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FACETS FOR POLYHEDRA ARISING IN THE DESIGN OF COMMUNICATION NETWORKS WITH LOW-CONNECTIVITY CONSTRAINTS*

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Abstract. This paper addresses the important practical problem of designing survivable fiber optic communication networks. This problem can be formulated as a minimum-cost network design problem with certain low-connectivity constraints. Previous work presented structural properties of optimal solutions and heuristic methods for obtaining "near-optimal" network designs. Some facet-inducing inequalities for the convex hull of the solutions to this problem are given. A companion paper describes computational results on real-world telephone network design problems with a cutting plane method based on this work. These computational results are summarized in the last section of this paper.

Key words. network design, network survivability, connectivity, polyhedral combinatorics

AMS(MOS) subject classifications. 05C40, 90C27, 90B12

1. Introduction. A recent trend in communication networks is the emergence of fiber optic technology as one of the major components in the "network of the future." This transmission medium is cost-effective and reliable, and provides very high transmission capacity. This combination promises to usher in new telecommunication services requiring large amounts of bandwidth. At the same time, the unique characteristics of this technology imply the need for new network design approaches. (See [CFLM] for more details.)

Survivability is an important factor in the design of communication networks. Network survivability is used here to mean the ability to restore service in the event of a catastrophic failure of a network component, such as the complete loss of a transmission link, or the failure of a switching node. Service could be restored by routing traffic through other existing network links and nodes, assuming that the design of the network has provided for this additional connectivity. Clearly, a higher level of redundant connectivity results in a greater network survivability and a greater overall network cost. This leads to the problem of designing a minimum-cost network that meets certain required connectivity constraints.

Survivability is a particularly important issue for fiber networks. The high capacity of fiber facilities results in much more sparse network designs with larger amounts of traffic carried by each link than is the case with traditional bandwidth-limited technologies. This increases the potential damage to network services due to link or node failures. It is necessary to trade off the potential for lost revenues and customer goodwill against the extra costs required to increase the network survivability. Recent works on methods for designing survivable fiber communication networks by [CMW] and [MS] conclude that (1) survivability is an important issue for fiber networks, (2) "two-connected" topologies provide a high level of survivability in a cost-effective manner, and (3) good heuristic methods exist for quickly generating "near-optimal"

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networks. In particular, it was determined that a network topology should provide for at least two diverse paths between certain "special" offices, thus providing for protection against any single link or single node failure for traffic between these offices. These special offices represent high revenue-producing offices and other offices that require a higher level of network survivability.

We now formalize the network design problems that are being considered in this paper. A set of **nodes** V is given that represents the locations of the switches (offices) that must be interconnected into a network in order to provide the desired services. A collection E of **edges** is also specified that represents the possible pairs of nodes between which a direct transmission link can be placed. We let $G = (V, E)$ be the (undirected) graph of possible direct link connections. The graph G may have parallel edges but contains no loops. (Thus we assume throughout this paper that all graphs considered are loopless. But they may have parallel edges. Graphs without parallel edges are called **simple.**)

Given a graph $G = (V, E)$ and $W \subseteq V$, the edge set $\delta(W) := \{ij \in E \mid i \in W, j \in V \backslash W\}$ is called the **cut** (induced by W). (We will write $\delta_G(W)$ to make clear—in case of possible ambiguities—with respect to which graph the cut induced by W is considered.)

For $W, W' \in V$ with $W \cap W' = \emptyset$ we define $[W : W'] := \{ij \in E \mid i \in W, j \in W'\}$. So $\delta(W) = [W : V \backslash W]$. For $W \subseteq V$, we denote by $G[W]$ the subgraph of G induced by W and by $E(W)$ its edge set $\{ij \in E \mid i, j \in W\}$. G/W is the graph obtained from G by contracting the nodes in W to a new node w (retaining parallel edges). We call the reverse operation of replacing the shrunk node w by the original node set W the **expansion** of w in G/W to G. We will denote by $G - v$ the graph obtained by removing the vertex v and all incident edges from G, and by $G - F$ the graph obtained by removing the edge set F from G (we write $G - f$ instead of $G - \{f\}$). If $G - v$ has more connected components than G for some node v, we will call v an **articulation node** of G. Similarly, if $G - e$ has more connected components than G, we will call edge e a **bridge** of G.

Each edge $e \in E$ has a **fixed cost** c_e of establishing the direct link connection. The cost of establishing a network $N = (V, F)$ consisting of a subset $F \subseteq E$ of edges is $c(F) := \sum_{e \in F} c_e$, i.e., it is the sum of the costs of the individual links contained in F. The goal is to build a minimum-cost network so that the required survivability conditions, which we describe below, are satisfied. We note that the cost here represents setting up the topology for the communication network and includes placing conduits in which to lay the fiber cables, placing the cables into service, and other related costs. We do not consider costs that depend on how the network is implemented, such as routing, multiplexing, and repeater costs. Although these costs are also important, it is usually the case that a topology is first designed and then these other costs are considered in a second stage of optimization.

For any pair of distinct nodes $s, t \in V$, an $[s, t]$-**path** P is a sequence of nodes and edges $(v_0, e_1, v_1, e_2, \cdots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident with the nodes v_{i-1} and v_i $(i = 1, \cdots, l)$, where $v_0 = s$ and $v_l = t$, and where no node or edge appears more than once in P. A collection $P_1, P_2, \cdots, P_k$ of $[s, t]$-paths is called **edge-disjoint** if no edge appears in more than one path, and is called **node-disjoint** if no node (except for s and t) appears in more than one path. (Remark: In order to be consistent with standard graph theory we do not consider two parallel edges as two node-disjoint paths.)

The **survivability conditions** require that the network satisfy certain edge- and node-connectivity requirements. In particular, each node $s \in V$ has an associated nonnegative integer r_s, which represents its **connectivity requirement**. This means that for each pair of distinct nodes $s, t \in V$, the network $N = (V, F)$ to be designed has to have at least

$$r(s,t) := \min\{r_s, r_t\}$$

edge-disjoint (or node-disjoint) $[s,t]$-paths. These conditions ensure that some communication path between s and t will survive a prespecified level of edge (or node) failures. The levels of survivability specified depend on the relative importance placed on maintaining connectivity between different pairs of offices.

The fiber optic network design problems that arise in practice and that we are addressing in this paper have three types of offices. The so-called "special" offices have connectivity requirement 2 while "ordinary" offices have connectivity requirement 1. An office with connectivity requirement 0 is called "optional" since it need not be part of the network to be designed.

Figure 1.1 shows an example network. Special offices are indicated by squares, ordinary offices by circles. Optional offices do not occur. The lines (thin, bold, and dashed) represent the possible direct links from which the minimum-cost survivable network must be designed. The network obtained by removing the dashed lines, i.e., the graph formed by the union of bold and thin lines, represents a feasible network. It consists of a two-connected part (the bold lines) containing all special nodes, in which every pair of nodes is linked by at least two node-disjoint paths and a collection of trees (the thin lines), which link the remaining nodes into the two-connected part.

Thus in the remainder of this paper we consider the case where the connectivity requirements satisfy

$$r_s \in \{0,1,2\} \quad \text{for all } s \in V.$$

Nodes of connectivity requirement 0 (respectively, 1, 2) will also be called nodes of **type 0** (respectively, **type 1, 2**). Let us define the **2ECON** problem (respectively, **2NCON** problem) to be the network design problem where between each pair of distinct nodes s and t at least $\min(r_s, r_t)$ edge-disjoint (respectively, node-disjoint) paths are required.

Given $G = (V, E)$, we extend the connectivity requirement function r to functions operating on sets by setting

$$\begin{aligned} r(W) &:= \max\{r_s \mid s \in W\} \text{ for all } W \subseteq V, \quad \text{and} \\ \operatorname{con}(W) &:= \max\{r(s,t) \mid s \in W, t \in V \backslash W\} \\ &= \min\{r(W), r(V \backslash W)\} \quad \text{for all } W \subseteq V, \quad \emptyset \neq W \neq V. \end{aligned}$$

Let us now introduce, for each edge $e \in E$, a variable x_e and consider the vector space $\mathbb{R}^E$. Every subset $F \subseteq E$ induces an **incidence vector** $\chi^F = (\chi_e^F)_{e \in E} \in \mathbb{R}^E$ by setting $\chi_e^F := 1$ if $e \in F$, and $\chi_e^F := 0$ otherwise. Vice versa, each 0/1-vector $x \in \mathbb{R}^E$ induces a subset $F^x := \{e \in E \mid x_e = 1\}$ of the edge set E of G. For any subset of edges $F \subseteq E$, we define $x(F) := \sum_{e \in F} x_e$. We can now formulate the 2NCON network design problem introduced above as the following integer linear

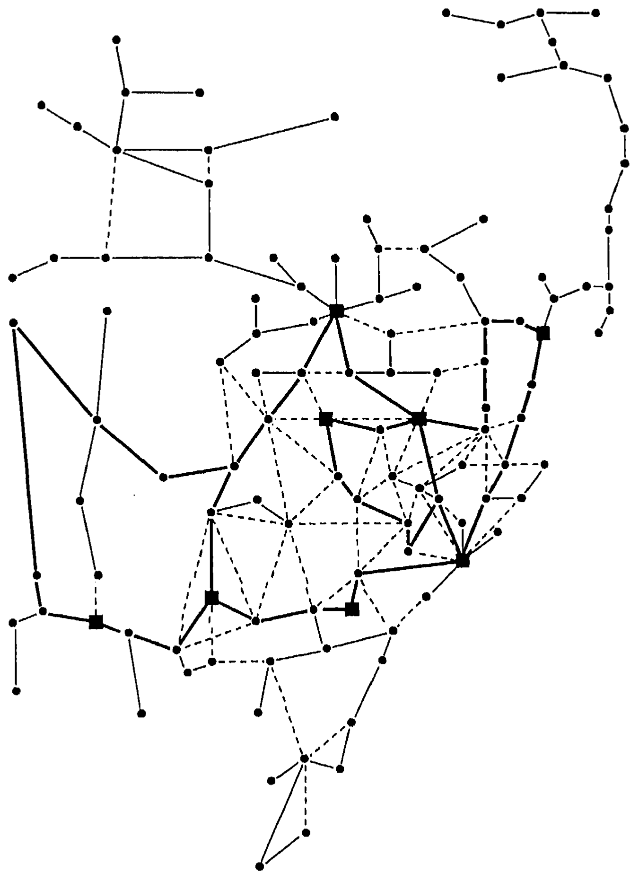


FIG. 1.1

program:

$$
(1.1)\qquad
\begin{array}{ll}
\multicolumn{2}{l}{\min \sum_{ij\in E} c_{ij}x_{ij}} \\
\multicolumn{2}{l}{\text{subject to}} \\
\text{(i) } x(\delta(W)) \geq \operatorname{con}(W) & \text{for all } W \subseteq V, \emptyset \neq W \neq V; \\
\text{(ii) } x(\delta_{G-z}(W)) \geq 1 & \text{for all } z \in V, \text{ and for all } W \subseteq V\backslash\{z\},\ \emptyset \neq W \neq \\
 & V\backslash\{z\} \text{ with } r(W) = 2 \text{ and } r(V\backslash(W \cup \{z\})) = 2; \\
\text{(iii) } 0 \leq x_{ij} \leq 1 & \text{for all } ij \in E; \\
\text{(iv) } x_{ij} \text{ integral} & \text{for all } ij \in E.
\end{array}
$$

It follows from Menger's theorem that, for every feasible solution x of (1.1), the subgraph $N = (V, F^x)$ of G defines a network satisfying the two-connected survivability requirements for the 2NCON problem. Removing (ii), we have an integer linear program for the 2ECON network design problem. (Note that in the case $r = \{0,1\}^V$, inequalities (i), (iii), and (iv) of (1.1) characterize the Steiner tree problem.) An inequality of type (i) is called a **cut inequality**, one of type (ii) is called a **node-cut inequality**, and one of type (iii) is called a **trivial inequality**.

The main objective of this paper is to study the 2ECON and 2NCON network design problems from a polyhedral point of view to see which inequalities are suitable choices for a cutting plane approach, i.e., we want to find a tighter LP-relaxation than the one obtained by dropping the integrality constraints (iv) of (1.1) for the 2ECON and 2NCON network design problems. To do this we define the following polytopes. Let $G = (V, E)$ be a graph and let $r \in \{0,1,2\}^V$ be given with $r_v = 2$ for at least two nodes. Then

$$
\begin{aligned}
2\mathrm{NCON}(G;r) &:= \operatorname{conv}\{x \in \mathbb{R}^E \mid x \text{ satisfies (i), (ii), (iii), (iv) of (1.1)}\},\\
2\mathrm{ECON}(G;r) &:= \operatorname{conv}\{x \in \mathbb{R}^E \mid x \text{ satisfies (i), (iii), (iv) of (1.1)}\}
\end{aligned}
$$

are the polytopes associated with the 2NCON and 2ECON network design problems. (Above, "conv" denotes the convex hull operator.) We say that $F \subseteq E$ is **feasible** for one of these polytopes if χ^F is.

Related problems have been investigated previously. A general integer linear programming approach to network design problems with connectivity requirements is presented in [GM] along with a preliminary study of these problems from a polyhedral point of view. We shall make several references to this work in what follows. [CFN] study the dominant of the 2ECON$(G;r)$ polytope in the special case where $r = \{2\}^V$. [MMP] study the 2ECON$(G;r)$ and 2NCON$(G;r)$ polytopes in the special case where $r = \{2\}^V$, and G is a complete graph with the edge weights satisfying the triangle inequality. They show that in this case the optimization problems are the same over both polytopes and then give a certain type of "characterization" of the optimal solutions.

Let us now introduce some connectivity functions and some notation concerning "essential" edges and dimension of polyhedra. Let $G = (V, E)$ and $r \in \{0,1,2\}^V$ be given; we say that $e \in E$ is **essential with respect to** 2ECON$(G;r)$ if 2ECON$(G - e; r) = \emptyset$; similarly we say e is **essential with respect to** 2NCON$(G;r)$ if 2NCON$(G - e; r) = \emptyset$. In other words, e is essential if its deletion results in a graph such that one of the survivability requirements cannot be satisfied. We denote the set of edges of E that are essential with respect to 2ECON$(G;r)$ by **2EES$(G;r)$**, and the set of edges that are essential with respect to 2NCON$(G;r)$ by **2NES$(G;r)$**. Clearly, for all subsets $F \subseteq E\backslash$2EES$(G;r)$, 2EES$(G;r)$ is contained in 2EES$(G - F; r)$ (similarly with 2NES$(G;r)$). Let $\dim(S)$ denote the **dimension** of a set $S \subseteq \mathbb{R}^n$, i.e., the maximum number of affinely independent elements in S minus 1. One of the results proved in [GM] says that the polyhedron 2ECON$(G;r)$ is full-dimensional if and only if 2EES$(G;r)$ is empty, and also that 2NCON$(G;r)$ is full-dimensional if and only if 2NES$(G;r)$ is empty.

