FACETS FOR POLYHEDRA ARISING IN THE DESIGN OF COMMUNICATION NETWORKS WITH LOW-CONNECTIVITY CONSTRAINTS

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Abstract. This paper addresses the important practical problem of designing survivable fiber optic communication networks. This problem can be formulated as a minimum-cost network design problem with certain low-connectivity constraints. Previous work presented structural properties of optimal solutions and heuristic methods for obtaining "near-optimal" network designs. Some exact and heuristic inequalities for the convex hull of the solutions to this problem are given. A companion paper describes computational results on real-world telephone network design problems with a cutting plane method based on this work. These computational results are summarized in the last section of this paper.

Keywords. Network design, network survivability, connectivity, polyhedral combinatorics

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1. Introduction. A recent trend in communication networks is the emergence of fiber optic technology as one of the major components in the "network of the future." This transmission medium is cost-effective and reliable, and provides very high transmission capacity. This combination promises to usher in new telecommunication services requiring large amounts of bandwidth. At the same time, the unique characteristics of this technology imply the need for new network design approaches. (See [CFLM] for more details.)

Survivability is an important factor in the design of communication networks. Network survivability is used here to mean the ability to restore service in the event of a catastrophic failure of a network component, such as the complete loss of a transmission link, or the failure of a switching node. Service could be restored by rerouting traffic through other existing network links and nodes, assuming that the design of the network has provided for this additional connectivity. Clearly, a higher level of redundant connectivity results in greater network survivability and a greater overall network cost. This leads to the problem of designing a minimum-cost network that meets certain required connectivity constraints.

Survivability is a particularly important issue for fiber networks. High capacity of fiber facilities results in much more sparse network designs with larger amounts of traffic carried by each link than is the case with traditional bandwidth-limited technologies. This increases the potential damage to network services due to link or node failures. It is necessary to trade off the potential for lost revenue and customer goodwill against the extra costs required to increase the network survivability. Recent works on methods for designing survivable fiber communication networks by [CMW] and [MS] conclude that (1) survivability is an important issue for fiber networks, (2) "two-connected" topologies provide a high level of survivability in a cost-effective manner, and (3) good heuristic methods exist for quickly generating "near-optimal"

networks. In particular, it was determined that a network topology should provide for at least two diverse paths between certain "special" offices, thus providing for protection against single link or single node failure, or traffic between these offices. These special offices represent high revenue-producing offices and other offices that require a higher level of network survivability.

We now formalize the network design problems that are being considered in this paper. A set of nodes \( V \) is given that represents the locations of the switches (offices) that must be interconnected into a network in order to provide the desired services. A collection \( E \) of edges is also specified that represents the possible pairs of nodes between which a direct transmission link can be placed. We let \( G = (V, E) \) be the (undirected) graph of possible direct link connections. The graph \( G \) may have parallel edges but contains no loops. (Thus we assume throughout this paper that all graphs considered are loopless. But they may have parallel edges. Graphs without parallel edges are called simple.)

Given a graph \( G = (V, E) \) and \( W \subseteq V \), the edge set \( \delta(W) := \{ (i, j) \in E \mid i \in W, j \notin W \} \) is called the cut (induced by \( W \)). We will write \( \delta_G(W) \) to make clear—in case of possible ambiguities—with respect to which graph the cut induced by \( W \) is considered.

For \( W, W' \subseteq V \) with \( W \cap W' = \emptyset \) we define \( |W \setminus W'| := \{ (i, j) \in E \mid i \in W, j \notin W \} \). So \( \delta_G(W) = W \setminus W' \).

For \( W \subseteq V, \) we denote by \( G(W) \) the subgraph of \( G \) induced by \( W \) and by \( E(W) \) its edge set \( \{ (i, j) \in E \mid i, j \in W \} \). \( G(W) \) is the graph obtained from \( G \) by contracting the nodes in \( W \) to a new node \( w \) (retaining parallel edges). We call \( \delta_G(W) \) the reverse operation of replacing the shrunk node \( w \) by the original node set \( W \) the expansion of \( W \) in \( G \). We will denote by \( G - w \) the graph obtained by removing the vertex \( w \) and all incident edges from \( G \), and by \( G - \{ w \} \) the graph obtained by removing the edge set \( \delta_G(W) \) from \( G \) (we write \( G - \{ w \} \) instead of \( G - \{ w \} \)). If \( G - v \) has more connected components than \( G \) for some node \( v \), we will call \( v \) an articulation node of \( G \). Similarly, if \( G - w \) has more connected components than \( G \), we will call \( w \) a bridge of \( G \).

Each edge \( e \in E \) has a fixed cost \( c_e \) of establishing the direct link connection. The cost of establishing a network \( N = (V, E) \) consisting of a subset \( S \subseteq E \) of edges is \( c(N) := \sum_{e \in S} c_e \). It is the sum of the costs of the individual links contained in \( S \).

The goal is to build a minimum-cost network so that the required survivability conditions, which we describe below, are satisfied. We note that the cost here represents setting up the topology for the communication network and includes placing conduits in which to put them. The emphasis is on service and other related costs. We do not consider costs that depend on how the network is implemented, such as routing, multiplexing, and repeater costs. Although these costs are also important, it is usually the case that in a fixed design and then these other costs are considered in a second stage of optimization.

For any pair of distinct nodes \( s, t \in V \), an \( s, t \)-path \( P \) is a sequence of nodes and edges \( (v_0, v_1, v_2, \ldots, v_{i-1}, v_i, \ldots, v_{k-1}, v_k) \), where \( e_i \) is incident with the nodes \( v_{i-1} \) and \( v_i \) (\( i = 1, \ldots, k \)) and where no node or edge appears more than once in \( P \). A collection \( P_1, P_2, \ldots, P_k \) of these \( s, t \)-paths is called edge-disjoint if no edge appears in more than one path, and is called node-disjoint if it has no node (except for \( s \) and \( t \)) appearing in more than one path. (Remark: In order to be consistent with standard graph theory we do not consider two parallel edges as two node-disjoint paths.)

The survivability conditions require that the network satisfy certain edge- and node-connectivity requirements. In particular, each node \( s \in V \) has an associated
nonnegative integer $r_{st}$, which represents its connectivity requirement. This means that for each pair of distinct nodes $s, t \in V$, the network $N = (V, E)$ to be designed has to have at least

$$r(s, t) := \min\{r_{st}, r_{ts}\}$$

edge-disjoint (or node-disjoint) $(s, t)$-paths. These conditions ensure that some communication path between $s$ and $t$ will survive a prespecified level of edge (or node) failures. The levels of survivability specified depend on the relative importance placed on maintaining connectivity between different pairs of offices.

The fiber optic network design problems that arise in practice and that we are addressing in this paper have three types of offices. The so-called "special" offices have connectivity requirement 2 while "ordinary" offices have connectivity requirement 1. An office with connectivity requirement 0 is called "optional" since it need not be part of the network to be designed.

Figure 1.1 shows an example network. Special offices are indicated by squares, ordinary offices by circles. Optional offices do not occur. The lines (thin, bold, and dashed) represent the possible direct links from which the minimum-cost survivable network must be designed. The network obtained by removing the dashed lines, i.e., the graph formed by the union of bold and thin lines, represents a feasible network. It consists of a two-connected part (the bold lines) containing all special nodes, in which every pair of nodes is linked by at least two edge-disjoint paths and a collection of trees (the thin lines), which link the remaining nodes into the two-connected part.

Thus in the remainder of this paper we consider the case where the connectivity requirements satisfy

$$r_{st} \in \{0, 1, 2\} \quad \text{for all } s, t \in V.$$ 

Nodes of connectivity requirement 0 (respectively, 1, 2) will also be called nodes of type 0 (respectively, type 1, 2). Let us define the 2ECON problem (respectively, 2NCON problem) to be the network design problem where between each pair of distinct nodes $s$ and $t$ at least $\min\{r_{st}, r_{ts}\}$ edge-disjoint (respectively, node-disjoint) paths are required.

Given $G = (V, E)$, we extend the connectivity requirement function $r$ to functions operating on sets by setting

$$r(W) := \max\{r_{st} \mid s \in W\} \quad \text{for all } W \subseteq V,$$

$$\text{con}(W) := \max\{r(s, t) \mid s \in W, t \in V \setminus W\},$$

and

$$\min\{r(W), r(V \setminus W)\} \quad \text{for all } W \subseteq V, \quad \emptyset \neq W \neq V.$$

Let us now introduce, for each edge $e \in E$, a variable $x_e$ and consider the vector space $\mathbb{R}^E$. Every subset $F \subseteq E$ induces an incidence vector $x_F = (x_e)_{e \in E} \in \mathbb{R}^E$ by setting $x_e := 1$ if $e \in F$, and $x_e := 0$ otherwise. Vice versa, each $0/1$-vector $x \in \mathbb{R}^E$ induces a subset $F_x := \{e \in E \mid x_e = 1\}$ of the edge set $E$ of $G$. For any subset of edges $F \subseteq E$, we define $x(F) := \sum_{e \in F} x_e$. We can now formulate the 2NCON network design problem introduced above as the following integer linear program:

\begin{align*}
\max & \quad \sum_{e \in E} r_e x_e \\
\text{subject to} & \quad \sum_{e \in F} x_e \geq r(F) \quad \text{for all } F \subseteq E, \quad \emptyset \neq F \neq E, \\
0 & \leq x_e \leq 1, \quad \forall e \in E.
\end{align*}
program:

\[
\begin{align*}
\min & \sum_{i,j} c_{ij} x_{ij} \\
\text{subject to} & \\
(i) & \pi(W) \geq \gamma(W) \quad \forall W \subseteq V, \emptyset \neq W \subseteq V; \\
(ii) & \pi(\emptyset_r(W)) \geq 1 \\
& \quad \forall z \in V, \text{for all } W \subseteq V \setminus \{z\}, \emptyset \neq W \subseteq V \setminus \{z\} \text{ with } r(W) = 2 \text{ and } r(V \setminus (W \cup \{z\})) = 2; \\
(iii) & 0 \leq x_{ij} \leq 1 \\
& \quad \forall i \in \bar{E}; \\
(iv) & x_{ij} \text{ integral} \\
& \quad \forall i \in \bar{E}.
\end{align*}
\]

It follows from Menger's theorem that, for every feasible solution \( x \) of (1.1), the subgraph \( N = (V, F^*) \) of \( G \) defines a network satisfying the two-connected survivability requirements for the 2CON problem. Removing (ii), we have an integer linear program for the 2ECON network design problem. (Note that in the case \( r = (0, 1)^V \), inequalities (i), (iii), and (iv) of (1.1) characterize the Steiner tree problem.) An inequality of type (i) is called a cut inequality, one of type (ii) is called a node-cut inequality, and one of type (iii) is called a trivial inequality.

