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ON THE SYMMETRIC TRAVELLING SALESMAN PROBLEM I: INEQUALITIES

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We investigate several classes of inequalities for the symmetric travelling salesman problem with respect to their facet-defining properties for the associated polytope. A new class of inequalities called comb inequalities is derived and their number shown to grow much faster with the number of cities than the exponentially growing number of subtour-elimination constraints. The dimension of the travelling salesman polytope is calculated and several inequalities are shown to define facets of the polytope. In part II ("On the travelling salesman problem II: Lifting theorems and facets") we prove that all subtour-elimination and all comb inequalities define facets of the symmetric travelling salesman polytope.

Key words: Linear Inequalities, Convex Polytopes, Facets, Travelling Salesman Problem.

0. Introduction and notation

The symmetric travelling salesman problem is the problem of finding the shortest hamiltonian cycle or tour in a weighted undirected finite graph without loops and multiple edges. In the most common interpretation of this problem the nodes of the graph represent cities, the edges of the graph direct travel routes between the cities and the weights the distances between pairs of cities. The problem appears to have been formulated some 45 years ago [24] and has been the object of intensive investigation during the past 25 years. The problem is of interest both from a practical and a theoretical point-of-view: Many problems in scheduling and production management can be formulated this way or by closely related models. On the other hand, as Karp [21] has shown, the travelling salesman problem belongs to the class of NP-complete or "hard" combinatorial optimization problems which to date cannot be solved by polynomially bounded algorithms.

In this paper and its sequel ("On the symmetric travelling salesman problem II: Lifting theorems and facets" [16]) we are concerned with the facial structure of the convex hull of tours of the n-city travelling salesman problem where a tour is regarded as a point in $\mathbf{R}^m$ with $m = \frac{1}{2}n(n-1)$. The problem has been formulated as a linear programming problem in zero—one variables this way by

Dantzig et al. [3] in 1954, who introduced the subtour-elimination constraints, a class of linear inequalities that are satisfied by all tours. A further class of such *valid* inequalities for the convex hull of tours are specializations of the matching constraints due to Edmonds [4] and the comb inequalities due to Chvátal [2]. We generalize these two types of inequalities to the class of (general) comb constraints and show that subtour-elimination and comb inequalities define *facets* of the convex hull of tours, i.e. belong to the class of inequalities that are *generated* by the tours considered as points in $\mathbf{R}^m$. In a way, our work is a continuation of much earlier work done by Heller [18, 19], Kuhn [22] and Norman [26] along these same lines, though this earlier work, to the extent that it is known to us, is restricted to studying the facial structure of the convex hull of tours for up to $n = 7$ cities in the symmetric case, see Gomory [7].

In Section 1 we define the concept of a *comb* in a graph and its associated inequality. We show that every comb inequality defines a proper face of the convex hull of tours and count the number of such inequalities. The number of comb inequalities is a combinatorial function of the number of cities and in Fig. 2 we have tabulated some of the respective values so as to allow a comparison to the number of subtour-elimination constraints. In Section 2 we calculate the dimension of the convex hull of tours for the n-city problem to be $m - n = \frac{1}{2}n(n-3)$. This result is known [22]; our proof is short and useful later on for other proofs as well. In Section 3 we prove that the inequalities given by the nonnegativity and upper bound conditions define "trivial" *facets* of the travelling salesman polytope, i.e. the convex hull of tours. Furthermore, we show that a certain comb inequality defines a facet for the travelling salesman polytope. These results are the necessary ingredients for the sequel [16]: In Sections 4 and 5 [16] we first prove four lifting theorems that permit us to "lift" facets from lower-dimensional to higher-dimensional travelling salesman polytopes. In particular, one of these results (Theorem 4.12) includes a related result obtained earlier by Maurras [23, Chapter 1] as a special case. In Section 6 of [16] we then prove that all subtour elimination and comb inequalities define *facets* of the travelling salesman polytope.