Let $G = (V, E)$ be a graph and $W \subseteq V$ with $|W| \geq 2$, and let $G' = (V, E')$ be the simple graph underlying G. We set

$\lambda(G, W) :=$ minimum cardinality of a subset F of E, such that two nodes of W are disconnected in $G - F$;

$\kappa(G, W) :=$ minimum cardinality of a set $S \cup F$, where $S \subseteq V$ and $F \subseteq E'$, such that two nodes of W are disconnected in $G' - (S \cup F)$.

If $|W| < 2$, then $\lambda(G, W)$ and $\kappa(G, W)$ are defined as ∞. If G with node set V_G is a subgraph of some graph H with node set V_H and $W \subseteq V_H$ we will also write $\lambda(G, W)$ instead of $\lambda(G, W \cap V_G)$. We will use these functions frequently in two special situations. To shorten notation in these cases, we introduce the following definitions:

$$\lambda_i(G) := \lambda(G, V_i), \qquad \kappa_i(G) := \kappa(G, V_i),$$

where $V_i := \{v \in V \mid r_v \geq i\}$, $i = 0, 1, 2$. So $\lambda_0(G)$ is nothing but the edge-connectivity of G, and $\kappa_0(G)$ is the node-connectivity of G.

Throughout this paper we make the following assumptions:

(1.2)
Let $G = (V, E)$ and $r \in \mathbb{Z}_+^V$ be given.
(i) $r \in \{0,1,2\}^V$ and at least two different nodes s, t satisfy $r_s = r_t = 2$;
(ii) if we consider the 2ECON problem we assume G to be two-node connected and $\lambda_2(G) \geq 3$;
(iii) if we consider the 2NCON problem we assume G to be two-node connected and $\kappa_2(G) \geq 3$.

We will say that (G, r) satisfies (1.2) and mean that the graph $G = (V, E)$ and the vector $r \in \mathbb{Z}_+^V$ of connectivity types satisfy conditions (i), (ii), and (iii). If (i) does not hold, then the 2ECON and the 2NCON problem reduces to the Steiner tree problem for which more specialized investigations can be (and have been) made. We want to exclude this case from the present investigation. It also does not occur in the practical problems we have in mind. One consequence of (ii) and (iii) of (1.2) is that the 2ECON or 2NCON problem contains no essential edges; hence the associated polyhedron is full-dimensional. We further justify assumptions (ii) and (iii) in §2.

In §§2 and 4 we present some decomposition and lifting results that simplify the later discussions. In §3 we investigate which of the basic inequalities given in (1.1) define facets for 2ECON, respectively, 2NCON. In §§5–8 we present several classes of facet-inducing inequalities for 2ECON and 2NCON. These include partition, node-partition, two-cover, and comb inequalities.

We will not discuss the separation problems associated with the classes of inequalities introduced in this paper. Let us just mention here that the cut and node-cut inequalities can be checked in polynomial time, but for all other classes of inequalities to be presented in this paper the separation problem is NP-hard (as is shown in [GMS]). Based on the polyhedral investigations presented in this paper we have designed cutting plane algorithms for the 2ECON and the 2NCON problems. A short summary of our computational results is given in §9. The details can be found in [GMS] and [S].

2. Decomposition. The problem of finding a cost-minimal network for the 2ECON problem can be decomposed into at least two independent problems if the underlying graph G contains an articulation node v disconnecting two nodes of type at least 1. The subproblems are solved on the two-node-connected components of G with the same cost function and the same connectivity types r; only the connectivity type of the articulation node v may have to be adjusted. The 2ECON problem may also be decomposed into independent subproblems if G contains two edges e, f, such that in $G-\{e,f\}$ two nodes of type 2 are disconnected. Another simple decomposition is possible for the 2NCON problem if the graph G contains two nodes u, v so that in $G-\{u,v\}$ two nodes of type 2 are disconnected. These and other more complicated decompositions are described in more detail in [GMS].

Observe that using the above decompositions, any 2ECON or 2NCON problem with essential edges may be decomposed into problems without essential edges. This is the reason why we restrict ourselves to graphs G and connectivity types r for which our general assumptions (ii) and (iii) of (1.2) hold. This implies also that 2ECON$(G;r)$ and 2NCON$(G;r)$ are full-dimensional [GM].

There is another (technical) reason why we restrict ourselves to full-dimensional polyhedra here. If polyhedra are not full-dimensional, proofs often become more involved technically and statements about nonredundancy of certain systems become quite ugly due to the necessity to exclude equivalent inequalities. This is also true in our case. It is not difficult to derive the results for the lower-dimensional cases from the results presented later. But the statements of these theorems are often rather complicated and we want to avoid unnecessary technicalities.

3. Basic facets. In this section we investigate under which conditions the cut inequalities (1.1)(i), the node-cut inequalities (1.1)(ii), and the trivial inequalities (1.1)(iii) define facets for 2ECON$(G;r)$, respectively, 2NCON$(G;r)$.

An inequality $a^Tx \le \alpha$ is **valid** with respect to a polyhedron P if $P \subseteq \{x \mid a^Tx \le \alpha\}$; the set $F_a := \{x \in P \mid a^Tx = \alpha\}$ is called the **face** of P defined by $a^Tx \le \alpha$. If $\dim(F_a) = \dim(P) - 1$ and $F_a \ne \emptyset$, then F_a is a **facet** of P and $a^Tx \le \alpha$ is called **facet-defining** or **facet-inducing**.

The following theorem follows from Theorem 3.3 in [GM] and characterizes which of the trivial inequalities (1.1)(iii) define facets.

THEOREM 3.1. *Let (G,r) satisfy (1.2).*

(a) *$x_e \le 1$ defines a facet of 2ECON$(G;r)$ and of 2NCON$(G;r)$ for all $e \in E$.*

(b) *$x_e \ge 0$ defines a facet of 2ECON$(G;r)$ (respectively, 2NCON$(G;r)$) for $e \in E$, if and only if for every edge $f \ne e$ the polytope 2ECON$(G-\{e,f\};r)$ (respectively, 2NCON$(G-\{e,f\};r)$) is nonempty.*

The next theorem characterizes the **cut inequalities** (1.1)(i) that define facets.

THEOREM 3.2. *Let (G,r) satisfy (1.2) and let $W \subseteq V$ with $\emptyset \ne W \ne V$.*

(a) *Suppose* con$(W) = 2$. *Then $x(\delta(W)) \ge 2$ defines a facet of 2ECON$(G;r)$ if and only if*

(a_1) *$G[W]$ and $G[V\backslash W]$ are connected;*

(a_2) *$\lambda_1(G[W]) \ge 2$ and $\lambda_1(G[V\backslash W]) \ge 2$;*

(a_3) *e is a bridge of $G[W]$. Then f is a bridge of $G[V\backslash W]$, U, U' are the node sets of the two components of $G[W]-e$, and $\bar U$, $\bar U'$ are the node sets of the two components of $G[V\backslash W]-f$; and if $r(U) = r(\bar U) = 2$ (implying $r(U') = r(\bar U') = 0$), then $|[U:\bar U]| \ge 1$.*

(b) *Suppose* con$(W) = 1$. *Then $x(\delta(W)) \ge 1$ defines a facet of 2ECON$(G;r)$ if and only if*

(b_1) *$G[W]$ and $G[V\backslash W]$ are connected;*

(b_2) *$\lambda_1(G[W]) \ge 2$ and $\lambda_1(G[V\backslash W]) \ge 2$;*

(b_3) *$\lambda_2(G[W]) \ge 3$ and $\lambda_2(G[V\backslash W]) \ge 3$.*

(c) *Suppose* con$(W) = 0$. *Then $x(\delta(W)) \ge 0$ does not define a facet of 2ECON $(G;r)$ or of 2NCON$(G;r)$.*

(d) *Suppose* con$(W) = 2$. *Then $x(\delta(W)) \ge 2$ defines a facet of 2NCON$(G;r)$ if and only if*

(d_1) *the conditions (a_1), (a_2), and (a_3) of (a) are satisfied;*

(d_2) *$\kappa_2(G[W]) \ge 2$ and $\kappa_2(G[V\backslash W]) \ge 2$;*

(d_3) *u is an articulation node of $G[W]$ and $\bar u$ is an articulation node of $G[V\backslash W]$, and if U and $\bar U$ are node sets of components of $G[W]-u$ and $G[V\backslash W]-\bar u$, respectively, such that $r(U) = r(\bar U) = 2$, then $|[U:\bar U]| \ge 1$, and (because of (d_2)) all other components of $G[W]-u$ and $G[V\backslash W]-\bar u$ do not contain nodes of type 2;*

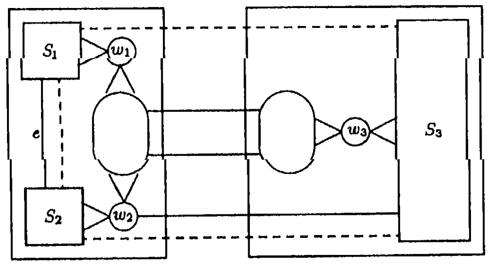
(d_4) *neither for $S = W$ nor for $S = V\backslash W$ does the following situation occur: There are an edge $e \in E(S)$ and nodes $w_1, w_2 \in S$ (not necessarily distinct) and a node $w_3 \in V\backslash S$ so that there exists a component (S_1,E_1) of $(G[S]-e)-w_1$, a component (S_2,E_2) of $(G[S]-e)-w_2$, and a component (S_3,E_3) of $G[V\backslash S]-w_3$ with $r(S_1) = r(S_2) = r(S_3) = 2$, $S_1 \cap S_2 = \emptyset$, such that in $G-e$ there is no edge between S_i and S_j for all $i \ne j$, $i,j \in \{1,2,3\}$ (see Fig. 3.1 for an illustration; dashed lines denote nonexisting edges);*

(d_5) *for $S = W$ and $S = V\backslash W$ the following has to hold: if $V\backslash S$ has exactly two neighbors in S, then one of these two nodes is the only node of type 2 in S.*

(e) *Suppose* con$(W) = r(W) = 1$. *Then $x(\delta(W)) \ge 1$ defines a facet of 2NCON $(G;r)$ if and only if*

(e_1) *the conditions (b_1), (b_2), and (b_3) of (b) are satisfied;*

(e_2) *$\kappa_2(G[V\backslash W]-e) \ge 2$ for all $e \in E(V\backslash W)$.*



W $\qquad$ $V\backslash W$

FIG. 3.1

Proof. We give a proof of (d). (The proofs of (a) in the general case, (b), and (e) use the same ideas and are thus omitted. (c) is trivial.)

We first show that if one of the conditions (d_1)–(d_5) is not satisfied, then the cut inequality $x(\delta(W)) \geq 2$ does not define a facet. Necessity of (d_1) is seen easily (see, e.g., Corollary 6.7 of [GM]). Suppose (d_2) is violated. Let u be an articulation node of $G[W]$, and let (S_1, E_1), (S_2, E_2) be two components of $G[S] - u$ with $r(S_1) = r(S_2) = 2$. Then $x(\delta(W)) \geq 2$ can be written as the sum of the node-cut inequalities $x(\delta_{G-u}(S_1)) \geq 1$ and $x(\delta_{G-u}(S_2)) \geq 1$ plus possibly some nonnegativity constraints. Therefore, $x(\delta(W)) \geq 2$ does not define a facet. If (d_3) is violated there are nodes u, $\bar{u}$ and node sets U, $\bar{U}$ with the indicated properties and $[U : \bar{U}] = \emptyset$. In this case the cut inequality can be written as the sum of two other node-cut inequalities $x(\delta_{G-u}(U)) \geq 1$ and $x(\delta_{G-\bar{u}}(\bar{U})) \geq 1$. Hence $x(\delta(W)) \geq 2$ does not define a facet.

Now suppose we have the situation excluded by (d_4) for $S = W$. In this case, it is not possible to construct a feasible solution with $x(\delta(W)) = 2$ and $x_e = 0$, because any feasible set not using e would either have node w_3 as an articulation node or use three edges of $\delta(W)$. Therefore, all feasible sets C with $|C \cap \delta(W)| = 2$ have to use e, so the face defined by $x(\delta(W)) \geq 2$ is contained in the face defined by $x_e \leq 1$. Since 2NCON$(G;r)$ is full-dimensional, these faces cannot be the same. Therefore, $x(\delta(W)) \geq 2$ does not define a facet.

Suppose (d_5) is violated. Let the two neighbor nodes of $V \backslash S$ in S be called u and v. If, in contradiction to (d_5), there is at least one node of type 2 in $S \backslash \{u, v\}$ or $r_u = r_v = 2$, then $x(\delta(W)) \geq 2$ can be written as the sum of the two node-cut inequalities $x(\delta_{G-u}(S \backslash \{u\})) \geq 1$ and $x(\delta_{G-v}(S \backslash \{v\})) \geq 1$.

Now let the conditions of (d) be satisfied for some inequality $a^T x := x(\delta(W)) \geq 2$. Let $b^T x \geq \beta$ be a facet-defining inequality such that the face F_a induced by $a^T x \geq 2$ in 2NCON$(G;r)$ is contained in the facet F_b induced by $b^T x \geq \beta$. Our aim is to show that b is a positive multiple of a, which implies that F_a is identical with the facet F_b.

Let us first state some conditions under which for a given e, $f \in \delta(W)$, the incidence vector of $C_{e,f} := E(W) \cup E(V \backslash W) \cup \{e, f\}$ is feasible for 2NCON$(G;r)$ and hence in $F_a \subseteq F_b$. Assume that both W and $V \backslash W$ contain more than one node of type 2. (In the other case, the proof has to be modified a little.) (1) If e, f are to induce a feasible $C_{e,f}$ they may not have a common endpoint (unless this is the only node of type 2 in W or $V \backslash W$, which we excluded). (2) If we denote the two endpoints of e and f in W with u and v, respectively, then for any node s of type 2 in W there must exist an $[s, u]$-path and an $[s, v]$-path that are node-disjoint; the same for $V \backslash W$.

