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SHORT COMMUNICATION

PARTIAL LINEAR CHARACTERIZATIONS OF THE ASYMMETRIC TRAVELLING SALESMAN POLYTOPE

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We consider the linear programming formulation of the asymmetric travelling salesman problem. Several new inequalities are stated which yield a sharper characterization in terms of linear inequalities of the travelling salesman polytope, i.e., the convex hull of tours.

In fact, some of the new inequalities as well as some of the well-known subtour elimination constraints are indeed facets of the travelling salesman polytope, i.e., belong to the class of inequalities that uniquely characterize the convex hull of tours to a n-city problem.

Let P_A^n denote the polytope of the $n \times n$ assignment problem, i.e.,

$$P_A^n = \{x \in \mathbf{R}^{n^2} : x \geqslant 0\,,\ \sum_{j=1}^{n} x_{ij} = 1 \text{ for } j = 1, \ldots, n\,,\ \sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, n\}\,,$$

where $x \in \mathbf{R}^{n^2}$ has been indexed as usually $x = (x_{11}, \ldots, x_{1n}, \ldots, x_{n1}, \ldots, x_{nn})$. Then every tour of the travelling salesman is known (4) to be a vertex of P_A^n, but not vice versa. Denote by P_S^n the polytope obtained from P_A^n by intersecting P_A^n with the totality of the halfspaces given by the subtour-elimination constraints (3), i.e., linear constraints of the form

$$(S_k) \qquad \sum_{(i,j) \in S \times S} x_{ij} \leqslant k\,, \qquad |S| = k + 1\,, \qquad S \subset N\,,$$

where $k = 0, 1, \ldots, [\frac{1}{2}n] - 1$ and $N = \{1, \ldots, n\}$. Denoting by P_T^n the convex hull of the tours of the travelling salesman it is well-known that $P_T^n = \text{conv}\,\{x \in P_S^n : x \text{ integer}\}$. Very little is known about the polytope P_T^n. (In (8) it has been established that the diameter of P_T^n equals two for all $n \geqslant 6$).

Denote by d_n the dimension of the polytope P_T^n. A *facet* F of P_T^n is a $(d_n - 1)$-dimensional face of P_T^n. As customary in the literature, we will use the term facet synonymously for the inequality $\pi x \leqslant \pi_0$ that generates a facet F. It is an easy consequence of the definition of P_T^n that we can assume that every facet $\pi x \leqslant \pi_0$ of P_T^n satisfies $\pi \geqslant 0$ and $\pi_0 > 0$. Hence, an inequality $\pi x \leqslant \pi_0$ is a facet of P_T^n if and only if

(i) $x \in P_T^n$ implies $\pi x \leqslant \pi_0$,

(ii) there exist d_n linearly independent vertices x^i of P_T^n satisfying $\pi x^i = \pi_0$ for $i = 1, \ldots, d_n$. In order to derive results concerning the facial structure of P_T^n, one has to determine first the number d_n specifying the dimension of P_T^n.

In (1) it was conjectured that $d_n = n^2 - 3n + 1$. In (5) we prove that $d_n = n^2 - 3n + 1$ and derive several new classes of inequalities that are *valid* over P_T^n and in fact, more than just validity, constitute *facets* of P_T^n.

More precisely, in (5) we prove the following theorems and results:

Theorem 1. (i) *The dimension of P_T^n equals $d_n = n^2 - 3n + 1$ for $n \geqslant 3$.*
(ii) *The subtour elimination constraints (S_k) define facets, i.e. $(d_n - 1)$-dimensional faces, of P_T^n for $k = 1$ and $k = 2$, and for all $n > 5$.*

It should be noted that we address ourselves here to the convex hull of tours and not to the convex hull of all integer points of the associated problem obtained by relaxing the equality stipulation in the definition of P_A^n to be less-than-or-equal-to inequalities which (in the equivalent formulation for the symmetric travelling salesman problem) is the object of study of Chvatal (2). The relaxed polytope is of full dimension, provided that the variables x_{ii} for $i = 1, \ldots, n$ are eliminated, whereas the convex hull of tours, i.e., P_T^n, is contained in an affine proper subspace of

$$\mathbf{R}^{n^2} \cap \{x \in \mathbf{R}^{n^2}.\ x_{ii} = 0 \text{ for } i = 1, \ldots, n\}\,.$$

After an intensive search of the literature related to the travelling salesman problem we found that the statement of the assertion (i) of Theorem 1 appears without proof in two abstracts by I. Heller and H. Kuhn, respectively.