Our interest in establishing this fact is twofold: Firstly, it is of mathematical interest to know which ones of the proposed inequalities *really matter* in defining this incredibly complex polytope. Secondly, facets are "strongest cutting planes" in an integer programming sense (see [6]) and it is thus natural to expect that such inequalities are of substantial computational value in the numerical solution of this hard combinatorial optimization problem. These expectations are substantiated by two accompanying computational studies: In [8] the solution to optimality of a 120-city problem on a complete graph is reported using only 96 inequalities defining facets. As far as we know this is the largest complete travelling salesman problem solved to optimality to date. In [27] the results of a computational study for a total of 74 travelling salesman problems of size varying from 15 cities to 318 cities on complete graphs are reported. In the case of the 318-city problem, 183 inequalities defining facets were generated (by a computer program) and the best tour that was obtained was proven to be at worst $\frac{1}{4}$% off the absolute optimum. In [27] also a statistical study is included which further supports the hypothesis that inequalities defining facets are of substantial computational value.

Some of the results reported here have been presented at several professional meetings in preliminary form [9, 10, 11, 12]. Related work on the asymmetric travelling salesman problem can be found in [8, 13, 14].

Our notation and terminology follows standard books such as [1, 6, 17, 29]. We use the following symbols in both this paper and its sequel [16]:

(1) $K_n = (V, E)$ is the *complete graph* on n nodes; V is the node set and the nodes are labelled $1, \ldots, n$; E is the edge set and consists of $\frac{1}{2}n(n-1)$ unordered pairs of indices i, j. We denote any element in E either by e or by $[i, j]$. The latter is used to specify the nodes i and j connected by the edge e. Whenever used as index, the brackets are dropped in $[i, j]$. If $W \subseteq V$, then $E(W)$ denotes the subset of edges of E with both endpoints in W. If $H \subseteq E$, then $V(H)$ denotes the subset of nodes of V defined by the edges in H. $\omega(v)$ for $v \in V$ denotes the star of v, i.e. the subset of edges of E that are incident with $v \in V$. If $U, W \subseteq V$, then $U - W = \{u \in U \mid u \notin W\}$. Instead of writing $W - \{w\}$ we simply write $W - w$. If $i_j \in V$ for $j = 1, \ldots, k$, then $[i_1, \ldots, i_k]$ denotes the chain connecting nodes $i_1, \ldots, i_k$ (i.e. i_1 to i_2, etc., i_{k-1} to i_k).

(2) $T \subseteq E$ is *a tour* if it is a (simple) cycle of length n in K_n. A cycle of length $k < n$ in K_n is called a subtour. A tour $T \subseteq E$ is also denoted as a cyclic permutation $\langle abc \cdots uvw \cdots s\rangle$ where a, b, c, etc. are nodes in V. In slight abuse of notation we use "$T = \langle abc \cdots uvw \cdots s\rangle$" to express the fact that the tour with edge-set T connects a to b, b to c, etc. and s to a. By T_n we denote the set of *all* tours in K_n.

(3) If $T \in T_n$, then x^T is the *incidence vector of* T, i.e.

$$x_e^T = \begin{cases} 1 & \text{if } e \in T, \\ 0 & \text{if not} \end{cases}$$

and has $m = \frac{1}{2}n(n-1)$ components. m is always used to designate $\frac{1}{2}n(n-1)$.

(4) The *matrix* A denotes the incidence matrix of K_n of size $n \times m$, i.e. the rows correspond to the nodes of K_n, columns to the edges of K_n.

(5) Q_T^n is the *travelling salesman polytope*, i.e. the convex hull of the incidence vectors of all tours in K_n:

$$Q_T^n = \text{conv}\{x^T \in \mathbf{R}^m \mid T \in T_n\}.$$

Q_λ^n is the linear programming relaxation of Q_T^n due to Dantzig et al. [3]:

$$Q_\lambda^n = \{x \in \mathbf{R}^m \mid Ax = \mathbf{2}, 0 \le x \le \mathbf{1}\}$$

where $\mathbf{2}$ is the vector with n components equal to two and $\mathbf{1}$ is the vector with m components equal to one. In particular, $Q_T^n \subseteq Q_\lambda^n$.

(6) For any $x \in \mathbf{R}^m$ and $H \subseteq E$ we denote $x(H) = \sum_{e\in H} x_e$. If $W \subseteq V$, then rather than writing $x(E(W))$, we simply write $x(W)$. If $U, W \subseteq V$ satisfy $U \cap W = \emptyset$ we denote $(U : W) = \{e \in E \mid e = [i, j]$ with $i \in U$ and $j \in W\}$, i.e. the set of edges that connect nodes in U to nodes in W.