The main objective of this paper is to study the 2ECON and 2CON network design problems from a polyhedral point of view to see which inequalities are suitable choices for a cutting plane approach, i.e., we want to find a tighter LP-relaxation than the one obtained by dropping the integrality constraints (iv) of (1.1) for the 2ECON and 2CON network design problems. To do this we define the following polytopes. Let \( G = (V, E) \) be a graph and let \( r = (0, 1)^V \) be given with \( r_e = r_v = 2 \) for at least two nodes.

\[
\begin{align*}
0_{2NCON}(G) & : \text{conv}\{x \in \mathbb{R}^E : x \text{ satisfies (i), (ii), (iii), (iv) of (1.1)}\}, \\
2\text{CON}(G) & : \text{conv}\{x \in \mathbb{R}^E : x \text{ satisfies (i), (ii), (iii) of (1.1)}\}.
\end{align*}
\]

are the polytopes associated with the 2CON and 2ECON network design problems. (Above, "conv" denotes the convex hull operator.) We say that \( F \subseteq E \) is feasible for one of these polytopes if \( x_F \) is.

Related problems have been investigated previously. A general integer linear programming approach to network design problems with connectivity requirements is presented in [GM] along with a preliminary study of these problems from a polyhedral point of view. We shall make several references to this work in what follows. [CFN] study the dominant of the 2CON(G;r) polytope in the special case where \( r = (2)^V \). [MPM] study the 2ECON(G;r) and 2CON(G;r) polytopes in the special case where \( r = (2)^V \) and \( G \) is a complete graph with the edge weights satisfying the triangle inequality. They show that in this case the optimization problems are the same over both polytopes and then give a certain type of "characterization" of the optimal solutions.

Let us now introduce some connectivity functions and some notation concerning "essential edges" and dimension of polyhedra. Let \( G = (V, E) \) and \( r = (0, 1)^V \) be given; we say that \( e \in E \) is essential with respect to \( 2\text{CON}(G;r) \) if \( 2\text{CON}(G - e; r) = \emptyset \); similarly we say \( e \) is essential with respect to \( 0_{2NCON}(G;r) \) if \( 0_{2NCON}(G - e; r) = \emptyset \). In other words, \( e \) is essential if its deletion results in a graph such that one of the survivability requirements cannot be satisfied. We denote the set of edges of \( E \) that are essential with respect to \( 2\text{CON}(G;r) \) by \( 2\text{EES}(G;r) \) and the set of edges that are essential with respect to \( 0_{2NCON}(G;r) \) by \( 2\text{NES}(G;r) \). Clearly, for
in \([GMS]\). Based on the polyhedral investigations presented in this paper we have designed cutting plane algorithms for the 2ECON and 2NCON problems. A short summary of our computational results is given in \([GMS]\). The details can be found in \([GMS]\) and \([S]\).

2. Decomposition. The problem of finding a cost-minimal network for the 2ECON problem can be decomposed into at least two independent problems if the graph \(G\) contains an articulation node \(v\) disconnecting two nodes of type at least 1. The subproblems are solved on the two-node-connected components of \(G\) with the same cost function and the same connectivity types \(r\) only if the connectivity type of the articulation node \(v\) may have to be adjusted. The 2ECON problem may also be decomposed into independent subproblems if \(G\) contains two edges \(e, f\), such that in \(G-(e, f)\) two nodes of type 2 are disconnected. Another simple decomposition is possible for the 2NCON problem if the graph \(G\) contains two nodes \(u, v\) so that in \(G-(u, v)\) two nodes of type 2 are disconnected. These and other more complicated decompositions are described in more detail in \([GMS]\).

Observe that using the above decompositions, any 2ECON or 2NCON problem with essential edges may be decomposed into problems without essential edges. This is the reason why we restrict ourselves to graphs \(G\) and connectivity types \(r\) for which our general assumptions (ii) and (iii) of (1.2) hold. This implies also that 2ECON\([G; r]\) and 2NCON\([G; r]\) are full-dimensional \([GM]\).

There is another (technical) reason why we restrict ourselves to full-dimensional polyhedra here. If polyhedra are not full-dimensional, proofs often become more involved technically and statements about nonredundancy of certain systems become quite ugly due to the necessity to exclude equivalent inequalities. This is also true in our case. It is not difficult to derive the results for the lower-dimensional cases from the results presented later. But the statements of these theorems are often rather complicated and we want to avoid unnecessary technicalities.

3. Basic facts. In this section we investigate under which conditions the cut inequalities (1.1)(i), the node-cut inequalities (1.1)(ii), and the trivial inequalities (1.1)(iii) define facets for 2ECON\([G; r]\), respectively 2NCON\([G; r]\).

An inequality \(a^T x \leq \alpha\) is valid with respect to a polyhedron \(P \subseteq \mathbb{R}^n\), if \(a^T x = \alpha\) is called the face of \(P\) defined by \(a^T x \leq \alpha\). If \(\text{dim}(P) = \text{dim}(F) - 1\) and \(F \neq \emptyset\), then \(F\) is a face of \(P\) and \(a^T x \leq \alpha\) is called facet-defining or facet-inducing.

The following theorem follows from Theorem 3.3 in \([G]\) and characterizes which of the trivial inequalities (1.1)(iii) define facets.

**Theorem 3.1.** Let \((G, r)\) satisfy (1.2).

(a) \(a^T x \leq \alpha\) defines a facet of 2ECON\([G; r]\) if and only if \(a^T x = \alpha\) for all \(x \in K\).

(b) \(a^T x \geq \alpha\) defines a facet of 2ECON\([G; r]\) (respectively, 2NCON\([G; r]\)) if and only if \(a^T x \geq \alpha\) for all \(x \in K\).

The next theorem characterizes the cut inequalities (1.1)(i) that dominate facets.

**Theorem 3.2.** Let \((G, r)\) satisfy (1.2) and let \(W \subseteq V\) with \(V \neq W\).

(a) Suppose \(\text{cut}(W) = 2\). Then \(z(\delta(W)) \geq 2\) defines a facet of 2ECON\([G; r]\) if and only if

- \(a_1\) \(G[W]\) and \(G[V\setminus W]\) are connected;
- \(a_2\) \(\lambda_1(G[W]) \geq 2\) and \(\lambda_1(G[V\setminus W]) \geq 2\);
- \(a_3\) \(e\) is a bridge of \(G[W]\).

(b) Suppose \(\text{cut}(W) = 1\). Then \(z(\delta(W)) \geq 2\) defines a facet of 2NCON\([G; r]\) if and only if

- \(b_1\) \(G[W]\) and \(G[V\setminus W]\) are connected;
- \(b_2\) \(\lambda_1(G[W]) \geq 2\) and \(\lambda_1(G[V\setminus W]) \geq 2\);
- \(b_3\) \(e\) is a bridge of \(G[W]\).

3.2.3. Proofs. We give a proof of (d). (The proofs of (a) in the general case, (b), and (c) use the same ideas and are thus omitted. (c) is trivial.)
We first show that if one of the conditions (d\_1) - (d\_3) is not satisfied, then the cut inequality \( \pi(\delta(W)) \geq 2 \) does not define a facet. Necessity of (d\_1) is seen easily (see, e.g., Corollary 6.7 of [GM]). Suppose (d\_2) is violated. Let \( u \) be an articulation node of \( GW' \), and let \((S\_1, E_1), (S\_2, E_2)\) be two components of \( G(S) - u \) with \( \pi(S\_i) = \pi(S\_j) = 2 \). Then \( \pi(\delta(W)) \geq 2 \) can be written as the sum of the node-cut inequalities \( \pi(S\_i - u) \geq 1 \) and \( \pi(S\_j - u) \geq 1 \) plus possibly some nonnegativity constraints.

Therefore, \( \pi(\delta(W)) \geq 2 \) does not define a facet. If (d\_3) is violated there are nodes \( u, v, v \) and node sets \( U, U \) with the indicated properties and \( U \cup \emptyset = \emptyset \). In this case the cut inequality can be written as the sum of two other node-cut inequalities \( \pi(S\_1 - U) \geq 1 \) and \( \pi(S\_2 - U) \geq 1 \). Hence \( \pi(\delta(W)) \geq 2 \) does not define a facet.

Now suppose we have the situation excluded by (d\_4) for \( S = W \). In this case, it is not possible to construct a feasible solution with \( \pi(\delta(W)) = 2 \) and \( x = 0 \), because any feasible set not using \( e \) would either have node \( w_u \) as an articulation node or use three edges of \( GW' \). Therefore, all feasible sets \( G \) with \( |C \cap \delta(W)| = 2 \) have to use \( e \), so the face defined by \( \pi(\delta(W)) = 2 \) is contained in the face defined by \( x = 0 \). Since 2CNCON(G;r) is full-dimensional, these faces do not define a facet. Therefore, \( \pi(\delta(W)) \geq 2 \) does not define a facet.

Suppose (d\_4) is violated. Let the two neighbor nodes of \( V(S) \) in \( S \) be called \( u, v \) and \( w, v \). In contradiction to (d\_4), there is at least one node of type 2 in \( S\_i \), say \( u \) or \( r_x = r_u = 2 \), then \( \pi(\delta(W)) \geq 2 \) can be written as the sum of the two node-cut inequalities \( \pi(S\_i - (u, v)) \geq 1 \) and \( \pi(S\_i - (u, v)) \geq 1 \).

Now let the conditions of (d\_4) be satisfied for some inequality \( \pi(\delta(W)) \geq 2 \). Let \( M \geq 2 \) be a facet-defining inequality such that \( t_1 \) and \( t_2 \) face \( F_2 \), induced by \( \pi(\delta(W)) \geq 2 \). The aim is to show that \( t \) is a positive multiple of \( M \), which implies that \( F_2 \) is identical with the facet \( F_2 \).

Let us first state some conditions under which for a given \( c, f \in GW' \), the incidence vector of \( C_{GW'} := \{(u,v) | u \in V(W), v \in f \} \) is feasible for 2CNCON(G;r) and hence in \( F_2 \subset F_0 \). Assume that both \( W \) and \( V(W) \) contain more than one node of type 2. (In the other case, the proof has to be modified a little.) 1. If \( e \) is a path or a path of \( c \) and \( f \) of \( W \), then \( \pi(c,W) \geq 1 \) and \( \pi(f,W) \geq 1 \), then any node \( x \) of type 2 in \( W \) must exist an \( x \)-path and an \( x \)-path that are node-disjoint; otherwise, \( \pi(c,W) \geq 1 \) and \( \pi(f,W) \geq 1 \).

We can rewrite these conditions in the following way: Let \( U \) denote a two-node-connected component of \( GW' \) containing some node of type 2 of \( GW' \). Note that by condition (d\_4), \( U \) must then contain all nodes of type 2 in \( U \). Now remove from \( U \) the set of all articulation nodes of \( GW' \). Let \( u \) be a node set \( U \) in \( GW' \) be defined in the same way as \( U \) in \( GW \). Condition (2) says that \( c \) and \( f \) may not be incident to the node \( u \). If two components of \( GW' \) and \( GW \) are the same, then \( \pi(c) = \pi(f) \). The maximum matching possible in this graph has size at least \( 2u \) otherwise, there are two nodes covering all edges in \( G' \), which translates to condition (1.2)(ii), (d\_1) or (d\_3) of Theorem 3.2 being violated.

Now we are ready to show that \( b_0 \) has the same value \( \gamma \) for all \( e \) of \( GW' \). Assume that both \( W \) and \( V(W) \) contain more than one node of type 2 in \( GW' \). A matching with three edges, say, \( e, f, g \). Since the incidence vector of \( f, g \) has a matching with three edges, \( (e, f, g) \) constitute a matching with \( (e, f, g) \). Therefore, the incidence vectors of both \( C_G(C, f, g) \), and \( C_G(C, f, g) \) in \( F_3 \), and we have \( b_0 = b_0 = \gamma \). This way we can prove \( b_0 = \gamma \) for all \( e \) of \( GW' \).

