is easy to see that in all four cases edge e must be in P, a contradiction.

Hence, property (i) is necessary, indeed.

(ii) Now, suppose there exists an edge $e \in \bigcap_{Q \in \mathcal{Q}} Q$. Since for each edge-minimal routing P with $a^T\chi^P = 1$, there exists a unique quasi path, we conclude that e is contained in every such routing. Thus, $I_a \subseteq \{x \in \mathrm{STP}(G, \mathcal{N}) | x_e^1 + x_e^2 = 1\}$, a contradiction.

(iv) Suppose F_1 and F_2 are not maximal with respect to properties (i) to (iii). Then, choose $F_1' \subseteq E\setminus F$ and $F_2' \subseteq E\setminus F$ such that $F_1 \cup F_2 \subset F_1' \cup F_2'$, and F_1' and F_2' are maximal with respect to properties (i)–(iii). Due to part 1 of the proof, we know that $(\chi^{E\setminus(F \cup F_1')}, \chi^{E\setminus(F \cup F_2')})^T x \geq 1$ defines a facet of $\mathrm{STP}(G, \mathcal{N})$. By summing up this facet-defining inequality together with the valid inequalities $x_e^1 \geq 0$ for all $e \in F_1'\setminus F_1$ and $x_e^2 \geq 0$ for all $e \in F_2'\setminus F_2$ we obtain $a^T x \geq 1$. Hence, $a^T x \geq 1$ does not define a facet of $\mathrm{STP}(G, \mathcal{N})$, a contradiction.

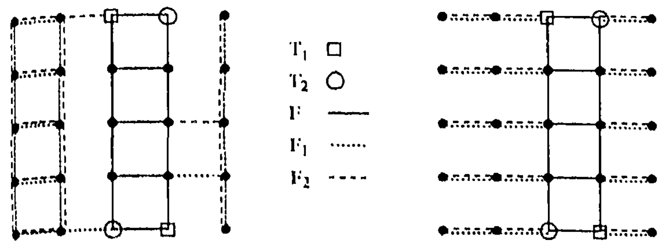
This completes the proof. □

Let us now illustrate Theorem 2.4 on two switchbox problems.

Example 2.5: Let $G = (V, E)$ be a complete $h \times b$ grid graph with $h \geq 3$, $b \geq 4$, and assume that the nodes of V are numbered such that $V = \{(i, j) | i = 1, \ldots, h, j = 1, \ldots, b\}$. Suppose $T_1 = \{(1, s), (h, s + 1)\}$ and $T_2 = \{(1, s + 1), (h, s)\}$ for some $s \in \{2, \ldots, b - 2\}$. Set $F := E(U)$, where $U := \{(i, j) | i = 1, \ldots, h, j = s, s + 1\}$.

- *Choosing $F_k := E\setminus(F \cup \delta(U)) \cup \{[(l_k, s - 1), (l_k, s)], [(r_k, s + 1), (r_k, s + 2)]\}$ for some $r_k, l_k \in \{1, \ldots, h\}$, $k = 1, 2$, meets the conditions of Theorem 2.4, and consequently, the corresponding grid inequality is facet-defining for $STP(G, \{T_1, T_2\})$ (see Figure 2.3 (a)).*
- *Choosing as F_1 and F_2 the set of all horizontal edges (i.e., edges $[(i, j), (i, j + 1)]$, $i \in \{1, \ldots, h\}$, $j \in \{1, \ldots, b - 1\}$) not included in F satisfies the conditions of Theorem 2.4, too. Therefore, the corresponding grid inequality is facet-defining for $STP(G, \{T_1, T_2\})$ (see Figure 2.3 (b)).*

Now we turn to the last class of inequalities we intend to describe in this paper. For a node set $W \subseteq V$, we define $S(W) = \{k \in \{1, \ldots, N\} | T_k \cap W \neq 0, T_k \cap (V\setminus W) \neq \emptyset\}$. We call a cut induced by a node set W *critical for* $(G, \mathcal{N})$, if $s(W) := |\delta(W)| - |S(W)| \leq 1$. If V_1, V_2, V_3 is a partition of V (that is V_1, V_2, V_3 are