We can rewrite these conditions in the following way: Let U denote a two-node-connected component of $G[W]$ containing some node of type 2 of $G[W]$. Note that by condition (d_2), U must then contain all nodes of type 2 in W. Now remove from U the set of all articulation nodes of $G[W]$. Let a node set $\bar{U}$ (in $G[V \backslash W]$) be defined in the same way as U in $G[W]$. Condition (2) says that e and f may not be incident to the same component of $G[W] - U$ and $G[V \backslash W] - \bar{U}$. All in all, e and f must constitute a matching of size 2 in the graph G' derived from G by shrinking all components of $G[W] - U$ and $G[V \backslash W] - U$ and deleting all edges except those in $\delta(W)$. The maximum matching possible in this graph has size at least 3, otherwise there are two nodes covering all edges in G', which translates to condition (1.2)(iii), or (d_3) or (d_5) of Theorem 3.2 being violated.

Now we are ready to show that b_e has the same value γ for all $e \in \delta(W)$. Assume that both W and $V \backslash W$ contain more than one node of type 2. G' has a matching with three edges, say, e, f, and g. Since the incidence vectors of $C_{e,f}$, $C_{f,g}$, and $C_{g,e}$ lie in F_b, we have $b_e = b_f = b_g = \gamma$. For any fixed edge $t \in \delta(W) \backslash \{e, f, g\}$ either $\{t, e\}$, $\{t, f\}$, or $\{t, g\}$ constitute a matching in G', say, $\{t, e\}$. Therefore, the incidence vectors of both $C_{t,e}$ and $C_{f,e}$ are in F_b, and we have $b_t = b_f = \gamma$. This way we can

prove $b_t = \gamma$ for all $t \in \delta(W)$.

To prove $b_e = 0$ for all $e \in E(W)$ we need to construct a set $C \subseteq E$ with $\chi^C \in F_a$ and $e \notin C$ for some fixed $e \in E(W)$. Since $\chi^{C \cup \{e\}}$ is also in F_a we know $b_e = 0$. Assuming again that both W and $V \backslash W$ have at least two nodes of type 2, we try for a given $e = v_1 v_2 \in E(W)$ to find f, $g \in \delta(W)$ constituting a matching of G', so that $C := C_{f,g} \backslash \{e\}$ is feasible for 2NCON$(G;r)$. If $\kappa_2(G[W] - e) \geq 2$, we can find such f, $g \in \delta(W)$ inducing a feasible $C_{f,g} \backslash \{e\}$ in G by similar arguments as above. Since the incidence vectors of $C_{f,g} \backslash \{e\}$ and $C_{f,g}$ are in F_b, we have $b_e = 0$.

Now suppose $\kappa_2(G[W] - e) = 1$. Consider the tree structure of the two-node-connected components and the articulation points of $G[W] - e$. Since $\kappa_2(G[W]) \geq 2$ and $\kappa_2(G[W] - e) = 1$, the endnodes v_1 and v_2 of e lie in two different two-node-connected components. Furthermore, there is a $[v_1, v_2]$-path in $G[W] - e$ that touches all two-node-connected components containing nodes of type 2 and all articulation nodes of type 2. Let w_1 be an articulation node of $G[W] - e$ so that the component of $(G[W] - e) - w_1$ containing one endnode v_1 of e also contains some node of type 2 (possibly $= v_1$), and so that the node set S_1 of this component is as small as possible with respect to this property. Similarly, find an articulation node w_2 and a component of $(G[W] - e) - w_2$ with node set S_2 containing v_2 and some node of type 2, so that $|S_2|$ is as small as possible. S_1 and S_2 are disjoint. Since G satisfies (1.2) there has to be some edge $f \in \delta(W)$ leaving S_1 and an edge $g \in \delta(W)$ leaving S_2. Since $V \backslash W$ has two nodes of type 2, condition (d_4) ensures that f and g may be chosen without common endpoint, such that in $G[V \backslash W]$ there is no articulation point separating the two endpoints of f and g from some node of type 2. Because S_1 and S_2 are connected in $G[W] - e$ by a path touching all two-node-connected components of $G[W] - e$ containing nodes of type 2, the set $C := C_{f,g} \backslash \{e\}$ defined above is feasible. Therefore, $b_e = 0$. So we have proved that $b = \gamma a$. Since F_b cannot define a facet if $b \leq 0$, we have $\gamma > 0$. So $x(\delta(W)) \geq 2$ and $b^T x \geq \beta$ define the same facet $F_a = F_b$. □

The following theorem characterizes which of the **node-cut inequalities** (1.1)(ii) define facets for 2NCON$(G;r)$.

THEOREM 3.3. *Let (G, r) satisfy (1.2) and let a node $z \in V$, and a set $W \subseteq V \backslash \{z\}$ with $\emptyset \neq W \neq V \backslash \{z\}$ and $r(W) = 2$, $r(V \backslash (W \cup \{z\})) = 2$ be given. Denote by V_i, $i = 1, 2$, the set of nodes in V of type at least i, and let $\bar{W} := V \backslash (W \cup \{z\})$.*

The node-cut inequality $x(\delta_{G-z}(W)) \geq 1$ defines a facet of 2NCON$(G;r)$ if and only if

(a) *$G[W]$ is connected;*

(b) *$\lambda(G[W \cup \{z\}], V_1 \cup \{z\}) \geq 2$;*

(c) *$\lambda_2(G[W]) \geq 2$;*

(d) *$u \in W$ is an articulation node of $G[W \cup \{z\}]$ separating two nodes of $V_2 \cup \{z\}$, and $S \subseteq W$ is the node set of a component of $G[W \cup \{z\}] - u$ with $r(S) = 2$; then $[W \backslash S : W] = \emptyset$ and $r(W \backslash (S \cup \{u\})) \leq 1$;*

(e) *the following situation does not occur: there are an edge $e \in E(W)$ and two nodes $w_1, w_2 \in W$ (not necessarily distinct), so that, for $i = 1, 2$, $(G[W \cup \{z\}] - e) - w_i$ has a component with node set S_i where $S_i \subseteq W$ and $r(S_i) = 2$; furthermore, $S_1 \cap S_2 = \emptyset$ and $e \in [S_1 : S_2]$ (see Fig. 3.2 for an illustration; dashed lines denote nonexisting edges);*

(f) *conditions (a), $\cdots$, (e) also hold when W is replaced by $\bar{W}$.*

Proof. The proof is analogous to the proof of Theorem 3.2. □

4. Lifting theorems. We now present conditions under which valid inequalities (respectively, facets) for the 2ECON and 2NCON polytopes on a graph $\hat{G}$ can be lifted

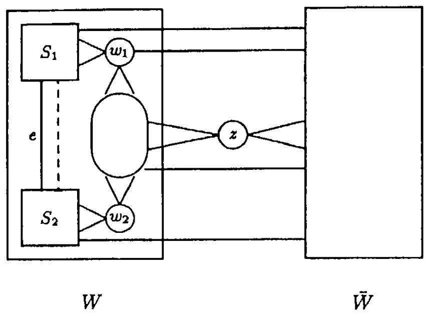


FIG. 3.2

to valid inequalities (respectively, facets) for higher-dimensional 2ECON and 2NCON polytopes on a graph G that contains $\hat{G}$ as a subgraph. These results simplify the proofs to be presented in the next sections.

Some of the results can be treated for the 2ECON and 2NCON polytopes simultaneously. Thus we introduce a slightly more general network design model that combines edge- **and** node-connectivity features. Let $G = (V, E)$ be a graph, $r \in \{0, 1, 2\}^V$ be the vector of connectivity types, and Z be some subset of V. (In this section we do not necessarily assume that $(G; r)$ satisfies (1.2).) We define the **2CON(Z)** problem to be the network design problem where between each pair of distinct nodes s and t at least $\min(r_s, r_t)$ edge-disjoint paths are required that have no node of $Z \setminus \{s, t\}$ in common. Note that for $Z = \emptyset$ only edge-disjoint paths are required, so in this case 2CON(Z) is the 2ECON problem. For $Z = V$ this is the 2NCON problem. This general model is introduced only for technical reasons. Throughout the rest of this paper we will be interested only in the cases $Z = \emptyset$ and $Z = V$.

The 2CON(Z) problem can be formulated as an integer linear program in the following way:

$$\min \sum_{ij \in E} c_{ij} x_{ij}$$

subject to

(4.1)

(i) $x(\delta(W)) \geq \text{con}(W)$	for all $W \subseteq V, \emptyset \neq W \neq V$;
(ii) $x(\delta_{G-z}(W)) \geq 1$	for all $z \in Z$, and for all $W \subseteq V \setminus \{z\}$, $\emptyset \neq W \neq V \setminus \{z\}$ with $r(W) = 2$ and $r(V \setminus (W \cup \{z\})) = 2$;
(iii) $0 \leq x_{ij} \leq 1$	for all $ij \in E$;
(iv) x_{ij} integral	for all $ij \in E$.

The polytope **2CON**$(G; Z; r)$ is then defined as the convex hull of all $x \in \mathbb{R}^E$ that satisfy (i),$\cdots$,(iv) of (4.1). As mentioned above, 2CON$(G; \emptyset; r)$ = 2ECON$(G; r)$ and 2CON$(G; V; r)$ = 2NCON$(G; r)$.

The polytope 2CON$(G; Z; r)$ is not necessarily full-dimensional. In the later sections we only apply the results of this section in the case $\dim(\text{2CON}(G; Z; r)) = |E|$. So we can avoid treating all the technicalities arising in the low-dimensional case, and we thus assume throughout this section that 2CON$(G; Z; r)$ has dimension $|E|$.

In Lemma 4.2 we derive valid inequalities for the 2CON$(G; Z; r)$ polytope from valid inequalities for the 2CON$(G/W; Z; r)$ polytope.

LEMMA 4.2. *Consider the* 2CON$(G; Z; r)$ *polytope and let* $W \subseteq V \setminus Z$. *Let the node* w *in* G/W *that represents node set* W *inherit its connectivity type from* W *by* $r_w := \text{con}(W)$. *If* $\hat{a}^T x \geq b$ *is a valid inequality for* 2CON$(G/W; Z; r)$ *where* $W \subseteq V \setminus Z$, *then* $a^T x \geq b$ *is valid for* 2CON$(G; Z; r)$, *where*

$$a_e = \hat{a}_e \quad \text{for } e \in E(G/W) \quad \text{and} \quad a_e = 0 \quad \text{for } e \in E(W).$$

We say that $a^T x \geq b$ *is obtained from* $\hat{a}^T x \geq b$ *by expanding* w *to* W.

Proof. We first remark that the lemma is true for any of the inequalities (i), (ii), or (iii) of (4.1). The reason is that the expansion of any inequality of type (i), (ii), or (iii) is again of the same type. (Note that since $Z \cap W = \emptyset$, a shrunk node w can never be chosen as a node z in a node-cut inequality (ii).)

Since 2CON$(G/W; Z; r)$ is the convex hull of the integral solutions of (i), (ii), and (iii) of (4.1) every valid inequality for 2CON$(G/W; Z; r)$ can be obtained by taking nonnegative combinations of the inequalities (i), (ii), and (iii), rounding the left- and right-hand sides up and recursively repeating this procedure. (This so-called cutting plane proof is described in [Chv]; see also [Sch, Cor. 23.2b].) It is easy to see that such a validity proof of $\hat{a}^T x \geq b$ from the inequalities (i), (ii), and (iii) of (4.1) for 2CON$(G/W; Z; r)$ yields a validity proof of $a^T x \geq b$ by applying the same nonnegative combinations and rounding operations to the associated expanded inequalities, since combining and rounding expanded inequalities produces expanded inequalities. □

The following lemma gives a technical condition for an "expanded" inequality derived by Lemma 4.2 to define a facet of 2CON$(G; Z; r)$.

LEMMA 4.3. *Consider the* 2CON(Z) *problem given by* (G, r) *satisfying* (1.2)(i) *and let* W *be a subset of* $V \setminus Z$ *with* $G[W]$ *connected. Let the node* w *in* G/W *(representing the node set* W*) inherit its connectivity type from* W *by* $r_w := \text{con}(W)$. *(Note that* $(G/W, r)$ *does not necessarily satisfy* (1.2)(i).*)*

Let the inequality $\hat{a}^T x \geq \alpha > 0$ *be valid for* 2CON$(G/W; Z; r)$ *and let* $a^T x \geq \alpha$ *be the inequality (valid for* 2CON$(G; Z; r)$*) obtained from* $\hat{a}^T x \geq \alpha > 0$ *by expanding node* $w \notin Z$ *to* $W \subseteq V(G)$.

Denote by F_a *the face of the polytope* $P :=$ 2CON$(G; Z; r)$ *induced by* $a^T x \geq \alpha$ *and by* $F_{\hat{a}}$ *the face of the polytope* $\hat{P} :=$ 2CON$(G/W; Z; r)$ *induced by* $\hat{a}^T x \geq \alpha$.

F_a *is a facet of* P *if and only if the following conditions hold:*

(a) *For any* $e \in E(W)$ *there exists a set* $\hat{C} \subseteq E(G/W)$ *with* $\chi^{\hat{C}} \in F_{\hat{a}}$ *so that the incidence vector of* $\hat{C} \cup E(W) \setminus \{e\}$ *lies in* F_a.

(b) *There exist* $s := |E(G/W)|$ *sets* $C_i \in E(G/W)$, $i = 1, \cdots, s$, *with* $\chi^{C_i} \in F_{\hat{a}}$ *so that*

(b$_1$) $\chi^{C_i \cup E(W)} \in F_a$, *and*

(b$_2$) *the* χ^{C_i} *are affinely independent.*

Proof. Suppose that (a) and (b) are satisfied. We want to show that F_a is a facet. (Note that (b) implies that $F_{\hat{a}}$ is a facet.) Let $b^T x \geq \beta$ define a facet F_b of P that contains F_a. For any $e \in E(W)$, condition (a) provides a set C with $e \notin C$ and $\chi^C \in F_b$. Therefore, $\chi^{C \cup \{e\}} \in F_b$ and $b_e = 0$ also. Condition (b) implies that vector b has to satisfy $b^T \chi^{C_i \cup E(W)} = \beta$ for $i = 1, \cdots, s$. Since we have just proved b^T to be $(0, \hat{b}^T)$ with $\hat{b} \in \mathbb{R}^{E(G/W)}$, this means $\hat{b}^T \chi^{C_i} = \beta$ for $i = 1, \cdots, s$. The affine independence of the $\dim(\hat{P})$ vectors χ^{C_i} implies that $\hat{b}^T x \geq \beta$ defines a facet of $\hat{P}$, necessarily the same as $F_{\hat{a}}$. Therefore, $(\hat{b}^T, \beta)$ is a positive multiple of $(\hat{a}^T, \alpha)$, and (b^T, β) is a positive multiple of (a^T, α). So F_a defines a facet.