To prove \( b_0 = 0 \) for all \( e \) in \( W \) we need to construct a set \( C = C \subseteq W \). In this case, CHSEX(\( e \)) is in \( F_0 \), we know \( b_0 = 0 \). Assuming again that both \( W \) and \( V(W) \) have at least two nodes of type 2, we try for a given \( e \) with \( e \) in \( W \) to find \( g, f \) in \( W \) constituting a matching of \( G(W), \) so that \( C = C_G(C, e) \) is feasible for 2CNCON(G;r). Then \( \pi(C, e) \geq 2 \), and we can find such \( f, g \in W \) inducing a feasible \( C_G(C, e) \) in \( G \) by similar arguments as above. Since the incidence vectors of \( C_G(C, e) \) and \( C_G(C, e) \) are in \( F_0 \), we have \( b_0 = 0 \).

Now suppose \( \pi(C, G') - e \) is a. Consider the tree structure of the two-node-connected components and the articulation points of \( GW' \) - e. Since \( \pi(C, G') \geq 2 \) and \( \pi(C, G') \leq 1 \), the endpoints \( v_1 \) and \( v_2 \) lie in two different two-node-connected components. Furthermore, there is a \( v_1, v_2 \)-path in \( GW' \) - e that touches all two-node-connected components and all articulation points of \( GW' \) - e. Since \( \pi(C, G') \geq 2 \) and \( \pi(C, G') \leq 1 \), the endpoints \( v_1 \) and \( v_2 \) lie in two different two-node-connected components.

4. Lifting theorems. We now present conditions under which valid inequalities (respectively, facets) for the 2ECON and 2CNCON polytopes on a graph \( G \) can be lifted
to valid inequalities (respectively, facets) for higher-dimensional 2CON and 2NCON polytopes on a graph $G$ that contains $G$ as a subgraph. These results simplify the proofs to be presented in the next sections.

Some of the results can be treated for the 2ECON and 2NCON polytopes simultaneously. Thus we introduce a slightly more general network design model that combines edge- and node-connectivity requirements. Let $G = (V, E)$ be a graph, $r \in \{0, 1, 2\}$ be the vector of connectivity types, and $Z$ be some subset of $V$. (In this section we do not necessarily assume that $(G; r)$ satisfies (1.2).) We define the 2CON($Z$) problem to be the network design problem where between each pair of distinct nodes $v$ and $w$ at least $\min(r(v), r(w))$ edge-disjoint paths are required that have no node of $Z(v, w)$ in common. Note that for $Z = \emptyset$ only edge-disjoint paths are required, so in this case 2CON($Z$) is the 2ECON problem. For $Z = V$ this is the 2CON problem. This general model is introduced only for technical reasons. Throughout the rest of this paper we will be interested only in the cases $Z = \emptyset$ and $Z = V$.

The 2CON($Z$) problem can be formulated as an integer linear program in the following way:

$$\begin{align*}
\min & \quad \sum_{e \in E} c_{ej} x_{ej} \\
\text{subject to} & \quad (4.1)(i) \quad \chi(x(W)) \geq \chi(W) \quad \text{for all } W \subseteq V, 0 \notin W \neq V; \\
& \quad (ii) \quad \chi(\delta_\pm(W)) \geq 1 \quad \text{for all } z \in Z, \text{ and for all } W \subseteq V \setminus \{z\}, 0 \notin W \neq \chi(\{z\}) = 2 \quad \text{and } r(W) = 2 \quad \text{and } r(V \cup \{z\}) = 2; \\
& \quad (iii) \quad 0 \leq x_{ij} \leq 1 \quad \text{for all } (i, j) \in E; \\
& \quad (iv) \quad x_{ij} \text{ integral} \quad \text{for all } (i, j) \in E.
\end{align*}$$

The polytope 2CON($G; Z; r$) is then defined as the convex hull of all $x \in \mathbb{R}^E$ that satisfy (i)-(iv) of (4.1). As mentioned above, 2CON($G; \emptyset; r$) $\subset$ 2CON($G; r$); and 2CON($G; V; r$) $\subset$ 2CON($G; r$).

The polytope 2CON($G; Z; r$) is not necessarily full-dimensional. In the later sections we only apply the results of this section in the line 2CON($G; Z; r$) $\subset$ $\mathbb{R}^E$.

So we can avoid treating all the technicalities arising in the low-dimensional case, and we thus assume throughout this section that 2CON($G; Z; r$) has dimension $|E|$.

In Lemma 4.2 we derive valid inequalities for the 2CON($G; Z; r$) polytope from valid inequalities for the 2CON($G; W; Z; r$) polytope.

**Lemma 4.2**. Consider the 2CON($G; Z; r$) polytope and let $W \subseteq V \setminus Z$. Let the node $u \in G(W)$ that represents node set $W$ inherit its connectivity type from $W$ by $r_u := \chi(W)$. If $\delta_\pm x \geq b$ is a valid inequality for 2CON($G(W); Z; r$) where $W \subseteq V \setminus Z$, then $\delta_\pm x \geq b$ is valid for 2CON($G; Z; r$), where

$$a_e = a_e \quad \text{for } e \in E(G(W)) \quad \text{and} \quad a_e = 0 \quad \text{for } e \in E(W).$$

We say that $\delta_\pm x \geq b$ is obtained from $\delta_\pm x \geq b$ by expanding $u$ to $W$.

**Proof**. We first remark that the lemma is true for any of the inequalities (i), (ii), or (iii) of (4.1). The reason is that the expansion of any inequality of type (i), (ii), or (iii) is of the same type. (Note that since $Z \cap W = \emptyset$, $u$ is a shrunk node $u$ can never be chosen as a node in a node-cut inequality (ii)).

Since 2CON($G(W); Z; r$) is the convex hull of the integral solutions of (i), (ii), and (iii) of (4.1) every valid inequality for 2CON($G(W); Z; r$) can be obtained by taking nonnegative combinations of the inequalities (i), (ii), and (iii), rounding the left- and right-hand sides up and recursively repeating this procedure. This so-called cutting plane proof is described in [Chv] see also [Sch, Cor. 23.2b]. It is easy to see that much a validity proof of $\delta_\pm x \geq b$ from the inequalities (i), (ii), and (iii) of (4.1) for 2CON($G(W); Z; r$) yields a validity proof of $\delta_\pm x \geq b$ by applying the same nonnegative combinations and rounding operations to the associated expanded inequalities, since combining and rounding expanded inequalities produces expanded inequalities.

The following lemma gives a technical condition for an "expanded" inequality derived by Lemma 4.2 to define a facet of 2CON($G; Z; r$).

**Lemma 4.3**. Consider the 2CON($Z$) problem given by $(G; r)$ satisfying (1.2) and let $W$ be a subset of $V \setminus Z$ with $|W|$ connected. Let the node $u \in G(W)$ (representing the node set $W$) inherit its connectivity type from $W$ by $r_u := \chi(W)$ (Note that $(G(W); r_u)$ does not necessarily satisfy (1.2)).

Let the inequality $\delta_\pm x \geq a \geq 0$ be valid for 2CON($G(W); Z; r$) and let $\delta_\pm x \geq a$ be the inequality (valid for 2CON($G; Z; r$)) obtained from $\delta_\pm x \geq a$ by expanding node $u \notin Z$ to $W \cup \{z\}$.

Denote by $F_u$ the face of the polytope $P := 2CON(G; Z; r)$ induced by $\delta_\pm x \geq a$ and by $F_W$ the face of the polytope $\hat{P} := 2CON(G(W); Z; r)$ induced by $\delta_\pm x \geq a$.

$F_u$ is a facet of $\hat{P}$ if and only if the following conditions hold:

(a) For any $e \in E(W)$ there exists a set $C \subseteq E(G(W))$ with $x^C \in F_u$ so that the incidence vector of $(C \cup \{W\}\backslash\{e\})$ lies in $F_W$.

(b) There exist $s := \{E(G(W))\}$ sets $G_i \subseteq E(G(W))$, $i = 1, \ldots, s$, with $x^C \geq F_u$ so that

$$\begin{align*}
& \quad (b_1) \quad x^G_i(W) \in F_u, \quad \text{and} \\
& \quad (b_2) \quad x^G_i \text{ are affinely independent.}
\end{align*}$$

**Proof**. Suppose that (a) and (b) are satisfied. We want to show that $F_u$ is a facet. (Note that (b) implies that $F_u$ is a facet.) Let $\delta_\pm x \geq a$ define a facet $F_u$ of $\hat{P}$ that contains $F_u$. For any $e \in E(W)$, condition (a) provides a set $C \subseteq E(G(W))$ with $x^C \in F_u$, and $x^C \in F_u$. Therefore, $x^{C \cup \{W\}\backslash\{e\}}(W) = a$ for $i = 1, \ldots, s$. Since we have just proved $\delta_\pm x \geq a$ to be $(0, b^\pm(W))$ with $b^\pm(W) \in \mathbb{R}^{|\hat{P}|}$, this means $\delta \chi^C = \chi^W = a$ for $i = 1, \ldots, s$. The affine independence of the dimension($\hat{P}$) vectors $x^C$ implies that $\delta_\pm x \geq a$ defines a facet of $\hat{P}$, necessarily the same as $F_u$. Therefore, $(\delta^\neg, \beta)$ is a positive multiple of $(\delta^\neg, \alpha)$, and $(\delta^+, \beta)$ is a positive multiple of $(\delta^+, \alpha)$. So $F_u$ defines a facet.
On the other hand, if we know that $a^2 \geq c$ defines a facet of $P$, then for each $e \in E(W)$ there must exist a set $C$ with $e \notin C$ and $\chi^C \subseteq F_1$; otherwise $F_2 \subseteq \{x \in P : x_e = 1\}$. If we shrink node set $W$ to node $w$ in the graph $G$ defined by $C$ we arrive at a set $\tilde{C} := C \cup E(W)$ whose incidence vector satisfies $a^2 \geq c$ and is feasible for $\tilde{P}$ because $con(C) = con(w)$. $w$ may be an articulation node in $G$, but this does not matter because $w \notin E$. The set $\tilde{C} \cup E(W) \{e\}$ is feasible for $P$ because it contains the feasible set $C$ and because $G(W)$ is connected. Therefore, (a) is satisfied.

If $F_3$ is a facet of $P$, there exist $\{E(W)\}$ affine independent vertices $\chi^C \subseteq F_2$, where $C \subseteq E$ is feasible for $P$, for $i = 1, \ldots, |E|$. We set $x_i := x^C_i$ for $i = 1, \ldots, |E|$. There must be a subset of $\{E(W)\}$ affine independent vertices among $\chi^C \subseteq F_2$, where $\chi^C$ is derived from $\chi^C_1, \ldots, \chi^C_\ell$ by deleting the components $e \in E(W)$. The $x_i$ for $i = 1, \ldots, |E|$ are feasible for $P$ because the deletion of the $E(W)$-components of a vector $x$ in $\{0,1\}^E$ is equivalent to the contraction of $W$ in the subgraph $(V, F_{2 \ell})$ of $G$ defined by $x$. So the affine independent subset of $\{x_i : i = 1, \ldots, |E|\}$ satisfies (b1) and (b2).