(a) (b)

Fig. 2.3.

pairwise disjoint node sets with $V_1 \cup V_2 \cup V_3 = V$) such that $\delta(V_1)$ us a critical cut and if $T_1 \cap V_1 = \emptyset$ and $T_1 \cap V_i \neq \emptyset$ for $i = 2, 3$, we call the inequality

$$x^1([V_2 : V_3]) \geq 1$$

a *critical cut inequality (with respect to T_1)*. Then, the following theorem holds.

Theorem 2.6: Let $G = (V, E)$ be a graph with $V = \{u, v, w\}$. Moreover, let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a net list such that all terminal sets are of cardinality two and $T_1 = \{u, v\}$. Set $E_{ij} := \{e \in E | e$ is incident to i and $j\}$ for $i, j \in V$ and $N_i = \{k \in \{1, \ldots, N\} | i \in T_k\}$ for $i \in V$. Assume that $|E_{uv}| \geq 2$, $N_w = \{2, \ldots, N\}$, $|E_{uw}| \geq |N_u| - 1$, $|E_{vw}| \geq |N_v| - 1$ and $|E_{uw}| + |E_{vw}| = N$; see Figure 2.4. Then, the inequality

$$x^1(E_{uv}) \geq 1$$

defines a facet for $STP(G, \mathcal{N})$.

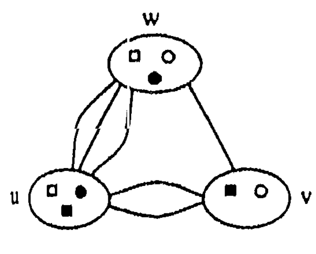


Fig. 2.4.

Proof: Let $a = (\chi^{E_{uv}}, 0, \ldots, 0) \in \mathbb{R}^{\mathcal{N} \times E}$. First, we show that the inequality is valid. Let S_1 be a Steiner tree for T_1 with $S_1 \cap E_{uv} = \emptyset$. Then, we know that $S_1 \cap E_{uw} \neq \emptyset$ and $S_1 \cap E_{vw} \neq \emptyset$. Thus, $|\delta(w)\backslash S_1| \leq |E_{uw}| + |E_{vw}| - 2 = N - 2$. Since $|N_w| = N - 1$ and $1 \notin N_w$, there cannot exist Steiner trees $S_2, \ldots, S_N$ such that $P = (S_1, \ldots, S_N)$ is a routing. So, we conclude that $a^T x \geq 1$ is valid.

Suppose, $b^T x \geq \beta$ is a facet-defining inequality of STP$(G, \mathcal{N})$ such that $I_a := \{x \in \text{STP}(G, \mathcal{N}) | a^T x = 1\} \subseteq I_b := \{x \in \text{STP}(G, \mathcal{N}) | b^T x = \beta\}$. In the following we show that b is a multiple of a. We prove this statement for the case $|E_{uw}| = |N_u|$ and $|E_{vw}| = |N_v| - 1$. The other case, $|E_{uw}| = |N_u| - 1$ and $|E_{vw}| = |N_v|$, can be shown accordingly (note that $|N_u| + |N_v| = N + 1$). For a routing $P = (S_1, \ldots, S_N)$ and an edge $e \in E$, we abbreviate $(S_1, \ldots, S_k \cup \{e\}, \ldots, S_N)$ by $P \cup_k e$.