On the other hand, if we know that $a^Tx \geq \alpha$ defines a facet of P, then for each $e \in E(W)$ there must exist a set C with $e \notin C$ and $\chi^C \in F_a$; otherwise $F_a \subseteq \{x \in P : x_e = 1\}$. If we shrink node set W to node w in the graph defined by C we arrive at a set $\hat{C} := C \backslash E(W)$ whose incidence vector satisfies $\hat{a}^Tx = \alpha$ and is feasible for $\hat{P}$ because $\text{con}(w) = \text{con}(W)$. w may be an articulation node in C, but this does not matter because $w \notin Z$. The set $\hat{C} \cup E(W)\backslash\{e\}$ is feasible for P because it contains the feasible set C and because $G[W]$ is connected. Therefore, (a) is satisfied.

If F_a is a facet of P, there exist $|E|$ affinely independent vertices χ^{C_i} in F_a, where $C_i \subseteq E$ is feasible for P, for $i = 1, \cdots, |E|$. We set $x_i := \chi^{C_i}$ for $i = 1, \cdots, |E|$. There must be a subset of $|E(G/W)|$ affinely independent vectors among $\hat{x}_1, \cdots, \hat{x}_{|E|}$, where $\hat{x}_i$ is derived from x_i for $i = 1, \cdots, |E|$ by deleting the components $e \in E(W)$. The $\hat{x}_i$, for $i = 1, \cdots, |E|$, are feasible for $\hat{P}$ because the deletion of the $E(W)$-components of a vector x in $\{0,1\}^E$ is equivalent to the contraction of W in the subgraph (V, F^x) of G defined by x. So the affinely independent subset of $\{\hat{x}_i : i = 1, \cdots, |E|\}$ satisfies (b_1) and (b_2). □

The conditions of Lemma 4.3 can be used to derive some conditions on $G[W]$ that are of a more graph-theoretical nature and sufficient for an "expanded" inequality to define a facet of $2\text{ECON}(G;r)$.

LEMMA 4.4. *Consider the* 2ECON *problem given by* (G,r) *satisfying* (1.2)(i). *Let* $W \subseteq V$ *with* $\emptyset \neq W \neq V$, *and let* w (*of type* $\text{con}(W)$) *be the node of* G/W *representing* W. *Consider an inequality* $\hat{a}^Tx \geq b$ *that is facet-defining for the polytope* $2\text{ECON}(G/W;r)$, *and consider the inequality* $a^Tx \geq b$ (*valid for* $2\text{ECON}(G;r)$) *derived from* $\hat{a}^Tx \geq b$ *by expanding node* w *to* W.

If $G[W]$ *is* $\max\{2, r(W)+1\}$*-edge-connected then* $a^Tx \geq b$ *defines a facet of* $2\text{ECON}(G;r)$.

Proof. Let F_a and $F_{\hat{a}}$ be defined as in Lemma 4.3. We will check conditions (a) and (b) of Lemma 4.3. The connectivity conditions on $G[W]$ imply that for any $e \in E(W)$ and $\hat{C} \subseteq E(G/W)$ that is feasible for $2\text{ECON}(G/W;r)$, the sets $\hat{C} \cup E(W)\backslash\{e\}$ and $\hat{C} \cup E(W)$ are feasible for $2\text{ECON}(G;r)$. Since $F_{\hat{a}}$ is a facet, there are enough affinely independent $\chi^{\hat{C}}$ to satisfy condition (b) of Lemma 4.3. □

Usually much weaker conditions on the edge-connectivity of $G[W]$ are already sufficient for an expanded inequality $a^Tx \geq b$ to define a facet of $2\text{ECON}(G;r)$. But this leads to further technicalities concerning assumptions on the structure of the graph and properties of $\hat{a}^Tx \geq b$; see, for instance, Theorem 3.2(a) and (b).

The next lemma gives a sufficient condition for an expanded inequality to define a facet of $2\text{NCON}(G;r)$. (Note that any inequality valid for $2\text{CON}(G;Z;r)$ is also valid for $2\text{NCON}(G;r)$.)

LEMMA 4.5. *Consider the* 2NCON *problem given by* (G,r) *satisfying* (1.2)(i). *Let* $Z \subseteq V$ *and* $W \subseteq V\backslash Z$ *with* $\emptyset \neq W \neq V$ *and* $r(W) = 1$, *and let* w (*of type* 1) *be the node of* G/W *representing* W. *Consider an inequality* $\hat{a}^Tx \geq b$ *that is valid for the polytope* $2\text{CON}(G/W;Z;r)$ *and facet-defining for* $2\text{NCON}(G/W;r)$.

If $G[W]$ *is two-edge-connected, then the inequality* $a^Tx \geq b$ *derived from* $\hat{a}^Tx \geq b$ *by expanding node* w *to* W *defines a facet of* $2\text{NCON}(G;r)$.

Proof. First, $a^Tx \geq b$ is valid for $2\text{CON}(G;Z;r)$ by Lemma 4.2 and hence for $2\text{NCON}(G;r)$. To prove that $a^Tx \geq b$ also defines a facet of $2\text{NCON}(G;r)$, we apply Lemma 4.3 with $\hat{P} := 2\text{CON}(G/W;V;r)$ and $P := 2\text{CON}(G;V;r)$. Conditions (a) and (b) are still sufficient for $a^Tx \geq b$ to define a facet of P, because of the fact that $w \notin Z$ is not used in the sufficiency part of Lemma 4.3. So we have to check (a) and (b) of Lemma 4.3, which is easy. □

Our final lifting result presents conditions under which a valid inequality for

$2\text{CON}(G;Z;r)$ on a complete graph $G = (V,E)$ can be extended to the graph with a new node w of type at least 1 added, along with all of the edges incident between w and V; we denote such a graph by $G + w$.

LEMMA 4.6. *Consider the* $2\text{CON}(Z)$ *problem given by a graph* $\hat{G}$ *and node types* r *satisfying* (1.2)(i), *where* $\hat{G} = (\hat{V}, \hat{E})$ *is a complete graph with two parallel edges* uv *for each* $u, v \in V$ *with* $u \neq v$. *Let* $\hat{a}^Tx \geq \hat{b}$ *be a valid inequality for* $2\text{CON}(\hat{G};Z;r)$ *with* $\hat{a} \geq 0$. *Let* $W \subseteq \hat{V}\backslash Z$ *be a node set with* $r(W) = 2$ *and* α *some nonnegative value so that either* $\hat{a}_e = \alpha$ *for all* $e \in \hat{E}(W)$ *or* $|W| = 1$.

We define an inequality $a^Tx \geq b$ *on the graph* $G := \hat{G} + w$ *with* $r_w := 1$ *by setting*

$$
\begin{aligned}
b &:= \hat{b} + \alpha, \\
a_e &:= \hat{a}_e \quad \textit{for all } e \in \hat{E}, \\
a_{uw} &:= \alpha \quad \textit{for all } u \in W, \\
a_{uw} &:= \beta_u := \max\{\alpha, \max\{\hat{a}_{uv},\ v \in W\}\} \quad \textit{for all } u \notin W.
\end{aligned}
$$

If $a_{uw} + a_{vw} \geq \alpha + a_{uv}$ *for all distinct nodes* $u, v \notin W$, *then* $a^Tx \geq b$ *is valid for* $2\text{CON}(G;Z;r)$.

Note that in Lemma 4.6 the restriction to complete graphs is no restriction at all, because any inequality valid for $2\text{CON}(G;Z;r)$, where G is a complete graph, is also valid if G is replaced by some subgraph (V,F). In the lemma we need completeness of $\hat{G}$ to compute the β_u correctly. Also, we can restrict ourselves without loss of generality to $\hat{a} \geq 0$ because it is easy to see that any inequality $\hat{a}^Tx \geq \hat{b}$ that is facet-defining for $2\text{CON}(\hat{G};Z;r)$ (except $-x_e \geq -1$) has nonnegative coefficients.

Proof. We will assume that $r_w = 1$, because validity of an inequality in this case implies its validity if $r_w = 2$. Assume further that $a^Tx \geq b$ is not valid, i.e., that there exists an edge set C that is feasible for $2\text{CON}(G;Z;r)$ and does not satisfy $a^T\chi^C \geq b$.

(1) If there is an edge $uw \in C$ with $u \in W$, we contract node set $\{u,w\}$ to node u. The resulting subgraph of $\hat{G}$ with edge set $C\backslash\{uw\}$ is feasible for $2\text{CON}(\hat{G};Z;r)$. Note that $\hat{a}_{vu} \leq a_{vw}$ for all $v \in V$. Therefore, $\hat{a}^T\chi^{C/\{u,w\}} \leq a^T\chi^C - a_{uw} < b - \alpha = \hat{b}$. But then $\hat{a}^Tx \geq \hat{b}$ is not valid for $2\text{CON}(\hat{G};Z;r)$, a contradiction.

If C uses no edge of $[W:\{w\}] \cap C$, we will show how to replace C by some set C' containing an edge in $[W:\{w\}]$, such that $a^T\chi^{C'} \leq a^T\chi^C < b$. So we can apply the argumentation above to derive a contradiction to the validity of $\hat{a}^Tx \geq \hat{b}$.

(2) Suppose all edges of $\delta(w) \cap C$ were bridges of (V,C). Since w is connected to W in C, there must be a bridge uw of (V,C), which separates w from some node $v \in W$. The set $C' := (C\backslash\{uw\}) \cup \{vw\}$ is feasible for $2\text{CON}(G;Z;r)$ and contains an edge of $[W:\{w\}]$. Moreover, $a^T\chi^{C'} = a^T\chi^C - a_{uw} + a_{vw} \leq a^T\chi^C < b$.

Now suppose there are edges of $\delta(w) \cap C$ that are not bridges of (V,C). Define U as the set of nodes that are incident to nonbridges of C (the so-called two-connected part of C). U must contain all nodes of type 2. By assumption, w belongs to U.

(3) Assume that w is not an articulation node of (V,C) disconnecting two nodes of type 2. The case that w is an articulation node is treated separately. Since $r(W) = 2$, there exists a node $s \in W$ of type 2, and since s and w are in U, there exist two edge-disjoint $[s,w]$-paths in C that do not coincide in any node $z \in Z$. Let $u, v \in U$ be the nodes adjacent to w on these two paths. If $u = v$, we eliminate one of the two wv-edges. This can be done without destroying feasibility of C because w is not an articulation node separating two nodes of type 2 and because $r_w = 1$. Also, $a^T\chi^C$ does not increase with this operation, since $a \geq 0$. Now we are either in the case that $\delta(w) \cap C$ contains only bridges of (V,C) (proceed with part (2) of the proof), or we construct two other $[s,w]$-paths that lead to different nodes $u \neq v$.

Now we show that $C' := (C \backslash \{uw, vw\}) \cup \{ws, uv\}$ is also feasible. Clearly C' is connected, so we only have to check for bridges and articulation nodes. Suppose that e is a bridge of (V, C') separating two nodes of type 2. In $C' \backslash \{e\}$, node s is connected to u and v by at least one of the two edge-disjoint paths and edge uv. If $e \neq ws$, all four nodes w, s, u, v lie in the same component (S, F) of $(V, C' \backslash \{e\})$. Since $C' \cap \delta(S) = C \cap \delta(S)$, edge e is also a bridge in (V, C) separating two nodes of type 2. So e must be ws. But $(V, C) - w$ is a subgraph of $(V, C') - w$, and w is not an articulation node of (V, C). Now suppose that $z \in Z$ is an articulation node that separates two nodes of type 2 in (V, C') but not in (V, C). $z = s$ need not be considered, because $s \notin Z$. The remaining cases lead to a contradiction similar to the case in which e is a bridge. So C' is feasible for $2\mathrm{CON}(\hat{G}; Z; r)$. C' also satisfies $a^T\chi^{C'} < b$ because $a^T\chi^{C'} = a^T\chi^C - (a_{uw} + a_{vw} - a_{uv}) + a_{ws} \leq a^T\chi^C - \alpha + \alpha$.

(4) The last remaining case in our transformation of C is the case in which w is an articulation node of (V, C) separating two nodes of type 2. Let $u, v \in U$ be nodes adjacent to w lying on different sides of $(V, C) - w$. Replace C by $C' = (C \backslash \{uw, vw\}) \cup \{uv\}$. C' is feasible, $a^T\chi^{C'} \leq a^T\chi^C$, and $(V, C') - w$ contains one component less than $(V, C) - w$. Ultimately, we reach a set C' where w does not separate any nodes of type 2, and we can apply one of the earlier cases. Thus, we have proved that if $a^Tx \geq b$ is not valid for $2\mathrm{CON}(G; Z; r)$, then also $\hat{a}^Tx \geq \hat{b}$ is not valid for $2\mathrm{CON}(\hat{G}; Z; r)$. □

The next theorem gives sufficient conditions for $a^Tx \geq b$ to define a facet.

THEOREM 4.7. *Consider the situation in Lemma 4.6, where we have an inequality $\hat{a}^Tx \geq \hat{b}$ valid for $2\mathrm{CON}(K_n; Z; r)$, and where (K_n, r) satisfies (1.2)(i). Let W, w with $r_w = 1$, $\alpha \geq 0$, be defined as in Lemma 4.6. Let $a^Tx \geq b$ be the inequality derived from $\hat{a}^Tx \geq \hat{b}$ by the formula in Lemma 4.6. Furthermore, let $G = (V, E)$ be a subgraph of K_{n+1} with $n + 1$ nodes, and define $\hat{G}$ as $G - w$.*

Then, for any $Z' \supseteq Z$, the inequality $a^Tx \geq b$ defines a facet of $2\mathrm{CON}(G; Z'; r)$ if the following conditions hold:

(a) *$\hat{a}^Tx \geq \hat{b}$ defines a facet of $2\mathrm{CON}(\hat{G}; Z'; r)$;*

(b) *for all $u \notin W$ with $uw \in E$ and $a_{uw} > \alpha$ there exists a node $v \in W$ with $a_{uw} = a_{uv}$ and $uv, vw \in E$;*

(c) *there exist two distinct nodes u, v with $a_{uv} = a_{vw} = a_{uw} = \alpha$, and uv, vw, $uw \in E$;*

(d) *all nodes u with $uw \in E$ and $a_{uw} = \alpha$ have type at least 1.*

Proof. First, note that $a^Tx \geq b$ is valid for $2\mathrm{CON}(G; Z'; r)$ because it is valid for $2\mathrm{CON}(G; Z; r)$ by Lemma 4.6. We prove the theorem by exhibiting $|E|$ affinely independent vectors in $F_a := \{x \in 2\mathrm{CON}(G, Z'; r) \mid a^Tx = b\}$. Let $F_{\hat{a}} := \{x \in 2\mathrm{CON}(\hat{G}, Z'; r) \mid \hat{a}^Tx = \hat{b}\}$.