The conditions of Lemma 4.3 can be used to derive some conditions on $G[W]$ that are of a more graph-theoretical nature and sufficient for an "expanded" inequality to define a facet of $2CON(G; r)$.

**Lemma 4.4.** Consider the 2CON problem given by $(G, r)$ satisfying (1.2(i)). Let $W \subseteq V$ with $\emptyset \neq W \neq V$, and let $w$ of type $con(W)$ be the node of $G[W]$ representing $W$. Consider an inequality $a^2 \geq b$ that is facet-defining for the polytope $2CON(G(W), r)$, and consider an inequality $a^2 \geq b$ that is facet-defining for $2CON(G, r)$. Let $F_1$ be a facet of $2CON(G, r)$ in $\{0,1\}^E$ and $F_2$ be a facet of $2CON(G(W), r)$ in $\{0,1\}^W$. Then for $i = 1, \ldots, |W|$, $x_i$ is derived from $x_{i'}$ for some $i' \in W$ by deleting the components $e \in E(W)$.

**Proof.** Let $F_1$ and $F_2$ be defined as in Lemma 4.3. We will check conditions (a) and (b) of Lemma 4.3. The connectivity conditions on $G[W]$ imply that for any $e \notin E(W)$ and $\emptyset \neq C \subseteq E(W)$ that is feasible for $2CON(G(W), r)$, the sets $\tilde{C} := C \cup E(W) \{e\}$ and $\tilde{C} := C \cup E(W) \{e\}$ are feasible for $2CON(G, r)$. Since $F_2$ is a facet, there are enough affine independent $x^C_i$ to satisfy condition (b) of Lemma 4.3. Usually only weaker conditions on the edge-connectivity of $G[W]$ are already sufficient for an expanded inequality $a^2 \geq b$ to define a facet of $2CON(G; r)$. But this leads to further technicalities concerning assumptions on the structure of the graph and properties of $a^2 \geq b$. See, for instance, Theorem 3.2(a) and (b).

The next lemma gives a sufficient condition for an expanded inequality to define a facet of $2CON(G, r)$. (Note that any inequality valid for $2CON(G, Z; r)$ is also valid for $2CON(G, r)$.)

**Lemma 4.5.** Consider the 2CON problem given by $(G, r)$ satisfying (1.2(i)). Let $Z \subseteq V$ and $W \subseteq V \setminus Z$ with $\emptyset \neq W \neq V$ and $r(W) = 1$, and let $w$ of type 1 in $G[W]$ representing $W$. Consider an inequality $a^2 \geq b$ that is valid for the polytope $2CON(G(W), r)$ and facet-defining for $2CON(G[W], r)$.

If $G[W]$ is two-edge-connected, then the inequality $a^2 \geq b$ is derived from $a^2 \geq b$ by expanding node $w$ to $w$ defines a facet of $2CON(G, r)$.

**Proof.** First, $a^2 \geq b$ is valid for $2CON(G, Z; r)$ by Lemma 4.2 and because for $G[W]$ with $P := 2CON(W, r)$ and $V := 2CON(V; r)$. (Conditions (a) and (b) are still sufficient for $a^2 \geq b$ to define a facet of $P$, because of the fact that $x$ is not used in the sufficient part of Lemma 4.3. So we have to check (a) and (b) of Lemma 4.3, which is easy.)

Our final lifting result presents conditions under which a valid inequality for $2CON(G; Z; r)$ on a complete graph $G = (V, E)$ can be extended to the graph with a new node $w$ of type at least 1 added, along with all of the edges incident between $w$ and $V$, we denote such a graph by $G + w$.

**Lemma 4.6.** Consider the 2CON problem given by a graph $G$ and node types $r$ satisfying (1.2)(i), where $G = (V, E)$ is a complete graph with two parallel edges $v$ for each $v, u \in V$ with $u \neq v$. Let $a^2 \geq b$ be a valid inequality for $2CON(G; Z; r)$ with $a \geq 0$. Let $W \subseteq V \cup \{z\}$ be a node set with $r(W) = 2$ and $a$ some nonnegative value so that either $\hat{a}_u = 0$ for all $e \in E(W)$ or $r(W) = 1$.

We define an inequality $a^2 \geq b$ on the graph $G := G + w$ with $r_w := 1$ by setting $b = \tilde{b} + \alpha$, $a_i = \tilde{a}_i$ for all $i \in \tilde{E}$, $u \leq a_i$ for all $u \in \tilde{E}$, $a_{uw} := \alpha$ for all $u \in W$, $a_{uw} := \alpha w$, $u = u_w$, $\hat{a}_u = \max(a, \max(a_{uw}, v \in W))$ for all $u \neq W$.

If $a_{uw} + a_{uw} \geq a + \alpha$ for all distinct nodes $u, w \neq W$, then $a^2 \geq b$ is valid for $2CON(G + w; Z; r)$.

Note that in Lemma 4.6 the restriction to complete graphs is not restriction at all, because any inequality valid for $2CON(G; Z; r)$, where $G$ is a complete graph, is also valid if $G$ is replaced by some subgraph $(V, F)$. In the lemma we need completeness of $G$ to compute the $\tilde{a}_i$ correctly. Also, we can restrict ourselves without loss of generality to $\alpha = 0$, because it is easy to see that any inequality $a^2 \geq b$ is facet-defining for $2CON(G; Z; r)$ (except $\tilde{a}_i^2 \geq a^2 \geq b$).

**Proof.** We will assume that $r_w = 1$, because validity of an inequality in this case implies its validity if $r_w = 2$. Assume further that $a^2 \geq b$ is not valid, i.e., that there exist an edge set $E = \emptyset$ that is feasible for $2CON(G + w; Z, r)$ and does not satisfy $a^2 \geq b$.

Let $d_{uw} \leq d_{uw}$ for all $u \in W$. Therefore, $a_{uw}^2 \leq a_{uw}^2 - \alpha u_w = a_{uw}^2 - b$. But then $a_{uw}^2 \geq b$ is not valid for $2CON(G + w; Z, r)$, a contradiction.

If $G$ uses no edge of $W \cup \{z\}$, then we will show how to replace $C$ by some set $C'$ containing an edge in $W \cup \{z\}$, such that $a_{uw}^2 \leq a_{uw}^2 < b$. So we can apply the argumentation above to derive a contradiction to the validity of $a^2 \geq b$.

(2) Suppose all edges of $d(u) \cap C$ were bridges of $(V, C)$. Since $c$ is connected to $W$ in $G$, there must be a bridge $w$ of $(V, C)$, which separates $w$ from some $u \in W$. The set $C' := (C \setminus \{w\}) \cup \{w\}$ is feasible for $2CON(G; Z; r)$ and contains an edge of $(W \cup \{z\})$.

Now suppose there are edges of $(d(u) \cap C)$ that are not bridges of $(V, C)$. Define $U$ as the set of nodes that are incident to nonbridges of $(V, C)$ (the so-called two- or three-connected part of $C$). $U$ must contain all nodes of type $2$. The case that $w$ is an articulation node of $(V, C)$ disconnecting two nodes of type 2.

(3) Assume that $w$ is not an articulation node of $(V, C)$, disconnecting two nodes of type 2. The case that $w$ is an articulation node of $(V, C)$ is treated separately. Since $(W, W) = 2$, there exists a node $e \in U$ of type 2, and since $u$ and $w$ are in $U$, there exist two edges incident in $u$: $u_{vu}$, $w_{uv}$, $w_{uw}$ in $C'$ that do not coincide in any node $u \in U$. Let $u_w \in U$ be the nodes adjacent to $w$ on those two paths. If $u_w$, we eliminate one of the two $u_{uv}$ edges. This can be done without destroying feasibility of $C$ because $w$ is not an articulation node separating two nodes of type 2 and because $r_w = 1$. Also, $a^2 \geq b$ does not increase with this operation, since $a \geq b$. Now we are either in the case that $\delta(u) \cap C$ contains only bridges of $(V, C)$ (proceed with part (2) of the proof), or we construct two other $u_{uv}$-paths that lead to different nodes $u \neq u_w$. 
Now we show that \( C' := (C')(u,w,v) \cup \{u,w,v\} \) is also feasible. Clearly \( C' \) is connected, so we only have to check for bridges and articulation nodes. Suppose that \( e \) is a bridge of \((V,C')\) separating two nodes of type 2. In \((V,C')\), node \( v \) is connected to \( u \) and by at least one of the two edge-disjoint paths and edge \( uv \). If \( v \neq w \), all four nodes \( u, v, w, u \) lie in the same component \((S,F)\) of \((V,C')(\{e\})\). Since \( C' \cap \delta(S) = C \cap \delta(S) \), edge \( e \) is also a bridge in \((V,C)\) separating two nodes of type 2. So \( e \) must be \( w \). But \((V,C) - w \) is a subgraph of \((V,C') - v \) and \( w \) is not an articulation node of \((V,C)\). Now suppose that \( e \in Z \) is an articulation node that separates two nodes of type 2 in \((V,C')\) but not in \((V,C)\). \( z = s \neq \alpha \) need not be considered, because \( s \notin Z \). The remaining cases lead to a contradiction similar to the case in which \( e \) is a bridge. So \( C' \) is feasible for \( 2\text{CON}(G;Z;r) \). This also satisfies \( \alpha^T C' \alpha < b \) because \( \alpha^T C' \alpha = C'' \alpha = C'' \alpha = \alpha^T \alpha = \alpha^T \alpha + \alpha^T wv = \alpha^T \alpha + \alpha^T wv < \alpha^T \alpha + \alpha^T \alpha < \alpha^T \alpha \).

(4) The last remaining case in our transformation of \( C \) is the case in which \( w \) is an articulation node of \((V,C)\), separating two nodes of type 2. Let \( u, v \in U \) be nodes adjacent to \( w \) lying on different sides of \((V,C) - w \). Replace \( C' \) by \( C'' := (C'(u,w,v)) \cup \{u,w,v\} \) is feasible, \( \alpha^T C'' \alpha \leq \alpha^T C' \alpha < \alpha^T \alpha \), and \((V,C') - w \) contains one component less than \((V,C) - w \). Ultimately, we reach a set \( C' \) where \( z \) does not separate any nodes of type 2, and we can apply one of the earlier cases. Thus, we have proved that if \( \alpha^T x \geq b \) is not valid for \( 2\text{CON}(G;Z;r) \), then also \( \alpha^T x \geq b \) is not valid for \( 2\text{CON}(G;Z;r) \).

The next theorem gives sufficient conditions for \( \alpha^T x \geq b \) to define a facet.

**Theorem 4.7.** Consider the situation in Lemma 4.6, where we have an inequality \( \alpha^T x \geq b \) valid for \( 2\text{CON}(K_n;Z;r) \), and where \((K_n;Z;r)\) satisfies (1.2). Let \( W \) with \( \tau_W = 1 \), \( \alpha \geq 0 \), be defined as in Lemma 4.6. Let \( \alpha^T x = b \) be the inequality derived from \( \delta S \geq \delta A \) by the formula in Lemma 4.6. Furthermore, let \( G = (V,E) \) be a graph of \( n + 1 \) nodes, and define \( G \) as \( G = V \). Then, for any \( Z \subseteq G \), the inequality \( \alpha^T x \geq b \) defines a facet of \( 2\text{CON}(G;Z;r) \).