(1) $b_e^k = 0$ for $e \in E_{uw}$ and $k = 1, \ldots, N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv}$. Furthermore, for every $k \in N_u\backslash\{1\}$, we set $S_k = \{e_k\}$ *with* $e_k \in E_{uw}\backslash\{e\}$ such that the edge sets S_k, $k \in N_u\backslash\{1\}$, are pairwise disjoint. Similarly, for every $k \in N_v\backslash\{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v\backslash\{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw}\backslash\{e\}| = |N_u| - 1$. Thus, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in I_a$. This yields $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(2) $b_e^k = 0$ for $e \in E_{uv}$ and $k = 2, \ldots, N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv}\backslash\{e\}$. This is possible, since $|E_{uv}| \geq 2$. Furthermore, for $k \in N_u\backslash\{1\}$, we choose $S_k = \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u\backslash\{1\}$, are pairwise disjoint. Analogously, for $k \in N_v\backslash\{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v\backslash\{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw}| = |N_u|$. Hence, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in I_a$, and we obtain that $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(3) $b_e^k = 0$ for $e \in E_{vw}$ and $k = 1, \ldots, N$.

Set $S_1 = \{f\}$ for some $f \in E_{uv}$. Furthermore, for $k \in N_u\backslash\{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u\backslash\{1\}$, are mutually disjoint. This is possible, since $|E_{uw}| = |N_u|$. Let $k_0 \in N_v$ and $\{f_1\} = E_{uw}\backslash\bigcup_{k \in N_u\backslash\{1\}} S_k$. This edge f_1 exists, since $|E_{uw}| = |N_u|$. Moreover, let $f_2 \in E_{uv}\backslash S_1$. This edge also exists, since $|E_{uv}| \geq 2$. Set $S_{k_0} = \{f_1, f_2\}$. For $k \in N_v\backslash\{1, k_0\}$, we choose $S_k = \{e_k\}$ with $e_k \in E_{vw}\backslash\{e\}$ such that the edge sets S_k, $k \in N_v\backslash\{1, k_0\}$, are pairwise disjoint. Again, this is possible, since $|E_{vw}\backslash\{e\}| = |N_v| - 2$. Thus, $P = (S_1, \ldots, S_N)$ and $P' = P \cup_k e$ are routings with $\chi^P, \chi^{P'} \in I_a$, and we conclude that $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(4) $b_e^1 = b_f^1$ for $e, f \in E_{uv}$.

Set $S_1 = \{f\}$ and $S_1' = \{e\}$. Furthermore, for $k \in N_u\backslash\{1\}$, we choose $S_k = \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u\backslash\{1\}$, are pairwise disjoint. Similarly, for $k \in N_v\backslash\{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that the edge sets S_k, $k \in N_v\backslash\{1\}$, are mutually disjoint. This is possible in both cases, since $|E_{vw}| = |N_v| - 1$ and $|E_{uw}| = |N_u|$. Thus, $P = (S_1, S_2, \ldots, S_N)$ and $P' = (S_1', S_2, \ldots, S_N)$ are routings with $\chi^P, \chi^{P'} \in I_a$. This yields $0 = b^T\chi^{P'} - b^T\chi^P = b_e^1 - b_f^1$.

Hence, we know that b is a multiple of a. To complete the proof we show that STP$(G, \mathcal{N})$ is fulldimensional. Otherwise, $a^T x \geq 1$ defines an equality of STP$(G, \mathcal{N})$. Due to (1)–(4) it suffices to construct a routing P with $a^T\chi^P \geq 2$. Choose $S_1 = E_{uv}$. Moreover, for $k \in N_u\backslash\{1\}$, we set $S_k = \{e_k\}$ with $e_k \in E_{uw}$ such that the edge sets S_k, $k \in N_u\backslash\{1\}$, are mutually disjoint. Similarly, for $k \in N_v\backslash\{1\}$, we choose $S_k = \{e_k\}$ with $e_k \in E_{vw}$ such that S_k, $k \in N_v\backslash\{1\}$, are pairwise disjoint. Then, $P = (S_1, \ldots, S_N)$ is a routing, and we have that $a^T\chi^P = |E_{uv}| \geq 2$. This completes the proof. □

We have introduced above several classes of inequalities for STP$(G, \mathcal{N})$ which define facets of special routing instances. Though in the first two theorems a complete graph and a disjoint net list is assumed, we can transform the results to grid graph routing problems by applying Lemma 2.2 and Lemma 2.3. Theorem 2.4 is directly applicable to grid graph instances, because a complete rectangular grid graph occurs as a subgraph of G. Theorem 2.6 can be interpreted as a polyhedral formulation of the so-called *cut condition* which says that $s(W)$ has to be non-negative for all $W \subset V$. This condition has been intensively studied in the literature especially for grid graphs (see for example [F90], [KM90], [OS81]).