Let $f = vw$ be an edge $a_f = \alpha$ and $r_v \geq 1$. This edge exists by condition (d). Then any set $\hat{C} \subseteq \hat{E}$ feasible for $F_{\hat{a}}$ can be enlarged to a set $C \subseteq E$ feasible for F_a by adding f. This way we can create $|\hat{E}|$ affinely independent vectors in F_a. Now we want to exhibit $|\delta(w)| - 1$ sets C_k with $\chi^{C_k} \in F_a$. The C_k are characterized by the fact that C_k contains an edge $e_k \in \delta(w) \backslash \{f\}$ that is not contained in any of the previous C_i, $i = 1, \cdots, k-1$. This fact implies that each χ^{C_k} will be affinely independent from all χ^{C_i}, $i = 1, \cdots, k-1$, and the $|\hat{E}|$ vectors already found in F_a. The C_k are constructed as follows. Order the edges in $\delta(w) \backslash \{f\}$ as e_1, e_2, etc., by increasing a_e-values, so that the edges e with $a_e = \alpha$ come first. Now, for an edge $e_k \in \delta(w) \backslash \{f\}$ with $a(e_k) = \alpha$ and $e_k = v_kw$, let $\hat{C} \subseteq \hat{E}$ be a set with incidence vector in $F_{\hat{a}}$ and set $C_k := \hat{C} \cup \{e_k\}$. For an edge $e_k = uw \in \delta(w) \backslash \{f\}$ with $a_{uw} > \alpha$, let uv be the edge with $a_{uv} = a_{uw}$, existing by condition (b). Let $\hat{C} \subseteq \hat{E}$ be a set with $\chi^{\hat{C}} \in F_{\hat{a}}$, which uses uv. The incidence vector of the set $C_k := (\hat{C} \backslash \{uv\}) \cup \{uw, wv\}$ is then in F_a. Therefore, we can create the proposed $|\delta(w)| - 1$ sets C_k.

We still have to exhibit one more vector in F_a that is independent from all the others. By condition (c) there is a triangle $uv, vw, uw \in E$ with $a_e = \alpha$ for all triangle edges. As before, there is a set $\hat{C}$ with $\chi^{\hat{C}} \in F_{\hat{a}}$ and $uv \in \hat{C}$. The incidence vector of $C := (C \backslash \{uv\}) \cup \{uw, wv\}$ is affinely independent of all the others already found in F_a, because these all satisfy $x([\{w\} : S]) = 1$ for $S = \{u \mid uw \in E, a_{uw} = \alpha\}$. So we have found $|E|$ affinely independent vectors in F_a. □

5. Partition inequalities for 2ECON and 2NCON. In this section we introduce a class of inequalities that is motivated by the partition inequalities for the connected subgraph polytope (see [GM]), and that generalizes cut inequalities.

DEFINITION 5.1. Let $G = (V, E)$ be a graph and $r \in \{0, 1, 2\}^V$. We call a collection $W_1, \cdots, W_p$ of subsets of V a **proper partition** of V if

- $W_i \neq \emptyset$, $i = 1, \cdots, p$,
- $W_i \cap W_j = \emptyset$, $1 \leq i < j \leq p$,
- $\bigcup_{i=1}^p W_i = V$,
- $r(W_i) \geq 1$, $i = 1, \cdots, p$.

The **partition inequality** induced by a proper partition $W_1, \cdots, W_p$ is given by

$$\frac{1}{2} \sum_{i=1}^{p} x(\delta(W_i)) \geq \begin{cases} p, & \text{if } r(W_i) = 2 \text{ for at least two node sets } W_i, \\ p - 1 & \text{otherwise.} \end{cases} \tag{5.2}$$

See Fig. 5.1 for an illustration of a partition inequality with four node sets $W_1, \cdots, W_4$. Here and in all following illustrations, node sets W with $r(W) = 2$ are depicted by big squares, and node sets W with $r(W) = 1$ are depicted by big circles. Nodes of types 2 and 1 are depicted by small squares and circles, respectively.

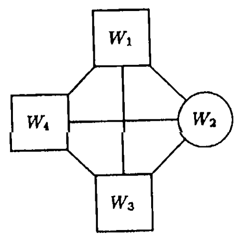


FIG. 5.1

The following observation follows immediately from the definition.

Remark 5.3. Any partition inequality (5.2) induced by a proper partition is valid for $2\mathrm{ECON}(G; r)$ and $2\mathrm{NCON}(G; r)$.

Note that a partition inequality induced by a proper partition with $p = 2$ is nothing but a cut inequality $x(\delta(W)) \geq \mathrm{con}(W)$. The next observation indicates that we cannot expect to obtain a useful characterization of those partition inequalities that define facets.

Remark 5.4. Checking whether a partition inequality supports 2ECON$(G;r)$ or 2NCON$(G;r)$ is NP-complete.

Proof. The problem is obviously in NP. Let $G = (V, E)$ be a graph and $r_v = 2$ for all $v \in V$. Then the sets $\{w\}$, $w \in V$, form a proper partition of V and the induced partition inequality reads $x(E) \geq |V|$. Thus there is a point in 2ECON$(G;r)$ or 2NCON$(G;r)$ that satisfies $x(E) \geq |V|$ with equality if and only if G is Hamiltonian. This implies the remark. □

We will now derive a sufficient condition for a partition inequality to define a facet.

THEOREM 5.5. *Let $G = (V, E)$ be a graph, $r \in \{0,1,2\}^V$, and let $W_1, \cdots, W_p$, $p \geq 3$, be a proper partition (see (5.1)). Let $\hat{G} = (\hat{V}, \hat{E})$ be the graph $G/W_1/\cdots/W_p$ where the W_i are shrunk to nodes w_i of connectivity type $\hat{r}(w_i) :=$ con(W_i) for $i = 1, \cdots, p$. Let V_1 be the set of nodes of type at least 1 in $\hat{G}$ and V_2 the set of nodes of type 2 in $\hat{G}$. The partition inequality* (5.2) *defines a facet of* 2ECON$(G;r)$ *if*

(a) $\kappa_2(\hat{G}) \geq 3$ *and* $\kappa_1(\hat{G}) \geq 2$;

(b) *in $\hat{G}$ every node of type 2 is adjacent to some node of type 1;*

(c) $\hat{G}[V_1 \backslash V_2]$ *is connected;*

(d) $\hat{G}[V_2]$ *is Hamiltonian;*

(e) $G[W_i]$ *is $(r(W_i) + 1)$-edge-connected for $i = 1, \cdots, p$.*

Proof. The partition inequality (5.2) can be written as $x(\hat{E}) \geq t$ for the graph $\hat{G}$, where $t = |\hat{V}|$ or $|\hat{V}| - 1$, according to whether $\hat{G}$ contains nodes of type 2 or not. If $\hat{G}$ contains only nodes of type 1 and $\hat{G}$ is two-node-connected (see condition (a)), the partition inequality $x(\hat{E}) \geq |\hat{V}| - 1$ defines a facet of the polytope of connected subgraphs of $\hat{G}$. This was shown in [GM]. By our lifting Lemma 4.4 and Theorem 5.5(e), we can expand all nodes w_i of $\hat{G}$ successively to node sets W_i, and thus obtain a facet of the 2ECON$(G;r)$ polytope.

Suppose that $\hat{G}$ contains nodes of type 2. First we show that conditions (a)–(d) are sufficient for $x(\hat{E}) \geq |\hat{V}|$ to define a facet F of 2NCON$(\hat{G};\hat{r})$. We do this by constructing $|\hat{E}|$ affinely independent vectors in F.

Take some Hamiltonian cycle C of $\hat{G}[V_2]$. Let $G' = (V', E')$ denote the graph $\hat{G}/V_2$. Any tree T spanning the nodes of the shrunk graph G' may be added to C, thus creating a set whose incidence vector is in F. There are at least $|E'|$ such trees with affinely independent incidence vectors. This is true because the inequality $x(E') \geq |V'| - 1$ defines a facet of the polytope of connected subgraphs of G' (see [GM]) if G' is two-node-connected. Note that G' is two-node-connected because $\hat{G}$ is two-node-connected by condition (a), and because $\hat{G}[V_1 \backslash V_2]$ is connected. Hence we can find $|E'|$ affinely independent vectors of the form $\chi^{C \cup T}$ in F.

Now take some cycle edge $e \in C$. With the help of conditions (b) and (c) we can construct a cycle not using e and spanning all nodes of type 2 in $\hat{G}$ by using the path $C \backslash \{e\}$ and a path in G'. This new cycle may be augmented by some trees to a feasible set with incidence vector in F. This vector is affinely independent of all other vectors constructed so far because these all satisfied $x_e = 1$. By applying this argument for each cycle edge successively we can construct $|E'| + |C|$ affinely independent vectors in F.

For any other edge $e = uv \in \hat{E}(V_2) \backslash C$ we want to construct a cycle spanning all nodes of type 2, and using e but no other edge of $\hat{E}(V_2) \backslash C$. This can be done easily by starting with u, going to v, running in some direction along the cycle C to the neighbor of u, taking a path in E' to the neighbor of v on C that is not already visited, and running along the other half of C to the starting point u. This cycle can be augmented to a set with incidence vector in F. This vector is affinely independent

of all the others exhibited so far because all of those satisfied $x_e = 0$. So we have $|E'| + |\hat{E}(V_2)| = |\hat{E}|$ affinely independent vectors in F. This proves that $x(\hat{E}) \geq |\hat{V}|$ defines a facet of 2NCON$(\hat{G};\hat{r})$, and hence of 2ECON$(\hat{G};\hat{r})$.

Our partition inequality (5.2) in G can be obtained from $x(\hat{E}) \geq |\hat{V}|$ by expanding successively the nodes w_i to node sets W_i according to the definition in Lemma 4.2. Because of Theorem 5.5(e) we can apply Lemma 4.4, and thus the partition inequality (5.2) defines a facet of 2ECON$(G;r)$. □

Remark 5.6. The partition inequality (5.2) defines a facet of 2NCON$(G;r)$ if G is complete and no node set W_i with $r(W_i) = 2$ contains exactly two nodes.

Proof. The proof is the same as for Theorem 5.5 except that in the end we use Lemma 4.3 instead of Lemma 4.4. □

In view of Theorem 3.2 (d), which gives quite complicated necessary and sufficient conditions for a cut inequality $x(\delta(W)) \geq 2$ to define a facet of 2NCON$(G;r)$, we did not further investigate necessary and sufficient conditions for a partition inequality (with $p > 2$) to define a facet of 2NCON$(G;r)$.

The next theorem shows which of the sufficient conditions of Theorem 5.5 are actually necessary for a partition inequality to define a facet of 2ECON$(G;r)$.

THEOREM 5.7. *Let (G, r) and a proper partition $W_1, \cdots, W_p$ with $p \geq 3$ be given, and let $\hat{G}$ and $\hat{r}$ be defined as in Theorem* 5.5. *The partition inequality* (5.2) *defines a facet of* 2ECON$(G;r)$ *only if*

(a) *conditions* (a) *and* (b) *of Theorem* 5.5 *are satisfied;*

(b) *$\hat{G}$ contains nodes of type 2; then $\hat{G}$ contains a cycle C containing all nodes of type 2;*

(c) $G[W_i]$ *is connected for $i = 1, \cdots, p$;*

(d) $\lambda_1(G[W_i]) \geq 2$ *for $i = 1, \cdots, p$.*

Proof. The necessity of condition (a) of Theorem 5.5 is easily seen. Suppose that condition (b) of Theorem 5.5 is violated and that $|W_i| = 1$ for all $i = 1, \cdots, p$. This implies that $\hat{G} = G$ and that there is a node $v \in V_2$ that is adjacent only to other nodes of type 2 in G. Then any set C that is feasible for 2ECON$(G;r)$ with $|C| = |V|$ has to use exactly two edges of $\delta(v)$. Otherwise C would have at least two cycles, and this would imply $|C| \geq |V| + 1$. So the face induced by the partition inequality $x(E) \geq |V|$ is contained in the face induced by $x(\delta(v)) \geq 2$. But since the partition was supposed to consist of at least three sets, the partition inequality does not define the same face as the cut inequality. If $|W_i| \geq 2$ for some i and if Theorem 5.5(b) is violated, one can argue similarly.

The necessity of conditions (b) and (c) of Theorem 5.7 is easily seen. As for (d), suppose that some $G[W_i]$ contains a bridge e so that $G[W_i] - e$ has two components with node sets U and W, with $r(U) \geq 1$ and $r(W) \geq 1$. In this case our partition inequality can be written as the sum of $x_e \leq 1$ and another partition inequality can be defined by the same partition as above, except that W_i is replaced by U and W. □

From an algorithmic point of view, Remark 5.4 seems to be bad news. Even worse, the separation problem for partition inequalities is NP-complete (see [GMS]). But in practice, using heuristic separation routines, the class of partition inequalities proved to be very useful in the cutting plane algorithm presented in [GMS]. Usually, partitions with a small number of node sets were used there, and for small p it is quite likely that—in our real-world examples—a partition inequality supports 2NCON$(G;r)$.

Moreover, checking the conditions of Theorem 5.7 is easy, and this helps to convert one partition inequality into another partition inequality that induces a face of higher dimension than the first one. Indeed, finding cutting planes that induce faces of dimension as high as possible is of importance in cutting plane algorithms. We noticed this clearly in our computational experiments (see [GMS]).

6. Node-partition inequalities. We now generalize node-cut inequalities to "node-partition inequalities" in the same way as we generalized cut inequalities to partition inequalities in the previous section. These new inequalities will only be valid for 2NCON$(G;r)$, but, in general, not for 2ECON$(G;r)$.

Let $G=(V,E)$ be a graph and $r\in\{0,1,2\}^V$. Let $z\in V$ and let $W_1,\cdots,W_p$ be a proper partition (see (5.1)) of $V\backslash\{z\}$ such that at least two node sets W_i contain nodes of type 2. The following **node-partition inequality** induced by z and $W_1,\cdots,W_p$ is given by

$$\text{(6.1)}\qquad \frac{1}{2}\Big(\sum_{i\in I_2} x(\delta_{G-z}(W_i)) + \sum_{i\in I_1} x(\delta_G(W_i)) + x([\{z\}:\cup_{i\in I_1}W_i])\Big)\geq p-1,$$

where $I_k:=\{i\in\{1,\cdots,p\}\mid r(W_i)=k\}$, $k=1,2$.