If the following conditions hold:

(a) \( \delta S \geq \delta A \) defines a facet of \( 2\text{CON}(G;Z;r) \);
(b) for all \( u \in W \) with \( wu \in E \) and \( u \neq w \), there exists a node \( w \in W \) with \( \alpha^T w = \alpha^T u = \alpha^T w = \alpha = \alpha \) and \( wu \in E \); and
(c) there exist distinct nodes \( u, v \) with \( \alpha^T u = \alpha^T v = \alpha^T w = \alpha \) in \( E \);
(d) all nodes \( u \) with \( wu \in E \) and \( \alpha^T u = \alpha \) have type at least 1.

Proof. First, note that \( \alpha^T x \geq b \) is valid for \( 2\text{CON}(G;Z;r) \) because it is valid for \( 2\text{CON}(G;Z;r) \) by Lemma 4.6. We prove the theorem by exhibiting \( B \) and \( (E) \) as \( \alpha^T x = b \) independent vectors in \( F = (x \in \text{CON}(G;Z;r) | \alpha^T x = b) \). Let \( F = \{ x \in \text{CON}(G;Z;r) | \alpha^T x = b \} \). Let \( f \) be an edge \( f = \alpha^T f \). Then any set \( C \subseteq E \) feasible for \( F \) can be enlarged to a set \( C' \subset E \) feasible for \( F' \) by adding \( f \). This way we can create \( B \) as \( \alpha^T x = b \) independent vectors in \( F' \). Now we have to exhibit \( (B) \cap \{i \in \{1, \ldots, n\} \} \) sets \( C_i \) with \( \alpha^T C_i = \alpha \). The \( C_i \) are characterized by the fact that \( C_i \) contains an edge \( e_i = \delta(e_i)(\{f\}) \) that is contained in any of the previous \( C_i \).

**Remark 3.5.** Any partition inequality (5.2) induced by a proper partition is valid for \( 2\text{CON}(G;r) \) and \( 2\text{CON}(G;Z;r) \).

Note that the partition inequality induced by a proper partition with \( p = 2 \) is nothing but a cut inequality \( \delta(e)(\{f\}) = \alpha \). The next observation indicates that we cannot expect to obtain a useful characterization of these partition inequalities that define facets.
Remark 5.4. Checking whether a partition inequality supports 2ECON(G;r) or 2CON(G;r) is NP-complete.

Proof. The problem is obviously in NP. Let \( G = (V,E) \) be a graph and \( r_v = 2 \) for all \( v \in V \). Then the sets \( s_v, e \in V \), form a proper partition of \( V \) and the induced partition inequality reads \( x(E) \geq |V| \). Thus there is a point in 2ECON(G;r) or 2CON(G;r) that satisfies \( x(E) \geq |V| \) with equality if and only if \( G \) is Hamiltonian. This implies the remark.

We will now derive a sufficient condition for a partition inequality to define a facet.

Theorem 5.5. Let \( G = (V,E) \) be a graph, \( r = \{0,1,2\} \), and let \( W_1, \ldots, W_p \) be proper partition of \( V \) and the induced partition inequality reads \( x(E) \geq |V| \). Thus there is a point in 2ECON(G;r) or 2CON(G;r) that satisfies \( x(E) \geq |V| \) with equality if and only if \( G \) is Hamiltonian.

\[ \text{Lemma 4.4 and Theorem 5.5(e)} \]

We can obtain all nodes of \( G \) from \( |W_i| \) to node sets \( W_i \), thus obtaining a facet of the polytope of connected subgraphs of \( G \). This is shown in [GM]. By our lemma Lemma 4.4 and Theorem 5.5(e), we can expand all nodes of \( G \) to successive node sets \( W_i \), and thus obtain a facet of the polytope of connected subgraphs of \( G \).

We will now derive a sufficient condition for a partition inequality to define a facet.

Theorem 5.6. Let \( G = (V,E) \) be a graph, \( r = \{0,1,2\} \), and let \( W_1, \ldots, W_p \) be proper partition of \( V \) and the induced partition inequality reads \( x(E) \geq |V| \). Thus there is a point in 2ECON(G;r) or 2CON(G;r) that satisfies \( x(E) \geq |V| \) with equality if and only if \( G \) is Hamiltonian.

\[ \text{Lemma 4.4 and Theorem 5.5(e)} \]

We can obtain all nodes of \( G \) from \( |W_i| \) to node sets \( W_i \), thus obtaining a facet of the polytope of connected subgraphs of \( G \). This is shown in [GM]. By our lemma Lemma 4.4 and Theorem 5.5(e), we can expand all nodes of \( G \) to successive node sets \( W_i \), and thus obtain a facet of the polytope of connected subgraphs of \( G \).

We will now derive a sufficient condition for a partition inequality to define a facet.

Theorem 5.7. Let \( G = (V,E) \) be a graph, \( r = \{0,1,2\} \), and let \( W_1, \ldots, W_p \) be proper partition of \( V \) and the induced partition inequality reads \( x(E) \geq |V| \). Thus there is a point in 2ECON(G;r) or 2CON(G;r) that satisfies \( x(E) \geq |V| \) with equality if and only if \( G \) is Hamiltonian.
6. Node-partition inequalities. We now generalize node-cut inequalities to "node-partition inequalities" in the same way as we generalized cut inequalities to partition inequalities in the previous section. These new inequalities will only be valid for \(2\text{CON}(G,r)\), but, in general, not for \(2\text{CON}(G,r)\).

Let \(G = (V,E)\) be a graph and \(r \in \{0,1,2\}^V\). Let \(z \in V\) and let \(W_1, \ldots, W_p\) be a proper partition (see (5.1)) of \(V \setminus \{z\}\) such that at least two node sets \(W_i\) contain nodes of type 2. The following node-partition inequality induced by \(z\) and \(W_1, \ldots, W_p\) is given by

\[
\frac{1}{2} \sum_{i=1}^p \left( \sum_{v \in W_i} x(\delta^+(W_i)) + \sum_{v \in W_i} x((\{z\} : v \in E, W_i)) \right) \geq p - 1,
\]

where \(I_k := \{i \in \{1, \ldots, p\} : r(W_i) = k\}, k = 1, 2, 2\).

In Fig. 6.1 a node partition inequality is depicted with three sets \(W_1\) with \(r(W_1) = 2\) and two sets \(W_2\) with \(r(W_2) = 1\). Edges with coefficient 0 are depicted by dashed lines; edges with coefficient 1 are depicted by solid lines.

![Diagram showing node partition inequality](image)

**Theorem 6.2.** The node partition inequality (6.1) is valid for \(2\text{CON}(G,r)\).

**Proof.** Consider first a node partition inequality induced by a node \(z\) and the partition consisting of all node sets \(\{v, v \in V \setminus \{z\}\). Suppose also that \(r_v = 2\) for all \(v \in V \setminus \{z\}\). This node partition inequality, \(x(E(V \setminus \{z\})) \geq |V| - 2\), is valid because after deletion of a node the rest of the network should still connect all nodes \(v \in V \setminus \{z\}\). Nodes of type 1 can be added successively to \(V \setminus \{z\}\) by applying Lemma 4.6 with \(Z := \{z\}, W := V \setminus \{z\}\), and \(\alpha := 1\). With Lemma 4.2 all nodes \(v \in V \setminus \{z\}\) can be expanded to node sets. In this way, every node partition inequality is proved to be valid.

The following theorem gives a sufficient condition for the node partition inequality (6.1) to define a facet of \(2\text{CON}(G,r)\).

**Theorem 6.3.** Consider a node partition inequality (6.1) induced by \(W_1, \ldots, W_p\), where the \(W_i\) are shrunk to nodes \(v_i\), \(i = 1, \ldots, p\). Let \(I_1\) and \(I_2\) be defined as in (6.1). The node partition inequality \(ax \geq p - 1\) defines a facet of \(2\text{CON}(G,r)\) if

- \(G\) is two-node-connected;
- \(G(W_i \cup \{z\}) = c \) is two-node-connected for all edges \(e \in G(W_i \cup \{z\})\) and for all \(i \in I_1\);
- \(G(W_i)\) is two-edge-connected for all \(i \in I_2\).

**Proof.** Let conditions (a), (b), and (c) be satisfied. We will show how to construct \(\{E\}\) affinely independent vectors in the face defined by the node partition inequality (6.1).

Let \(E'\) be the set of all edges whose coefficients in \(ax \geq p - 1\) are 0. By condition (a), the graph \(G = (V,E)\) contains \(E'\) spanning trees whose incidence vectors are affinely independent (see Theorem 4.10 in [GM]). Any such tree \(T\) of \(G\) can be augmented by \(E'\) to a feasible set \(C \subseteq E\) for \(2\text{CON}(G,r)\). Feasibility can be shown as follows. For any two nodes \(u, v \in G(W_i \cup \{z\})\) (where \(i \in I_2\)) there exist, by condition (b), two node-disjoint paths in \((V,C)\). For \(u \in W_i\) and \(v \in W_j\) (where \(i, j \in I_2\) and \(i \neq j\)), we construct the following two node-disjoint paths. In \((V,C) - z\), there exists a path from some node \(u' \in W_i\) to some node \(v' \in W_j\). Let \(u'\) and \(v'\) have the property that \(u'\) is the last node of \(W_i\) and \(v'\) is the first node of \(W_j\) encountered on this path. Since \(G(W_i \cup \{z\})\) is two-node-connected, it contains a \(u, u', v', v\)-path and \(u, [u', v]\)-path, which do not have a node except \(u\) in common. (If \(u' = u\), we only need one path, namely, the \([u', v]\)-path.) Similarly, \(G(W_j \cup \{z\})\) contains a \(v', v\)-path and a \(v, [v', w]\)-path, which are node-disjoint. From these paths we can construct two node-disjoint \([u, v]\)-paths in \((V,C)\). For all pairs \(u, v\) of nodes we can construct the required number of paths in \((V,C)\), which proves feasibility of \(G\). Feasibility is preserved even when some single \(e \in E'\) is deleted from \(C\).

The connectivity conditions given in (b) imply that if \(r(W_i) = 2\) for one of the node sets in the partition, then \(W_i\) must contain at least three nodes. This is not at all necessary. In fact, there exist facet-defining node-partition inequalities where all node sets in the partition contain exactly one node. Because we need it later on, we state this result as a lemma.

**Lemma 6.4.** Consider a \(2\text{CON}\) problem given by \((G,r)\) and let \(z\) be some node of \(G\). We suppose that \(G = (V,E)\) is a graph with at least four nodes and \(r_v = 2\) for all \(v \in V \setminus \{z\}\). The node-partition inequality (6.1) induced by the partition of \(\{W_i\}\) into node sets \(\{V \setminus \{z\}\}\) defines a facet of \(2\text{CON}(G,r)\) if \(z\) is adjacent to every node in \(G\).

**Proof.** This can be proved by considering trees of \(G - \{z\}\) augmented by certain edges of \(\{z\}\). Note that by (1.2)(iii) the graph \(G\) is supposed to be two-node-connected, so there exists a sufficient number of trees of \(G - \{z\}\).

Some necessary conditions for node-partition inequalities to define facets of \(2\text{CON}(G,r)\) can be derived from Theorem 3.3 for node-cut inequalities.