3 Separation

In this section we discuss the separation problem for two classes of inequalities introduced in the last section. The *separation problem* for a class of inequalities can be stated as follows. "Given a vector $y \in \mathbb{R}^{\mathcal{N} \times E}$, decide whether y satisfies all inequalities of the given class. If not, find an inequality of this class that is violated by y."

We first deal with the grid inequalities. Conditions (i) and (ii) are rather technical and we do not see how to handle them algorithmically. So, we decided to neglect these conditions as well as the assumption that C defines a cut in G. This way we obtain a larger class of inequalities and, certainly, a separation algorithm for this larger class may return an inequality that is not facet-defining. However, if the separation algorithm – provided it is exact – does not find any violated inequality, there are, in particular, no violated facet-defining grid inequalities.

We even go one step further and relax additional assumptions of Theorem 2.4, namely: $\hat{G}$ has to be a complete rectangular $h \times 2$ grid graph, $T_1 = \{(1, 1), (h, 2)\}$ and $T_2 = \{(1, 2), (h, 1)\}$. These three conditions are quite restrictive and are usually not satisfied by practical problem instances, even not by switchbox routing problems. We replace these properties by the condition that

(a) $H = (X, F)$ is a 2-node connected subgraph of G.
(b) $d_H(v) \leq 3$ for all $v \in X$ and $d_H(t) = 2$ for all terminals t (where $d_G(v)$ denotes the degree of node v in graph G).
(c) The terminal sets are located in an alternating sequence on the outer face of H, i.e., by walking around the outer cycle they appear in the sequence u, v, r, t or u, t, r, v where $u, r \in T_1$ and $v, t \in T_2$.

Obviously, $2 \times h$ grid graphs with the location of the terminals as in Theorem 2.4 satisfy these new conditions, but do these conditions still guarantee that the resulting inequality is valid? Suppose, this is not the case, then there exist two paths S_1, $S_2 \in F$ connecting T_1 and T_2, respectively. Since the terminals are located in an alternating sequence on the outer face of H and since each terminal has only degree 2 in H, there is a node v where both paths cross. This however implies that $d_H(v) \geq 4$ holds, a contradiction. This shows that the resulting inequality is valid, indeed.

Finally, we observe that the terminal sets need not necessarily lie on the outer face of H. It suffices that each terminal $t \in T_k$ is connected in $(V(F_k), F_k)$ to a node w_t in X playing the role of t on the outer face of H. Obviously, the inequality also remains valid if the terminal sets have cardinality greater or equal than 2.

Summing up, we search, given a vector $y \in \mathbb{R}^{\{1,2\} \times E}$, $y \geq 0$, and an instance $(G, \mathcal{N})$, for a subgraph $H = (X, F)$ and edge sets F_1, F_2 satisfying (a), (b) and the additional properties:

(c') There exist nodes $v_k^1, v_k^2 \in X$ that are connected to T_k in $(V(F_k), F_k)$ for $k = 1$, 2, and that satisfy (c).
(d) For all $u, v \in V(F)$, $u \neq v$, and $k \in \{1, 2\}$ there does not exist a path from u to v in (V, F_k) (see condition (iii) in Theorem 2.4).
(e) F_1 and F_2 are maximal with respect to the properties (a)–(d), see condition (iv) in Theorem 2.4.

Although these conditions appear more tractable than those of Theorem 2.4, we did not succeed in finding a polynomial time separation algorithm for these "generalized grid inequalites". Instead, we implemented a heuristic procedure that we outline now.