(7) An inequality $ax \le a_0$ is called *valid* (with respect to Q_T^n) if $Q_T^n \subseteq \{x \in \mathbf{R}^m \mid ax \le a_0\}$. A valid inequality $ax \le a_0$ *defines a proper face* of Q_T^n if $\emptyset \ne Q_T^n \cap \{x \in \mathbf{R}^m \mid ax = a_0\} \ne Q_T^n$. A valid inequality $ax \le a_0$ *defines a facet* of Q_T^n if its face is proper and there exist $\dim Q_T^n$ affinely independent points $x^i \in Q_T^n$ such that $ax^i = a_0$ for $i = 1, \dots, \dim Q_T^n$, where $\dim Q_T^n$ is the dimension of Q_T^n. Valid inequalities defining facets of Q_T^n are called *facetial* inequalities. A valid inequality $ax \le a_0$ is *dominated* if there exist valid inequalities $b^i x \le b_0^i$ and nonnegative μ_i for $i = 1, \dots, k$ such that a is not a multiple of b^i for $i = 1, \dots, k$, $ax \le \sum_{i=1}^k \mu_i b^i x$ for all $x \in Q_T^n$ and $\sum_{i=1}^k \mu_i b_0^i \le a_0$ hold. A dominated valid inequality is called redundant, because it is not needed in a linear description of Q_T^n by a minimal system of linear inequalities. Two valid inequalities $ax \le a_0$ and $bx \le b_0$ are called *equivalent* if

$$Q_T^n \cap \{x \in \mathbf{R}^m \mid ax = a_0\} = Q_T^n \cap \{x \in \mathbf{R}^m \mid bx = b_0\}.$$

1. Valid inequalities for Q_T^n

The best-known valid inequalities for the travelling salesman polytope Q_T^n are the subtour-elimination constraints due to Dantzig et al. [3]. The *subtour-elimination constraint on a node-set* W is given by

$$x(W) \le |W| - 1 \tag{1.1}$$

where $W \subset V$ satisfies $2 \le |W| \le n - 2$. The following proposition summarizes the known properties of subtour-elimination constraints (see e.g. [7]).

Proposition 1.1. (i) *Every subtour-elimination constraint* (1.1) *defines a proper face of* Q_T^n. (ii) *The subtour-elimination constraints on* W *and* $V - W$ *define the same face of* Q_T^n. (iii) *The number of subtour-elimination constraints defining distinct faces of* Q_T^n *is equal to*

$$\nu^S(n) = 2^{n-1} - n - 1. \tag{1.2}$$

Since for any pair $W, W' \subseteq V$ satisfying $W \cup W' \ne V$ one can readily find a tour in K_n satisfying the subtour-elimination constraint on W with equality and the subtour-elimination constraint on W' with inequality, it follows that the subtour-elimination constraints define exactly $\nu^S(n)$ distinct faces of Q_T^n. Furthermore, for $n \ge 5$, the trivial inequalities $x_e \ge 0$, $e \in E$, define distinct faces of Q_T^n. Consequently, we have $2^{n-1} + \frac{1}{2}n(n-3) - 1$ inequalities defining distinct faces of

Q_T^n. We discuss next a class of combinatorial inequalities whose total number apparently grows much faster with n than $O(2^{n-1})$.

Since every vertex of Q_T^n is a forteriori an integer vertex of Q_A^n, the 2-matching constraints due to Edmonds [4] constitute valid inequalities for Q_T^n. Chvátal [2] has generalized this class of inequalities to a wider class of inequalities which he called comb inequalities. Both classes of valid inequalities for Q_T^n like the subtour-elimination constraints have coefficients of zeros and ones only (except for the right-hand side constant) and are special cases of the following general *comb inequality*, which has coefficients equal to 0, 1 or 2: Let $W_i \subseteq V$ for $i = 0, 1, \dots, k$ satisfy