**Theorem 6.5.** The node-partition inequality (6.1) defines a facet of \(2\text{CON}(G,r)\) only if

- \(G(W_i)\) is connected for all \(i \in I_1\);
- \(\lambda_i(G(W_i \cup \{z\}) \geq 2\) for all \(i \in I_1\);
- \(\lambda_1(G(W_i)) \geq 2\) for all \(i \in I_2\);
- \(\lambda_2(G(W_i)) \geq 2\) for \(i = 1, \ldots, p\),

**Proof.** The proof is obvious.

The connectivity conditions given in Theorem 6.5 can be easily checked and are of some practical use in cutting plane algorithms to derive facets of higher dimension.

7. Lifted two-cover inequalities. The motivation for introducing and studying the next class of inequalities comes from the fact that the two matching...
equalities play an important role in solving the traveling salesman problem; see [GP] and [PG].

The roots, however, are Edmonds's results for b-matching polyhedra (see [8]) since a certain (complemented) b-matching problem provides an interesting relaxation of the ECON problem.

Let \( G = (V, E) \) be a graph and \( r \in \{0, 1, 2\}^V \). Every incidence vector of a feasible solution \( F \subseteq E \) to the 2ECON problem satisfies the "star inequalities" \( x(L(v)) \geq r_v \) for all \( v \in V \). And therefore the incidence vector of the complement \( F := E \setminus F \) of a feasible solution \( F \) to the 2ECON problem satisfies

\[
y(v) \leq b_v := |E(v)| - r_v \quad \text{for all } v \in V,
y(e) \leq 1 \quad \text{for all } e \in E.
\]

The convex hull of the integral solutions of (7.1) is the 1-capacitated b-matching polytope of \( G \), where \( b = (b_e)_{e \in V} \in \mathbb{Z}^V \). Let us set, for \( W \subseteq V \), \( b(W) := \sum_{e \in W} b_v \). Edmonds [8] has shown that a complete linear description of the 1-capacitated b-matching polytope of \( G \) is given by the following system

\[
y(v) \leq b_v \quad \text{for all } v \in V,
y(E(H)) + y(T) \leq 2|H| + |T| - 1 \quad \text{for all } H \subseteq V \text{ and } T \subseteq E(H) \text{ such that } |H| + |T| \text{ is odd,}
0 \leq y(e) \leq 1 \quad \text{for all } e \in E.
\]

Since \( \chi^F = 1 - \chi^E \), we can derive from (7.2) that every incidence vector of a feasible solution to the 2ECON problem satisfies

\[
x(E(H)) + x(E(H)) - x(T) \geq \sum_{e \in H} r_e - |T| + 1 \quad \text{for all } H \subseteq V \text{ and } T \subseteq D(H) \text{ such that } |H| + |T| \text{ is odd.}
\]

In the transformation from (7.2) to (7.3) we have also set \( T := D(H) \).

Since \( r \in \{0, 1, 2\}^V \), we call inequalities (7.3) two-cover inequalities. Note that it follows from Edmonds's result that the two-cover inequalities (7.3) plus the trivial constraints \( 0 \leq x_v \leq 1 \), for all \( v \in E \), gives a complete description of the two-cover polytope, which is the convex hull of all incidence vectors of edge sets \( F \subseteq V \) such that each node \( v \in V \) has at least \( r_v \) incoming edges.

From the two-cover inequalities we derive a larger class of inequalities as follows.

Let \( G = (V, E) \) be a graph and \( r \in \{0, 1, 2\}^V \). Let \( H \not\subseteq V \) be a node set, called the handle, and \( T \subseteq D(H) \) an edge set. For each \( e \in E \) we denote by \( T_e \), the set of the two endpoints of \( e \). The sets \( T_e \) are called teeth. For simplicity we also call the edges \( e \in T \) teeth in this section. If an edge \( e \in T \) is parallel to some edge \( f \in T \), we count \( T_e \) and \( T_f \) as two sets, even if \( T_e = T_f \). Let \( H_1, \ldots, H_p, \ p \geq 3 \) be a partition of \( H \) into nonempty disjoint node sets such that

- \( r(H_i) \geq 1 \) for \( i = 1, \ldots, p \);
- \( |T_i| \geq 3 \) and odd.

We call

\[
x(E(H)) - \sum_{i=1}^p x(E(H_i)) + x(T) \geq p - \left\lfloor \frac{|T|}{2} \right\rfloor
\]

the lifted two-cover inequality.

In Fig. 7.1 a handle with four node sets \( H_1, \ldots, H_4 \) and three teeth (drawn with dashed lines) is depicted, inducing a lifted two-cover inequality with right-hand side 3.

For the case in which \( r_v = 2 \) for all \( v \in V \), Mahjoub [M] has found the same class of inequalities (and calls them "odd wheel inequalities" using a quite different notation).

Note that a lifted two-cover inequality coincides with a two-cover inequality (7.3), if \( |H_1| = 1 \) and \( r(H_1) = 2 \) for \( i = 1, \ldots, p \). Note also that with each additional \( H_i \) with \( |H_i| = 1 \) and \( r(H_i) = 1 \) the right-hand side of a lifted two-cover inequality increases by 1, whereas the right-hand side of a two-cover inequality increases only by \( \frac{1}{2} \) (on the average). This implies that two-cover inequalities do not support \( 2ECON(G;r) \) if \( H \) contains nodes of type 1. Nevertheless, if the right-hand side of a two-cover inequality is increased appropriately, these inequalities define facets of \( 2ECON(G;r) \) in many cases. This odd behavior may be explained by the fact that in an edge-minimal solution to the two-cover problem the nodes of type 1 may lie on matching edges, whereas in an edge-minimal solution to the 2ECON problem they are connected by a tree (or they lie on some cycle).

Also, the class of lifted two-cover inequalities is not very useful for the 2CON problem, because they do not define facets in the case in which \( G \) is a complete graph and some \( H \) with incident tooth contains more than one node. In §8 we will introduce a class of inequalities for \( 2CON(G;r) \) that contain the lifted two-cover inequalities with \( |H_i| = 1 \) as a subcase, and define facets for complete \( G \) and \( |H_i| \geq 1 \). But these will be valid only for \( 2CON(G;r) \).

As in the previous sections, we will derive validity and facet results of lifted two-cover inequalities from validity and facet results of a special class of lifted two-cover inequalities, namely those with \( |H_i| = 1 \).

**Theorem 7.5.** A lifted two-cover inequality (7.4) is valid for \( 2ECON(G;r) \) (and hence for \( 2CON(G;r) \)).

**Proof.** First, assume that \( |H_i| = 1 \) and that all nodes in the handle are of type 2.

In this case, we have a two-cover inequality that is valid for the polytope of two-covers, hence for \( 2ECON(G;r) \). It is also easy to prove validity in this case by summing up
the inequalities:

\[ x(\delta(u)) \geq 2 \quad \text{for all } u \in H, \]

\[-x_{\alpha} \geq -1 \quad \text{for all } \alpha \in T, \]

\[x_{\alpha} \geq 0 \quad \text{for all } \alpha \in \delta(H) \setminus T, \]

dividing the result by 2 and rounding the right-hand side up.

Our next step is induction over the number of nodes of type 1 in the handle (but
still \(|H_{1}| = 1\)). This can be done with the help of Lemma 4.6 by setting \(W := H, \alpha := 1, \) and \(u\) as the new node of type 1. The result is a new valid inequality of the form (7.4).

Finally, using Lemma 4.2, we expand the nodes in the handle successively to node
sets \(H_{i}\) with coefficients 0 inside \(H_{i}\) to derive all inequalities of the form (7.4).

Note that when lifting a node \(u\) with incident \(w \in T\) to node set \(W,\) only one
edge of \(W \setminus \{u\}\) gets coefficient 0; all others have coefficient 1 in the lifted
two-cover inequality. (If all edges in \(W \setminus \{u\}\) had coefficient 0, the obtained inequality
would not be valid for 2ECON(G;r), but it would be valid for 2NCON(G;r); see
Theorem 3.2.)

Lifted two-cover inequalities are also valid if we allow an even number of teeth.
But they cannot define facets in this case, as can be seen easily.

The following theorem gives a necessary and sufficient condition for a special subclass of lifted two-cover inequalities to define facets of 2ECON(G;r).

Theorem 7.6. (a) A lifted two-cover inequality (7.4) with \(|H_{i}| = 1 \quad \text{for } i = 1, \ldots, k, \quad |H| = |T| = p, \) and \(|\delta(H) \setminus T| = 1, \) defines a facet of 2ECON(G;r) if and only if

\(G[H] \) is compatible (i.e., for each node \(v \in H\) there is a matching of \(G[H]\) that

is incident to all nodes in \(H\) except \(v\)).

(b) Let \(G[H] = (V, E)\) be a complete graph. Then any lifted two-cover inequality (7.4) with \(|H_{i}| = 1 \quad \text{for } i = 1, \ldots, k, \quad |H| = |T| = 3, \) and \(|\delta(H) \setminus T| = 1, \) defines a facet of 2ECON(G;r).

Proof. Let \(F\) be the face induced by the lifted two-cover inequality in question.

(a) Let \(F\) be contained in a facet \(F_{k}\) induced by some inequality \(M, \geq 2. \)

We want to prove that \(b\) is a scalar multiple of the left-hand side of the lifted two-cover inequality.

(b) Assume first that \(H\) contains only nodes of type 2 (with or without incident teeth). If nodes of type 2 without incident teeth are allowed in the handle, the restric-

tion of a feasible set \(C\) whose incidence vector is in \(E\) to the edge set \(E(H) \cup \delta(H)\cdot T\)

is something more complicated than a matching with additional edge. It is rather a collection of node-disjoint paths between pairs of nodes with incident teeth plus one

additional path connecting the last node with incident tooth to \(V\) or some other path. More exactly, if we set \(r_{1} := 2\) minus the number of incident teeth for \(v \in H\)

and \(r_{2} := 0\) for the node \(x \notin H,\) then \(C[T]\) meets each node \(v \in T\) with exactly

\(r_{1}\) edges, except for one node that is not by \(r_{2}\) edges. \(C[T]\) is a near-perfect

v-cover of \(E(H) \cup \delta(H)\cdot T\) (so to speak). To see this, add the \(r_{2}\) divide by two,

and compare this with the right-hand side of the lifted two-cover inequality. (But not

every near-perfect \(v\)-cover of \(E(T)\) plus \(T\) defines a feasible set, as there might be some

node-disjoint cycles.)

Since the structure of the feasible sets with incidence vector in \(E\) is somewhat

unpredictable, we wish to complete graphs in \(H\). Call this collection of paths \(C\). As

before, we can add any edge of \(\delta(v)\) (except the path edge \(C \cap \delta(v)\), plus all teeth,

and get a set with incidence vector in \(F_{k}\). This proves \(b_{\alpha} = 0\) for all \(\alpha \in \delta(v)\)\).

But we can construct another set \(C'\) the same way as before, only this time it uses

a different edge of \(\delta(v)\). So we have \(b_{\alpha} = n_{\alpha}\) for all \(\alpha \in \delta(v)\)\).