Algorithm 3.1: (Separation of grid inequalities)

(1) *Determine a 2-node connected subgraph $H = (X, F)$ of G.*

(2) *Let O be the set of outer face edges of H (note O defines a cycle in G), A and I the set of edges in E that are outside and inside the cycle O, respectively.*
(3) *Determine four nodes v_1^1, v_1^2, v_2^1, $v_2^2 \in V(O)$ satisfying* (c') *and let W_1^1, W_1^2, W_2^1, W_2^2 be the corresponding paths connecting these nodes to the terminal sets.*
(4) *If such nodes do not exist, STOP (no violated inequality was found).*
(5) *Add edges of I to O such that condition* (b) *remains satisfied.*
(6) *For $k = 1, 2$: Set $F_k = W_k^1 \cup W_k^2$, and add edges of A to F_k such that condition* (d) *remains satisfied.*
(7) *If $a = (\chi^{E \setminus (O \cup F_1)}, \chi^{E \setminus (O \cup F_2)})^T y < 1$, return the inequality $a^T x \geq 1$, otherwise the algorithm fails.*

Some comments concerning the implementation of Algorithm 3.1 are in order. In step (1) we look for a 2-node connected subgraph H in the graph $(V, \bar{E})$ where $\bar{E} := \{e \in E \mid y_e^1 + y_e^2 \geq 0.25\}$. The idea is that edges belonging to H get a coefficient of zero in the resulting inequality. Therefore, we choose edges whose corresponding variables have a high value in the LP-solution. The same ideas apply to the selection of the edges in steps (5) and (6). The nodes in step (3) are determined such that the weighted sum of the incident edges that do not belong to O is minimized.

Certainly, all these ideas are rather heuristic and there is no guarantee to find violated grid inequalities, whenever there exist such. However, it turns out that this algorithm finds several inequalities that help in solving practical problem instances (see next section).

Now we turn to the critical cut inequalities. We start with some important implications that can be derived from critical cuts.

Consider a node set $W \subseteq V$ whose induced cut $\delta(W)$ is critical, i.e., $s(W) \leq 1$. If $s(W) < 0$, we immediately conclude that the routing problem is infeasible and we stop the algorithm. In case $s(W) = 0$, we know that each Steiner tree S_k for T_k with $T_k \subseteq W$ cannot use edges of $\delta(W)$ and $E(V \setminus W)$. Thus, we can fix all variables x_e^k, $e \in \delta(W) \cup E(V \setminus W)$, to zero. Furthermore, since there always exists an optimum solution that is edge-minimal, the same arguments apply to the case $s(W) \leq 1$ and we can fix variables to zero correspondingly.

Note that such variables can be fixed at the very beginning of our branch and cut algorithm. These observations do not only help in reducing the size of the problems, but also help in separating critical cut inequalities in the following way. We call a cut $\delta(W)$ induced by a node set W a *Steiner cut* (with respect to some terminal set T_k) if $W \cap T_k \neq \emptyset$ and $T_k \not\subseteq W$. Obviously, each Steiner tree S_k for T_k contains at least one edge of $\delta(W)$ and thus the inequality $x^k(\delta(W)) \geq 1$ is valid.

Consider now the situation described in Theorem 2.6. The set $\delta(v)$ is a Steiner cut for T_1. Since $\delta(v) = E_{uv} \cup E_{vw}$ and since all variables x_e^1, $e \in E_{vw}$, are fixed to zero, the Steiner cut inequality $x^1(\delta(v)) \geq 1$ reads $x^1(E_{uv}) \geq 1$ which is the critical cut inequality of Theorem 2.6.

Therefore, critical cut inequalities can be automatically separated by separating Steiner cut inequalities. Steiner cut inequalities, however, can be separated in polynomial time by determining a minimum cut for each pair of terminals.

Thus, we are left with the task of finding critical cuts for a given instance $(G, \mathcal{N})$. Our procedure for finding such cuts applies to switchbox routing instances and is based on the observation stated in the following lemma.