In Fig. 6.1 a node partition inequality is depicted with three sets W_i with $r(W_i)=2$ and two sets W_i with $r(W_i)=1$. Edges with coefficient 0 are depicted by dashed lines; edges with coefficient 1 are depicted by solid lines.

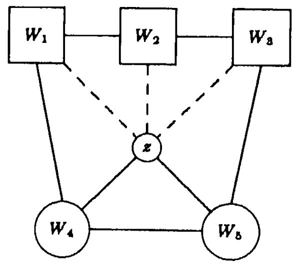


FIG. 6.1

THEOREM 6.2. *The node partition inequality* (6.1) *is valid for* 2NCON$(G;r)$.

Proof. Consider first a node partition inequality induced by a node z and the partition consisting of all node sets $\{v\}$, $v\in V\backslash\{z\}$. Suppose also that $r_v=2$ for all $v\in V\backslash\{z\}$. This node partition inequality, $x(E(V\backslash\{z\}))\geq |V|-2$, is valid, because after deletion of a node z the rest of the network should still connect all nodes $v\in V\backslash\{z\}$. Nodes of type 1 can be added successively to $V\backslash\{z\}$ by applying Lemma 4.6 with $Z:=\{z\}$, $W:=V\backslash\{z\}$, and $\alpha:=1$. With Lemma 4.2 all nodes $v\in V\backslash\{z\}$ can be expanded to node sets. In this way, every node partition inequality is proved to be valid. □

The following theorem gives a sufficient condition for the node partition inequality (6.1) to define a facet of 2NCON$(G;r)$.

THEOREM 6.3. *Consider a node partition inequality* (6.1) *induced by* $W_1,\cdots,W_p$. *Let* $\hat{G}$ *denote the graph* $(G-z)/W_1/\cdots/W_p$, *where the* W_i *are shrunk to nodes* w_i, $i=1,\cdots,p$. *Let* I_1 *and* I_2 *be defined as in* (6.1). *The node partition inequality* $a^Tx\geq p-1$ *defines a facet of* 2NCON$(G;r)$ *if*

(a) $\hat{G}$ *is two-node-connected;*

(b) $G[W_i\cup\{z\}]-e$ *is two-node-connected for all edges* $e\in G[W_i\cup\{z\}]$ *and for all* $i\in I_2$;

(c) $G[W_i]$ *is two-edge-connected for all* $i\in I_1$.

Proof. Let conditions (a), (b), and (c) be satisfied. We will show how to construct $|E|$ affinely independent vectors in the face defined by the node partition inequality (6.1).

Let E' be the set of all edges whose coefficients in $a^Tx\geq p-1$ are 0. By condition (a), the graph $\hat{G}=(\hat{V},\hat{E})$ contains $|\hat{E}|$ spanning trees whose incidence vectors are affinely independent (see Theorem 4.10 in [GM]). Any such tree T of $\hat{G}$ can be augmented by E' to a feasible set $C\subseteq E$ for 2NCON$(G;r)$. Feasibility can be shown as follows. For any two nodes u, $v\in G[W_i\cup\{z\}]$ (where $i\in I_2$) there exist, by condition (b), two node-disjoint paths in (V,C). For $u\in W_i$ and $v\in W_j$ (where $i,j\in I_2$ and $i\neq j$), we construct the following two node-disjoint paths. In $(V,C)-z$, there exists a path from some node $u'\in W_i$ to some node $v'\in W_j$. Let u' and v' have the property that u' is the last node of W_i and v' is the first node of W_j encountered on this path. Since $G[W_i\cup\{z\}]$ is two-node-connected, it contains a $[u,u']$-path and a $[u,z]$-path, which do not have a node except u in common. (If $u=u'$, we only need one path, namely, the $[u,z]$-path.) Similarly, $G[W_j\cup\{z\}]$ contains a $[v,v']$-path and a $[v,z]$-path, which are node-disjoint. From these paths we can construct two node-disjoint $[u,v]$-paths in (V,C). So for all pairs u, v of nodes we can construct the required number of paths in (V,C), which proves feasibility of C. Feasibility is preserved even when some single $e\in E'$ is deleted from C. □

The connectivity conditions given in (b) imply that if $r(W_i)=2$ for one of the node sets in the partition, then W_i must contain at least three nodes. This is not at all necessary. In fact, there exist facet-defining node-partition inequalities where all node sets in the partition contain exactly one node. Because we need it later on, we state this result as a lemma.

LEMMA 6.4. *Consider a* 2NCON *problem given by* (G,r) *and let* z *be some node of* G. *We suppose that* $G=(V,E)$ *is a graph with at least four nodes and* $r_v=2$ *for all* $v\in V\backslash\{z\}$. *The node-partition inequality* (6.1) *induced by the partition of* $V\backslash\{z\}$ *into node sets* $\{w\}$ *for* $w\in V\backslash\{z\}$ *defines a facet of* 2NCON$(G;r)$ *if* z *is adjacent to every node in* G.

Proof. This can be proved by considering trees of $G-z$ augmented by certain edges of $\delta(z)$. Note that by (1.2)(iii) the graph G is supposed to be three-node-connected, so there exists a sufficient number of trees of $G-z$. □

Some **necessary** conditions for node-partition inequalities to define facets of 2NCON$(G;r)$ can be derived from Theorem 3.3 for node-cut inequalities.

THEOREM 6.5. *The node-partition inequality* (6.1) *defines a facet of* 2NCON $(G;r)$ *only if*

(a) $G[W_i]$ *is connected for all* $i\in I$;

(b) $\lambda_1(G[W_i\cup\{z\}])\geq 2$ *for all* $i\in I_2$;

(c) $\lambda_1(G[W_i])\geq 2$ *for all* $i\in I_1$;

(d) $\lambda_2(G[W_i])\geq 2$ *for* $i=1,\cdots,p$.

Proof. The proof is obvious. □

The connectivity conditions given in Theorem 6.5 can be easily checked and are of some practical use in cutting plane algorithms to derive faces of higher dimension.

7. Lifted two-cover inequalities. The motivation for introducing and studying the next class of inequalities derives from the fact that the two-matching inequalities play an important role in solving the traveling salesman problem; see [GP] and [PG].

The roots, however, are Edmonds's results for b-matching polyhedra (see [E]) since a certain (complemented) b-matching problem provides an interesting relaxation of the ECON problem.

Let $G = (V, E)$ be a graph and $r \in \{0,1,2\}^V$. Every incidence vector of a feasible solution $F \subseteq E$ to the 2ECON problem satisfies the "star inequalities" $x(\delta(v)) \geq r_v$ for all $v \in V$. And therefore the incidence vector of the complement $\bar{F} := E \setminus F$ of a feasible solution F to the 2ECON problem satisfies

$$(7.1) \qquad \begin{array}{ll} y(\delta(v)) \leq b_v := |\delta(v)| - r_v & \text{for all } v \in V, \\ 0 \leq y_e \leq 1 & \text{for all } e \in E. \end{array}$$

The convex hull of the integral solutions of (7.1) is the **1**-capacitated b-matching polytope of G, where $b = (b_v)_{v \in V} \in \mathbb{Z}^V$. Let us set, for $W \subseteq V$, $b(W) := \sum_{v \in W} b_v$. Edmonds [E] has shown that a complete linear description of the **1**-capacitated b-matching polytope of G is given by the following system

$$(7.2) \qquad \begin{array}{ll} y(\delta(v)) \leq b_v & \text{for all } v \in V, \\ y(E(H)) + y(\bar{T}) \leq \frac{b(H)+|\bar{T}|-1}{2} & \text{for all } H \subseteq V \text{ and all } \bar{T} \subseteq \delta(H) \text{ such that } b(H) + |\bar{T}| \text{ is odd}, \\ 0 \leq y_e \leq 1 & \text{for all } e \in E. \end{array}$$

Since $\chi^F = \mathbf{1} - \chi^{\bar{F}}$, we can derive from (7.2) that every incidence vector of a feasible solution to the 2ECON problem satisfies

$$(7.3) \qquad x(E(H)) + x(\delta(H) \setminus T) \geq \frac{\sum_{v \in H} r_v - |T| + 1}{2}$$

for all $H \subseteq V$ and all $T \subseteq \delta(H)$ such that $\sum_{v \in H} r_v - |T|$ is odd. In the transformation from (7.2) to (7.3) we have also set $T := \delta(H \setminus \bar{T})$.

Since $r \in \{0,1,2\}^V$, we call inequalities (7.3) **two-cover inequalities**. Note that it follows from Edmonds's result that the two-cover inequalities (7.3) plus the trivial constraints $0 \leq x_e \leq 1$, for all $e \in E$, give a complete description of the two-cover polytope, which is the convex hull of all incidence vectors of edge sets $F \subseteq E$ such that each node $v \in V$ has at least r incident edges.

From the two-cover inequalities we derive a larger class of inequalities as follows. Let $G = (V, E)$ be a graph and $r \in \{0,1,2\}^V$. Let $H \neq V$ be a node set, called the **handle**, and $T \subseteq \delta(H)$ an edge set. For each $e \in T$ we denote by T_e the set of the two endnodes of e. The sets T_e, $e \in T$, are called **teeth**. For simplicity we also call the edges $e \in T$ teeth in this section. If an edge $e \in T$ is parallel to some edge $f \in T$, we count T_e and T_f as two sets, even if $T_e = T_f$. Let $H_1, \cdots, H_p$, $p \geq 3$ be a partition of H into nonempty disjoint node sets such that

— $r(H_i) \geq 1$ for $i = 1, \cdots, p$;
— $r(H_i) = 2$ if H_i is intersected by some tooth, $i = 1, \cdots, p$;
— no more than two teeth may intersect any H_i, $i = 1, \cdots, p$;
— $|T| \geq 3$ and odd.

We call

$$(7.4) \qquad x(E(H)) - \sum_{i=1}^{p} x(E(H_i)) + x(\delta(H)) - x(T) \geq p - \left\lfloor \frac{|T|}{2} \right\rfloor$$

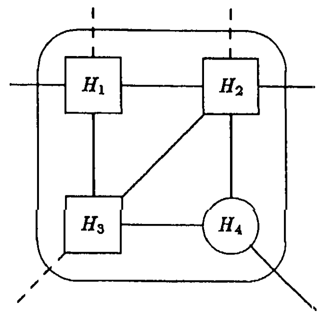


FIG. 7.1

the **lifted two-cover inequality**.

In Fig. 7.1 a handle with four node sets $H_1, \cdots, H_4$ and three teeth (drawn with dashed lines) is depicted, inducing a lifted two-cover inequality with right-hand side 3.

For the case in which $r_v = 2$ for all $v \in V$, Mahjoub [M] has found the same class of inequalities (and calls them "odd wheel inequalities" using a quite different notation).

Note that a lifted two-cover inequality coincides with a two-cover inequality (7.3), if $|H_i| = 1$ and $r(H_i) = 2$ for $i = 1, \cdots, p$. Note also that with each additional H_i with $|H_i| = 1$ and $r(H_i) = 1$ the right-hand side of a lifted two-cover inequality increases by 1, whereas the right-hand side of a two-cover inequality increases only by $\frac{1}{2}$ (on the average). This implies that two-cover inequalities do not support 2ECON$(G;r)$ if H contains nodes of type 1. Nevertheless, if the right-hand side of a two-cover inequality is increased appropriately, these inequalities define facets of 2ECON$(G;r)$ in many cases. This odd behavior may be explained by the fact that in an edge-minimal solution to the two-cover problem the nodes of type 1 may lie on matching edges, whereas in an edge-minimal solution to the 2ECON problem they are connected by a tree (or they lie on some cycle).

Also, the class of lifted two-cover inequalities is not very useful for the 2NCON problem, because they do not define facets in the case in which G is a complete graph and some H_i with incident tooth contains more than one node. In §8 we will introduce a class of inequalities for 2NCON$(G;r)$ that contain the lifted two-cover inequalities with $|H_i| = 1$ as a subclass, and define facets for complete G and $|H_i| \geq 1$. But these will be valid only for 2NCON$(G;r)$.

As in the previous sections, we will derive validity and facet results of lifted two-cover inequalities from validity and facet results of a special class of lifted two-cover inequalities, namely those with $|H_i| = 1$.

THEOREM 7.5. *A lifted two-cover inequality* (7.4) *is valid for* 2ECON$(G;r)$ (*and hence for* 2NCON$(G;r)$).

Proof. First, assume that $|H_i| = 1$ and that all nodes in the handle are of type 2. In this case, we have a two-cover inequality that is valid for the polytope of two-covers, hence for 2ECON$(G;r)$. It is also easy to prove validity in this case by summing up the inequalities:

$$\begin{aligned} x(\delta(v)) &\geq 2 \quad \text{for all } v \in H,\\ -x_e &\geq -1 \quad \text{for all } e \in T,\\ x_e &\geq 0 \quad \text{for all } e \in \delta(H)\backslash T, \end{aligned}$$

dividing the result by 2 and rounding the right-hand side up.

Our next step is induction over the number of nodes of type 1 in the handle (but still $|H_i| = 1$). This can be done with the help of Lemma 4.6 by setting $W := H$, $\alpha := 1$, and w as the new node of type 1. The result is a new valid inequality of the form (7.4).

Finally, using Lemma 4.2, we expand the nodes in the handle successively to node sets H_i with coefficients 0 inside H_i, to derive all inequalities of the form (7.4).

Note that when lifting a node w with incident $wv \in T$ to node set W, only one edge of $[W : \{v\}]$ gets coefficient 0; all others have coefficient 1 in the lifted two-cover inequality. (If all edges in $[W : \{v\}]$ had coefficient 0, the obtained inequality would not be valid for $2\text{ECON}(G;r)$, but it would be valid for $2\text{NCON}(G;r)$; see Theorem 8.2.) □

Lifted two-cover inequalities are also valid if we allow an even number of teeth. But they cannot define facets in this case, as can be seen easily.

The following theorem gives a necessary and sufficient condition for a special subclass of lifted two-cover inequalities to define facets of $2\text{ECON}(G;r)$.

THEOREM 7.6. (a) *A lifted two-cover inequality* (7.4) *with* $|H_i| = 1$ *for* $i = 1,\cdots,p$, $|H| = |T|(= p)$, *and* $|V\backslash H| = 1$, *defines a facet of* $2\text{ECON}(G;r)$ *if and only if* $G[H]$ *is hypomatchable (i.e., for each node* $v \in H$ *there is a matching of* $G[H]$ *that is incident to all nodes in* H *except* v).