Since \(\delta(v)\cdot T\) contains at least three edges, all edges in \(\delta(v)\cdot T\) have the same \(n_{\alpha}\)-value

of \(G[H] \setminus \{v\}\) is odd and \(|S|\) is odd, or both numbers are even. In any case, we

know that \(c_{\alpha}(G[H] \setminus \{v\}) - |S| \geq 2. \) So \(c_{\alpha}(G[H] \setminus \{v\})\), which is the same as \(c_{\alpha}(G[H] \setminus S)\), is still larger than \(|S|\).

For the sake of simplicity, we will rename \(S := S \cup \{v\}\). Let \(H_{1}\) be the node set of the first (odd or even) component of \(G \setminus S. \)

Let \(T_{0}\) denote the subset of teeth incident to \(H_{1}\) and let \(E_{0}\) denote the edge set

\(E(H) \setminus \delta(H_{1})\cdot T_{0}\). The \(T_{0}\) constitute a partition of \(T_{0}\) \(\delta(S),\) and the \(E_{0}\) constitute a partition of the edge set \(E(H) \setminus E(S) \cup (\delta(H) \setminus T_{0}),\)

\[x(E_{0}) \geq n_{H_{1}} := |H_{1}| - \frac{|T_{0}|}{2},\]

is a valid lifted two-cover inequality (this is even for an even number of teeth!).

If we take the sum of these inequalities plus the nonnegativity constraints for \(x \in E(S),\)

we achieve \(x(E) + x(\delta(H) \setminus T) \geq k,\) where \(k\) is the sum of the \(n_{H_{1}}.\) In the right-hand side, the \(|H_{1}|\) sum up to \(|H| - |S|,\) and the \(|T|/2\) sum up to \(\frac{|H|}{2} - |S| c_{\alpha}(G[H] \setminus S),\)

so the \(k\) sum up to

\[|H| - \frac{|T|}{2} + \frac{1}{2} c_{\alpha}(G[H] \setminus S) - |S| \geq |H| - \frac{|T|}{2} - \frac{1}{2} |S| c_{\alpha}(G[H] \setminus S),\]

Therefore, our lifted two-cover inequality can be written as the sum of at least two other valid inequalities; hence it does not define a facet.

(c) Assume that \(H\) contains only nodes of type 2 (with or without incident teeth).

If nodes of type 2 without incident teeth are allowed in the handle, the restric-
tion of a feasible set \(C\) whose incidence vector is in \(E\) to the edge set \(E(H) \cup \delta(H)\cdot T\)

is something more complicated than a matching with additional edge. It is rather a collection of node-disjoint paths between pairs of nodes with incident teeth plus one

additional path connecting the last node with incident tooth to \(V\) or some other path. More exactly, if we set \(r_{1} := 2\) minus the number of incident teeth for \(v \in H\)

and \(r_{2} := 0\) for the node \(x \notin H,\) then \(C[T]\) meets each node \(v \in T\) with exactly

\(r_{1}\) edges, except for one node that is not by \(r_{2}\) edges. \(C[T]\) is a near-perfect

v-cover of \(E(H) \cup \delta(H)\cdot T\) (so to speak). To see this, add the \(r_{2}\) divide by two,

and compare this with the right-hand side of the lifted two-cover inequality. (But not

every near-perfect \(v\)-cover of \(E(T)\) plus \(T\) defines a feasible set, as there might be some

node-disjoint cycles.)

Since the structure of the feasible sets with incidence vector in \(E\) is somewhat

unpredictable, we wish to complete graphs in \(H\). Call this collection of paths \(C\). As

before, we can add any edge of \(\delta(v)\) (except the path edge \(C \cap \delta(v)\), plus all teeth,

and get a set with incidence vector in \(F_{k}\). This proves \(b_{\alpha} = 0\) for all \(\alpha \in \delta(v)\)\).

But we can construct another set \(C'\) the same way as before, only this time it uses

a different edge of \(\delta(v)\). So we have \(b_{\alpha} = n_{\alpha}\) for all \(\alpha \in \delta(v)\)\).

Since \(\delta(v)\cdot T\) contains at least three edges, all edges in \(\delta(v)\cdot T\) have the same \(n_{\alpha}\)-value
\[ r_e = \alpha_e. \] Proving \( b_e = 0 \) for the teeth \( e \in T \) is easy, so we have that \( b \) is identical to the lifted two-cover inequality; therefore it defines a facet.

If \( H \) contains nodes of type 1, we use Theorem 4.7 for induction on the number of nodes of type 1 in \( H \) in the same way as we used Lemma 4.6 for proving validity of the lifted two-cover inequality.

Usually the feasible sets of \( 2 \text{CON}(G; r) \) whose incidence vectors satisfy the lifted two-cover inequality with equality are not feasible for \( 2 \text{CON}(G; r) \) if \( V \cup H \) consists of only one node, because this node may be an articulation node. But if \( V \cup H \) has sufficiently high connectivity, (7.4) may define a facet of \( 2 \text{CON}(G; r) \).

Remark 7.7. A lifted two-cover inequality (7.4) with \( |H_i| = 1 \) for \( i = 1, \ldots, p \), \( |H| = |T| = (p) \), defines a facet of \( 2 \text{CON}(G; r) \) if \( \Gamma \Gamma(V \cup H) \) is three-edge-connected, no two teeth are incident to the same node in \( (V \cup H) \), and no parallel edges exist.

Proof. The proof is analogous to the proof of Theorem 7.6.

But usually, as the following remark shows, lifted two-cover inequalities do not define facets for \( 2 \text{CON}(G; r) \) as soon as \( |H_i| \geq 2 \) for some \( H_i \) with an incident tooth.

Remark 7.8. A lifted two-cover inequality does not define a facet of \( 2 \text{CON}(G; r) \) if \( H \) is a node set \( H \) and a node \( v \in V \cup H \) so that \( (v) : H \) contains a tooth and a nontooth. (This is the case especially if \( G \) is complete and some \( H_i \) with incident tooth contains at least two nodes.)

Proof. It can be shown that a feasible set \( C \subseteq E \) with \( 2 \text{CON}(G; r) \) that satisfies such a lifted two-cover inequality with equality never uses the nontooth in \( (v) : H \).

But for the \( 2 \text{CON} \) problem we can use our lifting lemma of §4 to derive sufficient conditions for a lifted two-cover inequality with general \( H \) to define a facet of \( 2 \text{CON}(G; r) \).

Theorem 7.9. Given a lifted two-cover inequality (7.4), we will denote by \( \hat{\Gamma} \) the graph \( \hat{\Gamma}(H_i) \cup \cdots \cup \hat{\Gamma}(H) \).

(a) If \( \hat{\Gamma}(H) \) is hypomatchable (in the case \( p = |T| \) or complete (in the case \( p > |T| \), if the \( \hat{\Gamma}(H_i) \) for \( i = 1, \ldots, p \) are \( (H_i) + 1 \)-edge-connected, and if \( \hat{\Gamma}(V \cup H) \) is \( (\hat{\Gamma}(V \cup H)) + 1 \)-edge-connected, a lifted two-cover inequality defines a facet of \( 2 \text{CON}(G; r) \).

(b) If the lifted two-cover inequality is facet-inducing, then \( \hat{\Gamma}(H) \) and \( \hat{\Gamma}(H_i) \) are connected for \( i = 1, \ldots, p \), and \( \hat{\Gamma}(\hat{\Gamma}(H_i)) \geq 1 \) for \( i = 1, \ldots, p \). In fact, one can always find \( H, H_p \) with \( N(\hat{\Gamma}(H_i)) \geq 2 \) for \( i = 1, \ldots, p \) that induce the lifted two-cover inequality in question.

Proof. (a) Theorem 7.6 proves the lifted two-cover inequality to be facet-defining for \( 2 \text{CON}(G; r) \). With Lemma 4.4 we can lift this result to \( 2 \text{CON}(G; r) \).

(b) It is easy to see that the \( \hat{\Gamma}(H) \) must be connected for all \( i = 1, \ldots, p \). If \( \hat{\Gamma}(H_i) \) is not connected, we can split the handle \( H \) into two handles \( H' \) and \( H'' \) to derive two lifted two-cover inequalities whose sum gives the old one. So the old one cannot define a facet.

It remains to show that we can find \( H_1, \ldots, H_p \) with \( N(\hat{\Gamma}(H_i)) \geq 2 \) for \( i = 1, \ldots, p \), that induce our lifted two-cover inequality.

If \( H \) has no incident tooth and \( \hat{\Gamma}(\hat{\Gamma}(H_i)) = 1 \), then our lifted two-cover inequality can be written as the sum of another lifted two-cover inequality where \( H_i \) is split into at least two other sets plus one constraint \( x_e \leq 1 \). The same argument is possible if \( H \) has an incident tooth and \( \hat{\Gamma}(\hat{\Gamma}(H_i)) = 1 \). So in these cases our lifted two-cover inequality cannot define a facet.

Facets for Polyhedra Related to Low-Connected Networks

It remains to check the case in which \( H_i \) has an incident tooth \( e \) and \( x_e \hat{\Gamma}(H_i) = 1 \). In this case \( \hat{\Gamma}(H_i) \) has a bridge \( f \) so that \( \hat{\Gamma}(H_i) - f \) decomposes into two components \( U \) and \( W \) with \( r(U) = 1 \) and \( r(W) \geq 1 \). The interesting case is the one where the tooth \( e \) is incident to \( U \), because then we cannot simply split \( H_i \) into \( U \) and \( W \) to derive a stronger lifted two-cover inequality. But we can replace \( H_i \) by \( H_i - U \) and the tooth \( e \) by the bridge \( f \) to derive another lifted two-cover inequality of the same form as the old one. By repeating this procedure of reducing \( H_i \), we can assume that \( \hat{\Gamma}(\hat{\Gamma}(H_i)) \geq 2 \) for all \( i = 1, \ldots, p \).

8. Comb Inequalities. The following constraints were motivated, on the one hand, by the comb inequalities for the traveling salesman problem (see [GP]), and on the other hand, they were motivated by the fact that the lifted two-cover inequalities do not generally define facets for the \( 2 \text{CON} \) problem (see Remark 7.8). We wanted to find a facet containing the face induced by a lifted two-cover inequality in the case in which \( G \) is a complete graph and the \( H_i \) contain more than one node.

The class of inequalities we came up with in this case are valid for \( 2 \text{CON}(G; r) \), but not generally for \( 2 \text{CON}(G; r) \). We will call this class comb inequalities for \( 2 \text{CON}(G; r) \). These inequalities allow a further generalization using the concept of clique trees. But we will not discuss this here.

Let \( H, T_1, \ldots, T_k \) be subsets of \( V \) and let \( x_e \in T_i, H, i = 1, \ldots, k, \) be not necessarily distinct nodes (\( H \) is called the handle, the sets \( T_1, \ldots, T_k \) are the teeth, and the \( z_1, \ldots, z_k \) the special nodes) that satisfy the following conditions:

- \( r \geq 3 \) and odd;
- two teeth have at most one node in common; if \( \cap T_i \neq \emptyset \), then \( \cap T_i = \{ z_i \} = \{ z' \} \);
- each tooth \( T_i \) intersects the handle \( H \) in exactly one node; we denote this node by \( t_i \), for \( i = 1, \ldots, k \);
- \( \cap t_i = 2 \) for \( i = 1, \ldots, k \);
- \( \cap t_i \geq 1 \) for all \( v \in H \cup \{ x_{t_i} \} \).