Lemma 3.2: Let G be a complete rectangular grid graph and $\mathcal{N}$ a net list such that all terminal sets lie on the outer face. Let $W \subset V$, $\emptyset \neq W \neq V$, be a set of nodes and let the cut induced by W be critical with respect to the given instance $(G, \mathcal{N})$. Then, one of the following statements is true.

(i) *There exists a node $w \in V$ such that $\delta(w)$ is a critical cut with respect to $(G, \mathcal{N})$.*
(ii) *There exists a horizontal or vertical cut which is critical with respect to $(G, \mathcal{N})$.*
(A cut F is called horizontal *if there exists some $i \in \{1, \ldots, h-1\}$ such that $F = \{uv \in E | u = (i, j)$ and $v = (i+1, j)$ for some $j \in \{1, \ldots, b\}\}$; a vertical cut is defined accordingly.)*

This lemma can be proved by a thorough case analysis (see [M92]).

In our implementation we check all cuts that are horizontal, vertical or induced by some node $v \in V$ for being critical. Thus, we are sure that the algorithm finds a critical cut if there exists one. In order to find as many critical cuts as possible we extend each node inducing a critical cut in all four possible directions as long as the resulting node set induces a critical cut. Practical experiences show that this procedure is effective (see next section).

4 Computational Results

In this section we evaluate our separation algorithms for the class of critical cut and grid inequalities and show how the overall performance of the branch and cut algorithm is improved by taking these two classes of inequalities into account. Some features of the branch and cut algorithm are discussed, too. We have tested our algorithm on switchbox routing problems that are discussed in the literature. Table 1 summarizes the data.

Column 1 presents the name used in the literature. In column 2 and 3 the height and width of the underlying grid graph is given. Column 4 contains the number of nets. Columns 5 shows the resulting number of 0/1 variables. Finally, the last column states the reference to the paper the example is taken from. In all examples the edge weights are identical to one.

Table 1.

Example	height	width	nets	variables	Ref.
difficult switchbox	15	23	24	15648	[BP83]
more difficult switchbox	15	22	24	14952	[CH88]
terminal intensive switchbox	16	23	24	16728	[L85]
dense switchbox	17	15	19	9082	[L85]
augmented dense switchbox	18	16	19	10298	[L85]
modified dense switchbox	17	16	19	9709	[CH88]
pedagogical switchbox	16	15	22	9878	[CH88]

One of the main components of a branch and cut algorithm is certainly the LP solver. We use CPLEX 2.1 for the runs of our algorithm on the above switchbox instances. The linear programs we encounter appear to be quite difficult. One of the reasons for this is that these linear programs have many alternative optimum solutions and are simultaneously primally and dually degenerate (typically more than 90% of the CPU-time is spent solving the LPs). To overcome these difficulties we perturbe the right hand sides of the linear programs appropriately guaranteeing, however, that every optimum solution to the perturbed program remains optimum for the original program (see [GMW92b] for details). Other important features of our code are an LP based primal heuristic, preprocessing and initial cuts, detection of branching variables, determining the "tailing off" and certainly the interplay between all the separation routines that we designed and implemented. The latter issue is discussed now. We have implemented separation algorithms for the four classes of inequalities:

(1) alternating cycle inequalities,
(2) Steiner partition inequalities (see [GMW92b] for a formal definition),
(3) critical cut inequalities,
(4) grid inequalities.

In the following we test the performance of the branch and cut algorithm by using the two strategies: (a) separation of (1) and (2) only; (b) separation of (1), (2), (3) and (4). In a first test series we measure the CPU time that is required to obtain certain values of the lower bound by using strategy (a) or (b), respectively. More precisely, for each of the examples we have chosen two integer values, v_1, v_2, say and measured the CPU time of the branch and cut algorithm that was

Table 2.

Example	Lower Bound	Strategy (a)	Strategy (b)
difficult switchbox	460 464	237:54 1338:22	90:12 688:49
more difficult switchbox	450 452	324:37 854:26	164:25 530:11
terminal intensive switchbox	530 535	562:38 2841:48	154:52 686:45
dense switchbox	434 438	96:50 2204:06	21:38 226:28
augmented dense switchbox	463 465	304:27 1184:23	66:25 269:20
modified dense switchbox	450 452	151:13 4441:43	44:04 387:03
pedagogical switchbox	328 331	75:00 306:01	15:04 117:55

Table 3.