(b) *Let* $G[H]$ *be a complete graph. Then any lifted two-cover inequality* (7.4) *with* $|H_i| = 1$ *for* $i = 1,\cdots,p$, $|H| \geq |T| \geq 3$, *and* $|V\backslash H| = 1$, *defines a facet of* $2\text{ECON}(G;r)$.

Proof. Let F be the face induced by the lifted two-cover inequality in question.

(a) Let F be contained in a facet F_b induced by some inequality $b^T x \geq \beta$. We want to prove that b is a scalar multiple of the left-hand side of the lifted two-cover inequality.

Pick some $v \in H$. Any perfect matching M of $G[H\backslash\{v\}]$ can be enlarged to a set C whose incidence vector is in the face F by adding some edge $e \in \delta(v)\backslash T$ along with all tooth edges. The resulting set $C \cup \{e\} \cup T$ is two-edge connected and $C \cup \{e\}$ contains $|H| - \lfloor\frac{|T|}{2}\rfloor = \lceil\frac{|T|}{2}\rceil$ edges. By varying $e \in \delta(v)\backslash T$, we achieve $b_e = \alpha_v$ for all $e \in \delta(v)\backslash T$ and some constant α_v. Since $G[H]$ is connected, α_v is the same for all nodes $v \in H$. ($G[H]$ is connected if $G[H]$ is hypomatchable.)

Now we prove that $b_e = 0$ for $e \in T$. Let u be the node in H incident to e and let v be some node in H adjacent to u. The incidence vector of a perfect matching of $G[H\backslash\{v\}]$ plus edge uv plus $T\backslash\{e\}$ lies in F_b. Since adding edge e does not change the right-hand side, we know $b_e = 0$. Therefore, our lifted two-cover inequality defines a facet.

Suppose now that $E(H)$ is not hypomatchable. With the help of Tutte's theorem we will find a separation of $E(H) \cup (\delta(H)\backslash T)$ into edge sets $E_1, E_2, \cdots, E_s$ so that $x(E_i) \geq k_i$ is valid for $2\text{ECON}(G;r)$ and the sum of the k_i is at least $|H| - \lfloor\frac{|T|}{2}\rfloor$. This is done as follows: since for some node $v \in H$ the graph $G[H\backslash\{v\}]$ has no perfect matching, by Tutte's theorem there exists a node set $S \subseteq H\backslash\{v\}$ so that the number of odd components $c_o(G[H\backslash\{v\}] - S)$ of $G[H\backslash\{v\}] - S$ is strictly larger than $|S|$. Since $H\backslash\{v\}$ is an even node set ($|H| = |T|$ is odd), either the number of odd components of $G[H\backslash\{v\}] - S$ is odd and $|S|$ is odd, or both numbers are even. In any case, we know that $c_o(G[H\backslash\{v\}] - S) - |S| \geq 2$. So $c_o(G[H] - (S \cup \{v\}))$, which is the same as $c_o(G[H\backslash\{v\}] - S)$, is still larger than $|S \cup \{v\}|$. For the sake of simplicity, we will rename $S := S \cup \{v\}$. Let H_i be the node set of the ith (odd or even) component of $G - S$. Let T_i denote the subset of teeth incident to H_i and let E_i denote the edge set $E(H_i) \cup (\delta(H_i)\backslash T_i)$. The T_i constitute a partition of $T\backslash\delta(S)$, and the E_i constitute a partition of the edge set $(E(H) - E(S)) \cup (\delta(H)\backslash T)$.

$$x(E_i) \geq k_i := |H_i| - \left\lfloor\frac{|T_i|}{2}\right\rfloor$$

is a valid lifted two-cover inequality (this is valid also for an even number of teeth!). If we take the sum of these inequalities plus the nonnegativity constraints for $e \in E(S)$, we achieve $x(E(H)) + x(\delta(H)\backslash T) \geq k$, where k is the sum of the k_i. In the right-hand side, the $|H_i|$ sum up to $|H| - |S|$, and the $\lfloor|T_i|/2\rfloor$ sum up to $\frac{1}{2}(|T| - |S| - c_o(G[H] - S))$, so the k_i sum up to

$$|H| - \frac{|T|}{2} + \frac{1}{2}\big(c_o(G[H] - S) - |S|\big) \geq |H| - \left\lfloor\frac{|T|}{2}\right\rfloor.$$

Therefore, our lifted two-cover inequality can be written as the sum of at least two other valid inequalities; hence it does not define a facet.

(b) Assume first that H contains only nodes of type 2 (with or without incident teeth). If nodes of type 2 without incident teeth are allowed in the handle, the restriction of a feasible set C whose incidence vector is in F to the edge set $E(H) \cup \delta(H)\backslash T$ is something more complicated than a matching with additional edge. It is rather a collection of node-disjoint paths between pairs of nodes with incident teeth plus one additional path connecting the last node with incident tooth to $V\backslash H$ or to some other path. More exactly, if we set $\bar{r}_v := 2$ minus the number of incident teeth for $v \in H$ and $\bar{r}_z := 0$ for the node $z \notin H$, then $C\backslash T$ meets each node $v \in V$ with exactly $\bar{r}_v$ edges, except for one node that is met by $\bar{r}_v + 1$ edges. $C\backslash T$ is a near-perfect $\bar{r}$-cover of $E(H) \cup (\delta(H)\backslash T)$ (so to speak). To see this, add the $\bar{r}_v$, divide by two, and compare this with the right-hand side of the lifted two-cover inequality. (But not every near-perfect $\bar{r}$-cover of $E\backslash T$ plus T defines a feasible set, as there might be some node-disjoint cycles.)

Since the structure of the feasible sets with incidence vector in F is somewhat unwieldy, we switch to complete graphs. Let F_b be a facet containing F, induced by some valid inequality $b^T x \geq \beta$. First we show $b_e = \alpha_v$ for all $e \in \delta(v)\backslash T$ and all $v \in H$. The connectedness of $G[H]$ will imply that the α_v are the same for all $v \in V$. If $v \in H$ has an incident tooth, construct node-disjoint paths in $G[H]$ connecting pairs of nodes with incident teeth and meeting all nodes of H except v. To this set add any edge $e \in \delta(v)\backslash T$ and T. Since we have freedom in choosing e, we can prove $b_e = \alpha_v$ for all nodes $v \in H$ with incident teeth. If $v \in H$ has no incident tooth, construct node-disjoint paths in $E(H)$ between pairs of nodes with incident teeth plus one path (node-disjoint from all others) between v and the last leftover node with an incident tooth. These paths should meet all nodes in H. Call this collection of paths C. As before, we can add any edge of $\delta(v)$ (except the path edge $C \cap \delta(v)$), plus all teeth, and get a set with incidence vector in F_a. This proves $b_e = \alpha_v$ for all $e \in \delta(v)\backslash C$. But we can construct another set C' the same way as before, only this time it uses a different edge of $\delta(v)$. So we have $b_e = \gamma_v$ for all $e \in \delta(v)\backslash C'$ and some value γ_v. Since $\delta(v)\backslash T$ contains at least three edges, all edges in $\delta(v)\backslash T$ have the same a_e-value

$\gamma_v = \alpha_v$. Proving $b_e = 0$ for the teeth $e \in T$ is easy, so we have that b is identical to the lifted two-cover inequality; therefore it defines a facet.

If H contains nodes of type 1, we use Theorem 4.7 for induction on the number of nodes of type 1 in H in the same way as we used Lemma 4.6 for proving validity of the lifted two-cover inequality. □

Usually the feasible sets of 2ECON$(G;r)$ whose incidence vectors satisfy the lifted two-cover inequality with equality are not feasible for 2NCON$(G;r)$ if $V\backslash H$ consists of only one node, because this node may be an articulation node. But if $V\backslash H$ has sufficiently high connectivity, (7.4) may define a facet of 2NCON$(G;r)$.

Remark 7.7. A lifted two-cover inequality (7.4) with $|H_i| = 1$ for $i = 1, \cdots, p$, $|H| = |T| (= p)$, defines a facet of 2NCON$(G;r)$ if $G[H]$ is hypomatchable, $G[V\backslash H]$ is three-edge–connected, no two teeth are incident to the same node (in $V\backslash H$), and no parallel edges exist.

Proof. The proof is analogous to the proof of Theorem 7.6. □

But usually, as the following remark shows, lifted two-cover inequalities do not define facets for 2NCON$(G;r)$ as soon as $|H_i| \geq 2$ for some H_i with an incident tooth.

Remark 7.8. A lifted two-cover inequality does not define a facet of 2NCON$(G;r)$ if there is a node set H_i and a node $v \in V\backslash H$ so that $[\{v\} : H_i]$ contains a tooth and a nontooth.

(This is the case especially if G is complete and some H_i with incident tooth contains at least two nodes.)

Proof. It can be shown that a feasible set $C \subseteq E$ with 2NCON$(G;r)$ that satisfies such a lifted two-cover inequality with equality never uses the nontooth in $[\{v\} : H_i]$. □

But for the 2ECON problem we can use our lifting lemmas of §4 to derive sufficient conditions for a lifted two-cover inequality with general H_i to define a facet of 2ECON$(G;r)$.

THEOREM 7.9. *Given a lifted two-cover inequality* (7.4), *we will denote by $\hat{G}$ the graph $G/H_1/\cdots/H_p$.*

(a) *If $\hat{G}[H]$ is hypomatchable (in the case $p = |T|$) or complete (in the case $p > |T|$), if the $G[H_i]$ for $i = 1, \cdots, p$ are $(r(H_i)+1)$-edge-connected, and if $G[V\backslash H]$ is $\max\{2, r(V\backslash H)+1\}$-edge-connected, a lifted two-cover inequality defines a facet of* 2ECON$(G;r)$.

(b) *If the lifted two-cover inequality is facet-inducing, then $\hat{G}[H]$ and $G[H_i]$ are connected for $i = 1, \cdots, p$, and $\lambda_1(G[H_i]) \geq 1$ for $i = 1, \cdots, p$. In fact, one can always find $H_1, \cdots, H_p$ with $\lambda_1(G[H_i]) \geq 2$ for $i = 1, \cdots, p$ that induce the lifted two-cover inequality in question.*

Proof. (a) Theorem 7.6 proves the lifted two-cover inequality to be facet-defining for 2ECON$(\hat{G};r)$. With Lemma 4.4 we can lift this result to 2ECON$(G;r)$.

(b) It is easy to see that the $G[H_i]$ must be connected for all $i = 1, \cdots, p$.

If $\hat{G}[H]$ is not connected, we can split the handle H into two handles H' and H'' to derive two lifted two-cover inequalities whose sum gives the old one. So the old one cannot define a facet.

It remains to show that we can find $H_1, \cdots, H_p$ with $\lambda_1(G[H_i]) \geq 2$ for $i = 1, \cdots, p$ that induce our lifted two-cover inequality.

If H_i has no incident tooth and $\lambda_1(G[H_i]) = 1$, then our lifted two-cover inequality can be written as the sum of another lifted two-cover inequality where H_i is split into at least two other sets plus one constraint $x_e \leq 1$. The same argument is possible if H_i has an incident tooth and $\lambda_2(G[H_i]) = 1$. So in these cases our lifted two-cover inequality cannot define a facet.

It remains to check the case in which H_i has an incident tooth e and $\lambda_1(G[H_i]) = 1$. In this case $G[H_i]$ has a bridge f so that $G[H_i] - f$ decomposes into two components U and W with $r(U) = 1$ and $r(W) \geq 1$. The interesting case is the one where the tooth e is incident to U, because there we cannot simply split H_i into U and W to derive a stronger lifted two-cover inequality. But we can replace H_i by $H_i\backslash U$, H by $H\backslash U$, and the tooth e by the bridge f to derive another lifted two-cover inequality of the same form as the old one. By repeating this procedure of reducing H_i, we can assume that $\lambda_1(G[H_i]) \geq 2$ for all $i = 1, \cdots, p$. □

8. Comb inequalities.

The following constraints were motivated, on the one hand, by the comb inequalities for the traveling salesman problem (see [GP]), and on the other hand, they were motivated by the fact that the lifted two-cover inequalities do not generally define facets for the 2NCON problem (see Remark 7.8). We wanted to find a facet containing the face induced by a lifted two-cover inequality in the case in which G is a complete graph and the H_i contain more than one node.

The class of inequalities we came up with in this case are valid for 2NCON$(G;r)$, but not generally for 2ECON$(G;r)$. We will call this class comb inequalities for 2NCON$(G;r)$. These inequalities allow a further generalization using the concept of clique trees. But we will not discuss this here.

Let $H, T_1, \cdots, T_t$ be subsets of V and let $z_i \in T_i\backslash H$, $i = 1, \cdots, t$, be not necessarily distinct nodes (H is called the **handle**, the sets $T_1, \cdots, T_t$ are the **teeth**, and the $z_1, \cdots, z_t$ the **special nodes**) that satisfy the following conditions:

— $t \geq 3$ and odd;

— two teeth have at most one node in common;

— if $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j = \{z_i\} = \{z_j\}$;

— each tooth T_i intersects the handle H in exactly one node; we denote this node by t_i for $i = 1, \cdots, t$;

— $r_{t_i} = 2$ for $i = 1, \cdots, t$;

— $r_v \geq 1$ for all $v \in H \cup (\cup_{i=1}^{t}(T_i\backslash\{z_i\}))$.

We denote by V_2 the set of nodes of type 2 in G. The **special comb inequality** is given by

$$
\begin{aligned}
&x(E(H)) + x(\delta(H)) + \textstyle\sum_{i=1}^{t} x(E(T_i)) \\
&\quad + \textstyle\sum_{i=1}^{t} x([T_i\backslash(H \cup \{z_i\}) : V\backslash T_i]) - \sum_{i=1}^{t} x([\{t_i\} : T_i]) \\
&\quad - \textstyle\sum_{i=1}^{t} x([\{z_i\} : T_i \cap V_2]) \geq |H| + \sum_{i=1}^{t}(|T_i| - 2) - \lfloor \frac{t}{2} \rfloor .
\end{aligned}
\tag{8.1}
$$

The (general) **comb inequality** is derived from the special comb inequality (8.1) by expanding all nodes $w \in H$ that are not in $\{z_1, \cdots, z_t\}$ to node sets W (see Lemma 4.2). Figure 8.1 gives an illustration of a comb inequality with a handle H consisting of four node sets and three teeth T_i, $i = 1, \cdots, 3$, which has right-hand side 6. Edges with coefficient 0 are drawn with dashed lines, edges with coefficient 1 with solid lines, and edges with coefficient 2 with bold lines.