For any $K_i \subseteq N$, $i = 0, 1, \dots, k$, satisfying $|K_0 \cap K_i| = 1$ for $i = 1, \dots, k$, let $K^1 = \bigcup_{i=0}^{k} (K_i \times K_i)$. Then the inequality

$$\text{(UC1)} \qquad \sum_{(i,j)\in K^1} x_{ij} \leqslant s(K) := |K_0| + \sum_{h=1}^{k} (|K_h| - 1) - \langle \tfrac{1}{2}k \rangle ,$$

where $\langle r \rangle$ denotes the smallest integer greater than or equal to r, is a valid inequality for P_T^n. (In fact, (UC1) is a straight-forward generalization of the comb-inequality (2) derived by Chvatal for the symmetric travelling salesman problem). Let $K_i \subseteq N$, $i = 0, 1, \dots, k$, and K^1 be defined as before and let $p \neq q$, p, q $N - \bigcup_{i=0}^{k} K_i$. Define

$$K^2 = K^1 \cup \{(p, j) : j \in K_0\} \cup \{(j, q) : j \in K_0\} \cup \{(p, q)\}.$$

Then the inequality

$$\text{(UC2)} \qquad \sum_{(i,j)\in K^2} x_{ij} \leqslant s(K) + 1$$

is a valid inequality for P_T^n. Moreover, we have the following theorem (see (5)):

Theorem 2. *If* $k = 1$, $|K_0| = 1$ *and either*
(i) $|K_1| = n - 2$, *or*
(ii) $|K_1| = 2$ *in the definition of the inequality* (UC2),
then (UC2) *defines a facet of* P_T^n *for all* $n \geqslant 4$ (*in case* (i)) *and for all* $n \geqslant 6$ (*in case* (ii)).

Let $\{i_1, i_2, \dots, i_k, i_{k+1}\} \subseteq N$ be any $k + 1$ distinct indices, where $2 \leqslant k \leqslant n - 2$. Then the inequality

$$(\mathrm{D}_k) \qquad \sum_{j=1}^{k} x_{i_j i_{j+1}} + x_{i_{k+1} i_1} + 2\sum_{j=2}^{k} x_{i_j i_1} + \sum_{j=3}^{k} \sum_{h=2}^{j-1} x_{i_j i_h} \leqslant k$$

is a valid inequality for P_T^n. Moreover, for $k = 2$, the inequality (D_2) is equivalent to the UC2-inequality of Theorem 2, case (i), and hence defines a facet of P_T^n. (Two valid inequalities are termed equivalent if they are satisfied with equality by the same set of vertices of P_T^n; obviously, since $\dim P_T^n = n^2 - 3n + 1 < n^2$, for every facet F of P_T^n there exists an infinity of linear inequalities $ax \leqslant a_0$ such that $F = P_T^n \cap \{x \in \mathbb{R}^{n^2} : ax = a_0\}$.) Concerning the D_k-inequalities we prove furthermore:

Theorem 3. *The inequality* (D_k) *defines a facet of* P_T^n *for all* $n \geqslant 5$ *and* $k = 3$.

A further valid inequality for P_T^n is obtained in the following way: Let $\{i_1, i_2, i_3, i_4\} \subseteq N$ be any four distinct indices. We then prove in (5) that the inequality

$$(\mathrm{E}_3) \qquad 2x_{i_1 i_2} + x_{i_1 i_4} + 2x_{i_2 i_1} + x_{i_2 i_3} + x_{i_3 i_1} + x_{i_4 i_2} \leqslant 3$$

provides a facet for P_T^n with $n \geqslant 5$. Furthermore, it is shown in (5) that for sufficient large n ($n \geqslant 6$ will suffice) the facets furnished by the inequalities stated in the above theorems as well as by (E_3) are all distinct. We furthermore count the number of such inequalities and the number of tours satisfied by them with equality. Obviously, the polytope obtained by intersecting P_S^n with the totality of halfspaces defined by the inequalities (UC1), (UC2), (E_3) and (D_k) for $k = 2, \dots, n - 2$ is contained in P_S^n, hence we have a sharper partial linear characterization of P_T^n than the one given by the inequalities of P_S^n alone. In general, however, one needs more inequalities than the ones stated above in order to completely characterize P_T^n by linear inequalities. The use of the above new inequalities in a linear programming approach to the travelling salesman problem is currently under investigation.

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