Example	10 minutes		100 minutes		1000 minutes	
	(a)	(b)	(a)	(b)	(a)	(b)
difficult switchbox	197	194	195	192	21	6
more difficult switchbox	180	178	176	175	22	0
terminal intensive switchbox	185	180	176	169	165	39
dense switchbox	45	45	44	42	41	3
augmented dense switchbox	9	7	9	6	5	3
modified dense switchbox	62	62	62	2	60	0
pedagogical switchbox	126	122	121	12	0	0

necessary to obtain a lower bound of value v_1 or v_2 by using the two alternative separation strategies. The values v_1 and v_2 are shown in column 2 of Table 2. The time required to obtain a lower bound of v_1 and v_2 is given in column 3, if strategy (a) is used and in column 4, if strategy (b) is used. Our second test series compares the gap between lower bound and upper bound after the branch and cut code has run for 10, 100 and 500 minutes by using strategy (a) or (b), respectively. Table 3 summarizes the results. The entries in columns 2, 4 and 6 correspond to the gap between lower and upper bound after the code ran for 10, 100 and 500 minutes, respectively, by using strategy (a). Correspondingly, columns 3, 5 and 7 give the gap if strategy (b) is applied and again 10, 100 or 500 minutes have been spent by the algorithm.

In all these tests and for all examples, strategy (b) is significantly superior than strategy (a), since simultaneously the values of the lower and upper bound improve by adding violated critical cut and grid inequalities to the current LPs. This indicates that those inequalities do not only contribute to improving the value of the lower bound, but seem to give structure to the LP solutions as well that help the LP based primal heuristic in finding better feasible solutions. At least for our test instances, separating Steiner partition inequalities and alternating cycle inequalities only is rarely sufficient to solve these instances to optimality even when thousands of CPU minutes are available for the computations. For this reason we present our final results in Table 4 for the test instances when using strategy (b).

Column 2 of Table 4 gives the best feasible solution we have obtained with a primal heuristic. The entries in column 3 are the objective function values of the linear program (rounded up to the next integer) when no further violated con-

Table 4.

Example	Best Sol.	LP Value	Gap	Iter.	B & C	CPU-Time
difficult switchbox	464	464	0.0%	69	3	1564:15
more difficult switchbox	452	452	0.0%	53	1	983:23
terminal intensive switchbox	537	536	0.0%	163	13	3755:44
dense switchbox*	441	438	0.7%	119	4	1017:43
augmented dense switchbox*	469	467	0.4%	105	1	4561:41
modified dense switchbox	452	452	0.0%	51	1	387:03
pedagogical switchbox	331	331	0.0%	77	5	251:58

straints are found, i.e., when branching is performed for the first time. This values are lower bounds for the whole problem. In column 4 the percental deviation of the best solution from the lower bound is given. Column 5 (resp. 6) gives the number of cutting plane iterations (resp. the number of nodes in the branching tree). Finally, the last column reports on the running times. The values are stated in minutes obtained on a SUN 4/50GX workstation.

For the two examples "dense switchbox" and "augmented dense switchbox" marked with an asterisk, the execution of the branch and cut algorithm was stopped after the time given in the last column, because no further progress could be achieved. We believe that the values given in column 2 are optimal, but we are not yet able to prove this with the cutting plane algorithm. All other problem instances are solved to optimality. The running times in the last column are surely quite high. This is due to the fact that we were interest in finding an optimal solution. On the other hand, a provable quality guarantee of 5% can be given after at most 5 minutes for all these problem instances, which shows that our methodology is approaching practical usability. In fact, standard routing algorithms are rarely able to provide any quality guarantee at all.

Acknowledgement: This research was supported by Science Program SC1-CT91-620 of the EEC.

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Received: June 1993
Revised version received: July 1994