We denote by \( V_0 \) the set of nodes of type 2 in \( G \). The special comb inequality is given by

\[ \pi(H) + \pi(H) + \sum_{t_i} \pi(E(T_i)) \]

(8.1)

The (general) comb inequality is derived from the special comb inequality (8.1) by expanding all nodes \( w \in H \) that are not in \( \{ x_{t_1}, \ldots, x_{t_k} \} \) to node sets \( W \) (see Lemma 4.4). Figure 8.1 gives an illustration of a comb inequality with a handle \( H \) consisting of four node sets and three teeth \( T_i, i = 1, \ldots, 3 \), which has right-hand side 6. Edges with coefficients 0 are drawn with dashed lines, edges with coefficient 1 with solid lines, and edges with coefficient 2 with bold lines.

We note that the comb inequality becomes a lifted two-cover inequality with sets \( H = \{ t_i \} \) if \( |T| = 2 \) and \( 2 \text{CON}(G; r) \).

We will prove validity and facet results only for special comb inequalities. With the help of Lemma 4.2, 4.5 we can easily derive validity and facet results for general comb inequalities.

Theorem 8.2. A comb inequality (8.1) is valid for \( 2 \text{CON}(G; r) \) with \( Z = \{ z_1, z_2, \ldots, z_k \} \), and hence it is valid for \( 2 \text{CON}(G; r) \).
with \( W := T_1 \) and \( \alpha := 1 \); this is done in the same way as in the validity proof for node-partition inequalities.

Note that the comb inequality (8.1) is also valid if the number of teeth \( t \) is even.

But in this case it does not define a facet, as it can be written as the sum of a comb inequality and node-partition inequality (or a nonnegativity constraint).

Note also that if \( H \cup \cup_{i=1}^t T_i = V \) and \( z_1 = z_3 = \cdots = z_t = \frac{1}{2} \) for all \( i \), the special comb inequality with right-hand side \( |H| - \frac{1}{2} \) may degenerate into a node-partition inequality with right-hand side, namely, \( |H| - 1 \). In this case the special comb inequality cannot define a facet.

**Theorem 8.3.** The special comb inequality (8.1) defines a facet of \( 2\text{CON}(G,r) \) if \( r = 2 \) for all nodes \( v \in V \), if the \( z_i \) are all distinct, and if \( G \) is the complete graph minus all edges with coefficient 2 in (8.1).

**Proof.** The restriction to nodes of type 2 has only technical reasons, mainly because of Lemma 6.4. The restriction to edges with coefficients 0 and 1 is also introduced only for technical reasons. Once we have proved an inequality to define a facet only on a subset of edges of the complete graph, it is easy to prove it to be facet-defining on the complete graph.

Let \( \mathcal{F} \) be the face induced by the comb inequality in question, and let \( F \) be contained in the face \( \mathcal{F} \) induced by some valid inequality \( \sum e_x \geq \beta \). First we prove \( t_i = \alpha_i \) for all edges in \( E(T_i) \cup (\{t_i\} : H) \) with coefficient 1 and some \( \alpha_i \). We do this (without loss of generality) for tooth \( T_i \). Suppose that \( |T_i| > 3 \). (For “small” teeth that consist of only one edge, the following proof has to be modified somewhat.) Construct a collection \( \mathcal{P} \) of node-disjoint paths in \( G[H] \) between pairs of nodes \( t_j \) and \( t_k \) and some others. Those paths should meet every node in \( H \) except \( t_i \). To this collection of paths \( \mathcal{P} \), we may add certain trees in the teeth \( T_i \) that are constructed as follows:

(1) For \( T_i \), we take any feasible edge set whose incidence vector lies in the face of \( 2\text{CON}(G((V(T_i)) \cup \mathcal{P})) \) induced by a certain node-partition inequality on \( T_i \), namely, the one with node \( z_i = z_i \) and node sets \( \{u\} \) for all nodes \( u \) in \( T_i \) and \( \{u\} \) for the shrunk node standing for \( V(T_i) \) (cf. (2) used in the validity proof in Theorem 8.2). Those objects are mainly trees on \( T_i \). Note also that the face of \( 2\text{CON}(G((V(T_i)) \cup \mathcal{P})) \) induced by the node-partition inequality is a facet of \( \mathcal{F} \).

(2) For \( T_i \), we take any feasible edge set whose incidence vector lies in the face of \( 2\text{CON}(G((V(T_i)) \cup \mathcal{P})) \) induced by (3) or (4) of the validity proof in Theorem 8.2. Those objects are mainly trees on \( T_i \). Note also that the face of \( 2\text{CON}(G((V(T_i)) \cup \mathcal{P})) \) induced by the node-partition inequality is a facet of \( \mathcal{F} \).

Finally, we add all edges \( z_i z_j \) to this construction.

We claim that this combination of paths in \( G[H] \) and trees of \( T_i \) is feasible. This can be easily checked. Secondly, we claim that its incidence vector lies in the face induced by the comb inequality in the validity proof of the comb inequality are satisfied with equality except one.

Since we have some freedom in the choice of the “tree” in \( T_i \), and we know that the node-partition inequality used for the construction of these “trees” defines a facet of \( 2\text{CON}(G((V(T_i)) \cup \mathcal{P})) \), we know that \( b_i = \alpha_i \) for all nonzero edges in this node-partition inequality, and \( b_i = 0 \) for all zero edges \( e \). This can be done for all teeth \( T_i \) in the same way as shown for tooth \( T_i \).

Now we prove that all edges inside the handle have the same \( b_i \)-value. This value must be the same as \( \alpha_1, \alpha_2 \), etc. Thus, we know that all edges with coefficient 1 in the comb inequality have the same \( b_i \)-value and all edges \( e \) with coefficient 0 in the
comb inequality have $b_u = 0$.

To prove $b_u = a_u$ for all $e \in \delta_2 \mathcal{H}(v)$ and $v \in H$, we just vary our construction of paths in the beginning. This is done in exactly the same way as in the proof of Theorem 7.6(b). To give an example: if $v \in H$, then we construct paths between $t_1$ and $v$, $t_2$ and $v$, etc. that are all node-disjoint. These paths must meet all nodes in $G(H)$. In addition to this collection $P$ of paths we construct trees in $T$ according to point (2) above. Now we can add any edge $e \in (\delta_2(v)) \cap E(H)$ not already in some path to achieve a feasible solution whose incidence vector lies in the face $F_2$. So $b_u = b_y$ for all $e \in (\delta_2(v)) \cap E(H)$. To prove $b_u = b_y$ for all $e \in (\delta_2(v)) \cap E(H)$, we just choose a collection of paths using another edge of $F(v)$.

It is easy to see that the $a_u$ value for the $e$ of zero coefficient in the comb inequality is also 0.

So inequality $b^T x \geq 0$ is identical to the comb inequality (8.1) except for scalar multiplication. Therefore, it defines a facet of 2CONC($G$).

The question naturally arises whether there are also "comb" inequalities valid for 2CONC($G$-r). We know of such a class, but the validity proof is somewhat ugly. In such a "comb" inequality we have two types of teeth: "small" teeth consisting of only one edge with coefficient 0, and "large" teeth $T$ with coefficients 0 on edges in $T \setminus H$, and coefficients 1 on the edges leading from $T \setminus H$ to $T \setminus H$ and to the "outside." The edges in the handle have coefficients 2. This seems to be more symmetric, and therefore, in a way, nicer than the comb inequality (8.1).

Also, some other odds and ends of inequalities that do not fit into any of the presented classes are known to us. Some of these are published in Snow's dissertation [5].

9. Computational results. The theory presented here for the 2CONC and 2CONC polytopes was developed in order to solve problems of the type and size that arise in the design of survivable telephone networks in fiber optic technology. The idea was to design and implement a cutting plane algorithm that uses the inequalities introduced above.

As mentioned before, it unfortunately turned out that—except for the cut and node inequalities—the separation problem for all other classes of inequalities presented here is NP-hard. This means that we can use these classes of inequalities only heuristically. We had to make an experimental investigation of the relative benefit of running various heuristics that determine, for a given point $y$, an inequality of some class of valid inequalities that is violated by $y$.

The outcome of our computational study was a cutting plane code that uses exact separation routines for cut and node-cut inequalities and separation heuristics for partition, node-partition, and lift-and-project inequalities. The code was written in Fortran and generates the minimum number of cuts necessary to solve the problem. Our results show that the cutting plane algorithm is very effective in solving realistic problems of the type and size that arise in the design of survivable telephone networks.

The design and implementation of a practically efficient cutting plane algorithm is a rather tricky and time-consuming task. Its success is based on the proper combination of many details. Some of these are discussed in [GM] and [S]. We are unable to outline these here. Our final code showed the following computational characteristics on our test problems.
CONVERGENCE OF BROYDEN'S METHOD IN BANACH SPACES

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Abstract. This paper proves new convergence theorems for convergence of Broyden's method when applied to nonlinear equations in Banach spaces. The convergence is in the norm of the Banach space itself, rather than in the norm of some Hilbert space that contains the Banach space. It is shown that the norms in which superlinear convergence takes place are determined by the smoothing properties of the error in the Fréchet derivative approximation and not by the linear product in which Broyden's method is implemented. Among the consequences of the results in this paper are a proof of superlinear local convergence when Broyden's method is applied to integral equations with continuous kernels, global superlinear convergence of the Broyden iterates for singular and nonsingular linear compact fixed point problems in Banach space, and a new method for finding derivatives with spaces kernels, and superlinear convergence for a new method for integral equations when part of the Fréchet derivative can be explicitly computed. Partitioned variants of the methods and the "bad" Broyden method are also discussed.

Key words. Broyden's method, q-superlinear convergence, quasi-Newton update, Banach space

AMS(MOS) subject classifications. 65J15, 47H07, 49D15

1. Introduction. This paper considers the solution of equations in Banach space by Broyden's method. We write our equations as

\[(1.1) \quad F(u) = 0,\]

where \(F\) is a Lipschitz continuously differentiable map between Banach spaces \(X\) and \(Y\). We will consider both linear and nonlinear equations. Broyden's method is a variation of Newton's method in which an approximation, \(B\), to the Fréchet derivative, \(F'(u^*)\), is maintained along with an approximation \(u^*\) to a solution \(u^*\).

This method has been used with success for discretizations of infinite-dimensional problems in integral equations [23, 26], fluid mechanics [10], and optimal control [26]. In this paper we take the position that analysis of the convergence properties of the method should not be done for the discrete finite-dimensional problems alone because the results of such a finite-dimensional analysis can hide features that may depend on how the level of discretization is refined and may lead to conclusions that are valid only in finite dimensions. The papers [31, 32], and [24] present examples of these kinds of problems. Direct analysis of the infinite-dimensional problem can also lead to effective preconditioning strategies that produce good convergence properties even for finite-dimensional approximations.

The purpose of this paper is to extend convergence results for Broyden's method in infinite-dimensional spaces to Hilbert spaces and to the Banach space setting and thereby sharpen convergence results. In particular, our results show how the inner product used in the implementation of Broyden's method and the topology in which convergence takes place are related. The significant consequences of this Banach space analysis for problems considered previously in [23, 30], and [26] and [27] are to make the convergence estimates more precise, for example, to provide uniform convergence...