We note that the comb inequality becomes a lifted two-cover inequality with sets $H_i := \{t_i\}$ if $|T_i| = 2$ and $|E(T_i)| = 1$.

We will prove validity and facet results only for special comb inequalities. With the help of Lemmas 4.2–4.5 one can easily derive validity and facet results for general comb inequalities.

THEOREM 8.2. *A comb inequality* (8.1) *is valid for* 2CON$(G;Z;r)$ *with* $Z = \{z_1, z_2, \cdots, z_t\}$, *and hence it is valid for* 2NCON$(G;r)$.

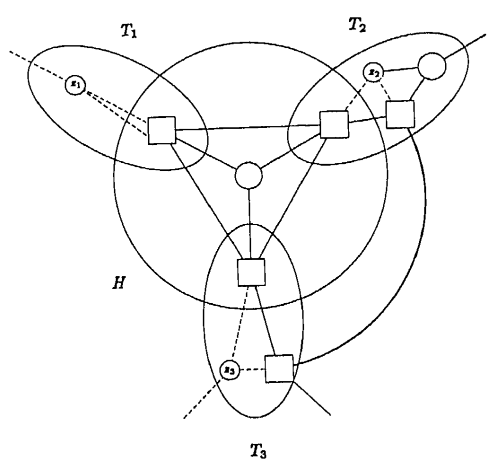


FIG. 8.1

Proof. Assume that all nodes in $(H \cup (\bigcup_{i=1}^t T_i)) \setminus \{z_1, \cdots, z_t\}$ are nodes of type 2. Then the left-hand side of the comb inequality (8.1) can be written as $\frac{1}{2}$ times the sum of the following inequalities with subsequent rounding:

(1) for all $v \in H\setminus(\bigcup_{i=1}^t T_i)$: the cut inequality $x(\delta(v)) \geq 2$;

(2) for all teeth T_i: the node-partition inequality (6.1) induced by z_i and the partition $\{V\setminus T_i, \{v\}$ for all $v \in T_i\setminus\{z_i\}\}$; the right-hand side is $|T_i| - 1$;

(3) for all teeth T_i with $r(T_i\setminus\{t_i, z_i\}) = 2$: the node-partition inequality (6.1) induced by z_i and the partition $\{V\setminus(T_i\setminus\{t_i\}), \{v\}$ for all $v \in T_i\setminus\{z_i, t_i\}\}$; the right-hand side is $|T_i| - 2$;

(4) for all teeth T_i with $r(T_i\setminus\{t_i, z_i\}) = 1$: the partition inequality (5.2) induced by the partition $\{(V\setminus T_i) \cup \{t_i, z_i\}, \{v\}$ for $v \in T_i\setminus\{t_i, z_i\}\}$; its right-hand side is $|T_i| - 2$;

(5) some nonnegativity constraints.

The sum of $(\frac{1}{2})$ times the right-hand sides of these inequalities is:

$$\begin{aligned} |H\setminus (\textstyle\bigcup_{i=1}^t T_i)| &+ \textstyle\sum_{i=1}^t (|T_i| - \frac{3}{2}) \\ &= |H| - t + \textstyle\sum_{i=1}^t |T_i| - \frac{3t}{2} \\ &= |H| + \textstyle\sum_{i=1}^t (|T_i| - 2) - \frac{t}{2}. \end{aligned}$$

Rounding this up gives the right-hand side of (8.1) exactly.

If the handle contains nodes of type 1, we apply Lemma 4.6 inductively with $W := H$ and $\alpha := 1$. If a tooth T_i contains nodes of type 1, we apply Lemma 4.6 with $W := T_i$ and $\alpha := 1$; this is done in the same way as in the validity proof for node-partition inequalities. □

Note that the comb inequality (8.1) is also valid if the number of teeth t is even. But in this case it does not define a facet, as it can be written as the sum of a comb inequality and node-partition inequality (or a nonnegativity constraint).

Note also that if $H \cup (\bigcup_{i=1}^t T_i) = V$ and $z_1 = z_2 = \cdots = z_t$ and $|T_i| = 2$ for all i, the special comb inequality with right-hand side $|H| - \lfloor \frac{t}{2} \rfloor$ may degenerate into a node-partition inequality with higher right-hand side, namely, $|H| - 1$. In this case the special comb inequality cannot define a facet.

THEOREM 8.3. *The special comb inequality* (8.1) *defines a facet of* 2NCON$(G; r)$ *if $r_v = 2$ for all nodes $v \in V$, if the z_i are all distinct, and if G is the complete graph minus all edges with coefficient* 2 *in* (8.1).

Proof. The restriction to nodes of type 2 has only technical reasons, mainly because of Lemma 6.4. The restriction to edges with coefficients 0 and 1 is also introduced only for technical reasons. Once we have proved an inequality to define a facet only on a subset of edges of the complete graph, it is easy to prove it to be facet-defining on the complete graph.

Let F be the face induced by the comb inequality in question, and let F be contained in the face F_b induced by some valid inequality $b^T x \geq \beta$. First we prove $b_e = \alpha_i$ for all edges in $E(T_i) \cup [\{t_i\} : H]$ with coefficient 1 and some α_i. We do this (without loss of generality) for tooth T_1. Suppose that $|T_1| \geq 3$. (For "small" teeth that consist of only one edge, the following proof has to be modified somewhat.) Construct a collection P of node-disjoint paths in $G[H]$ between pairs of nodes t_j, say, between t_2 and t_3, t_4 and t_5, etc. Those paths should meet every node in H except t_1. To this collection of paths P, we may add certain trees in the teeth T_i that are constructed as follows:

(1) For T_1 we take any feasible edge set whose incidence vector lies in the face of 2NCON$(G/(V\setminus T_1); r)$ induced by a certain node-partition inequality on T_1, namely, the one with node $z = z_1$ and node sets $\{v\}$ for all nodes v in T_1 and $\{w\}$ for the shrunk node standing for $V\setminus T_1$ (cf. (2) used in the validity proof in Theorem 8.2). These sets are trees on $T_1\setminus\{z_1\}$ plus certain edges of $\delta(z_1)$ plus some edge leading from T_1 to $V\setminus T_1$. Note also that the face of 2NCON$(G/(V\setminus T_1); r)$ induced by the node-partition inequality is a facet by Lemma 6.4.

(2) For T_i with $i \neq 1$, we take any feasible edge set whose incidence vector lies in the face of 2NCON$(G/((V\setminus T_i) \cup \{t_i\}); r)$ induced by (3) or (4) of the validity proof in Theorem 8.2. These objects are mainly trees on $T_i\setminus\{z_i, t_i\}$ plus certain edges in $[\{z_i\} : T_i]$. If $|T_i| = 2$, we just take the edge of tooth T_i.

Finally, we add all edges $z_i z_j$ to this construction.

We claim that this combination of paths in $G[H]$ and trees of T_i is feasible. This can be easily checked. Secondly, we claim that its incidence vector lies in the face induced by the comb inequality; this is true because all inequalities used in the validity proof of the comb inequality are satisfied with equality except one.

Since we have some freedom in the choice of the "tree" in T_1, and we know that the node-partition inequality used for the construction of these "trees" defines a facet of 2NCON$(G/(V\setminus T_i); r)$, we know that $b_e = \alpha_1$ for all nonzero edges in this node-partition inequality, and $b_e = 0$ for all zero edges e. This can be done for all teeth T_i in the same way as shown for tooth T_1.

Now we prove that all edges inside the handle have the same b_e-value. This value must be the same as α_1, α_2, etc. Thus, we know that all edges with coefficient 1 in the comb inequality have the same b_e-value and all edges e with coefficient 0 in the comb inequality have $b_e = 0$.

To prove $b_e = \alpha_v$ for all $e \in \delta_{G[H]}(v)$ and $v \in H$, we just vary our construction of paths in the beginning. This is done in exactly the same way as in the proof of Theorem 7.6(b). To give an example: If $v \in H \setminus T$, then we construct paths between t_1 and v, t_2 and t_3, etc. that are all node-disjoint. These paths should meet all nodes in $G[H]$. In addition to this collection P of paths we construct trees in T_i according to point (2) above. Now we can add any edge $e \in \delta(v) \cap E(H)$ not already in some path to achieve a feasible solution whose incidence vector lies in the face F_a. So $b_e = b_f$ for all $e, f \in (\delta(v) \setminus P) \cap E(H)$. To prove $b_e = b_f$ for all $e, f \in \delta(v) \cap E(H)$, we just choose a collection of paths using another edge of $\delta(v)$.

It is easy to prove that the b_e-value for the e of zero coefficient in the comb inequality is also 0.

So inequality $b^T x \geq \beta$ is identical to the comb inequality (8.1) except for scalar multiplication. Therefore, it defines a facet of $2\text{NCON}(G; r)$. □

The question naturally arises whether there are also "comb" inequalities valid for $2\text{ECON}(G; r)$. We know of such a class, but the validity proof is somewhat ugly. In such a "comb" inequality we have two types of teeth: "simple" teeth consisting of only one edge with coefficient 0, and "large" teeth T with coefficients 0 on edges in $T \setminus H$, and coefficients 1 on the edges leading from $T \setminus H$ to $T \cap H$ and to the "outside." The edges in the handle have coefficients 2. This seems to be more symmetric, and therefore, in a way, nicer than the comb inequalities (8.1).

Also, some other odds and ends of inequalities that do not fit into any of the presented classes are known to us. Some of these are published in Stoer's dissertation [S].

9. Computational results. The theory presented here for the 2ECON and 2NCON polytopes was developed in order to solve problems of the type and size that arise in the design of survivable telephone networks in fiber optic technology. The idea was to design and implement a cutting plane algorithm that uses the inequalities introduced above.

As mentioned before, it unfortunately turned out that—except for the cut and node-cut inequalities—the separation problem for all other classes of inequalities presented here is NP-hard. This means that we can use these classes of inequalities only heuristically. We had to make an experimental investigation of the relative benefit of running various heuristics that determine, for a given point y, an inequality of some class of valid inequalities that is violated by y.

The final outcome of our computational study was a cutting plane code that uses exact separation routines for cut and node-cut inequalities and separation heuristics for partition, node-partition, and lifted two-cover inequalities. For the type and size of practical problems used as our test cases, the other classes of inequalities were of no significant help. We expect, however, that for larger problem sizes and graphs of higher density further inequalities will be needed to achieve satisfactory computational performance. But that will make a more thorough design and investigation of separation heuristics for the other classes of inequalities necessary.

The design and implementation of a practically efficient cutting plane algorithm is a rather tricky and time-consuming task. Its success is based on the proper combination of many details. Some of these are described in [GMS] and [S]. We are unable to outline these here. Our final code showed the following computational characteristics on our test problems.

We obtained the data of seven real networks (nodes, possible direct links, costs of establishing links) from network designers at Bell Communications Research. The sizes ranged from 36 nodes and 65 edges to 116 nodes and 173 edges. For all networks, 2NCON and 2ECON solutions had to be found, but in only one case did these solutions differ. So we had eight test problems available. According to the network designers, these data represent the range of typical practical applications in this area.

We ran our cutting plane algorithm (using a research version of Bixby's LP-code (see [Bix]) and Jünger's Branch and Cut framework (unpublished)) on a SUN 3/60, a 3 MIPS machine. Five of the eight problems were solved to optimality in the cutting plane phase in less than 10 seconds. In the remaining three cases the cutting plane phase finished after at most 31 seconds with an integrality gap of less than 1 percent. In the subsequent branch and cut phases no more than 20 nodes were generated in the branching tree and at most an additional $1\frac{1}{2}$ minutes were needed to find an optimal solution and prove optimality. Further cases, run subsequently, showed similar computational performance. (See [GMS] for more details.)

Considering these computational results, we feel confident in saying that all survivable network design problems of the type and size arising at Bellcore can be solved to optimality with our code in at most a few minutes on a 3 MIPS machine. Thus the theoretical investigation presented here has helped (and helps further) to solve typical instances of a combinatorial optimization problem of significant practical importance.

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CONVERGENCE OF BROYDEN'S METHOD IN BANACH SPACES*

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Abstract. This paper proves new convergence theorems for convergence of Broyden's method when applied to nonlinear equations in Banach spaces. The convergence is in the norm of the Banach space itself, rather than in the norm of some Hilbert space that contains the Banach space. It is shown that the norms in which q-superlinear convergence takes place are determined by the smoothing properties of the error in the Fréchet derivative approximation and not by the inner product in which Broyden's method is implemented. Among the consequences of the results in this paper are a proof of sup-norm local q-superlinear convergence when Broyden's method is applied to integral equations with continuous kernels, global q-superlinear convergence of the Broyden iterates for singular and nonsingular linear compact fixed point problems in Banach space, a new method for integral equations having derivatives with sparse kernels, and q-superlinear convergence for a new method for integral equations when part of the Fréchet derivative can be explicitly computed. Partitioned variants of the methods and the "bad" Broyden method are also discussed.

Key words. Broyden's method, q-superlinear convergence, quasi-Newton update, Banach space

AMS(MOS) subject classifications. 65J15, 47H17, 49D15

1. Introduction.

This paper considers the solution of equations in Banach space by Broyden's method. We write our equations as

$$F(u) = 0, \tag{1.1}$$

where F is a Lipschitz continuously differentiable map between Banach spaces X and Y. We will consider both linear and nonlinear equations. Broyden's method is a variation of Newton's method in which an approximation, B, to the Fréchet derivative, $F'(u^*)$, is maintained along with an approximation u to a solution u^*. This method has been used with success for discretizations of infinite-dimensional problems in integral equations [23], [20]; fluid mechanics [10]; and optimal control [26]. In this paper we take the position that analysis of the convergence properties of the method should not be done for the discrete finite-dimensional problems alone because the results of such a finite-dimensional analysis can hide features that may depend on how the level of discretization is refined and may lead to conclusions that are valid only in finite dimension. The papers [11], [31], and [24] present examples of these kinds of problems. Direct analysis of the infinite-dimensional problem can also lead to effective preconditioning strategies that produce good convergence properties even for finite-dimensional approximations.

The purpose of this paper is to extend convergence results for Broyden's method in finite-dimensional spaces and Hilbert spaces to the Banach space setting and thereby sharpen convergence results. In particular, our results show how the inner product used in the implementation of Broyden's method and the topology in which convergence takes place are related. The significant consequences of this Banach space analysis for problems considered previously in [23], [20], [26], and [27] are to make the convergence estimates more precise, for example, to provide uniform convergence

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