$$|W_0 \cap W_i| \geq 1 \quad \text{for } i = 1, \dots, k. \tag{1.3}$$

$$|W_i - W_0| \geq 1 \quad \text{for } i = 1, \dots, k. \tag{1.4}$$

$$|W_i \cap W_j| = 0 \quad \text{for } 1 \leq i < j \leq k. \tag{1.5}$$

$$k \text{ odd.} \tag{1.6}$$

Then we call $C = \bigcup_{i=0}^{k} E(W_i)$ a *comb* in K_n; W_0 is called the *handle* and the W_i for $i = 1, \dots, k$ are called the *teeth* of the comb C. (See Fig. 1 for the graphical configuration that defines a comb. Every oval is a complete subgraph of K_n and these complete subgraphs overlap in the way defined by (1.3) through (1.5). The *comb inequality* corresponding to a comb C in K_n is given by

$$a^c x := \sum_{i=0}^{k} x(W_i) \leq |W_0| + \sum_{i=1}^{k} (|W_i| - 1) - \langle \tfrac{1}{2}k \rangle, \tag{1.7}$$

where as usual $\langle \alpha \rangle$ denotes the smallest integer greater than or equal to α. The right-hand side constant of inequality (1.7) is denoted $s(C)$ and referred to as the *size* of the comb C.

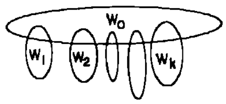

Fig. 1. Graphical configuration of a comb.

A comb C with $k = 1$ and $|W_0| = 1$ is a subtour-elimination constraint. A comb inequality is a 2-matching constraint [4] if the inequality in both (1.3) and (1.4) holds as an equality. A comb inequality is a *Chvátal-comb* [2] if the requirement (1.5) is dropped and the inequality (1.3) is required to hold as an equality. Chvátal [2] also permits k in (1.6) to be even. The proof of Proposition 1.3 below shows that (1.7) is valid for even k as well, but then the inequality (1.7) does not involve any integerization and is trivially seen to be dominated. Before proving validity of comb inequalities for Q_T^n we show that the requirement (1.5) does not exclude any Chvátal-combs that are essential for Q_T^n.

Proposition 1.2. (i) *A Chvátal–comb C satisfying $|W_0| = 1$ and $k \geq 2$ is dominated.* (ii) *A Chvátal–comb C satisfying $k = 1$ and $|W_0| \geq 2$ is dominated.* (iii) *A Chvátal–comb C satisfying $\bigcap_{j\in K} W_j \neq \emptyset$ for some $K \subseteq \{1, \dots, k\}$ with $|K| \geq 2$ is dominated.*

Proof. To prove (i), let $W_0 = \{v\}$ and $k \geq 2$. Then one verifies that

$$\sum_{e\in C} x_e \leq x(\omega(v)) + \sum_{i=1}^{k} x(W_i - v) \leq 1 + \sum_{i=1}^{k} (|W_i| - 1) - (k-1) \leq s(C).$$

To prove (ii), let $\{v\} = W_0 \cap W_1$ and $|W_0| \geq 2$. Then one verifies that

$$\begin{aligned}\sum_{e\in C} x_e &\leq x(\omega(v)) + x(W_0 - W_1) + x(W_1 - W_0)\\ &\leq 2 + |W_0| - 2 + |W_1| - 2 = s(C).\end{aligned}$$

To prove (iii), suppose that $K = \{1, \dots, p\}$ with $2 \leq p \leq k$ and let $\{v\} = \bigcap_{j=0}^{p} W_j$. If $|W_0| = 1$, the assertion follows from part (i). We can thus assume that $|W_0| \geq 2$ and due to part (ii), that $k \geq 3$. If $p = k$, then one verifies that

$$\sum_{e\in C} x_e \leq x(\omega(v)) + \sum_{i=0}^{k} x(W_i - v) \leq 2 + \sum_{i=0}^{k} (|W_i| - 2) \leq s(C).$$

If $2 \leq p < k$, define a new comb C' by $W_0' = W_0 - v$ and $W_{i-p}' = W_i$ for $i = p+1, \dots, k$. Then one verifies that

$$\begin{aligned}\sum_{e\in C} x_e &\leq x(\omega(v)) + \sum_{i=1}^{p} x(W_i - v) + \sum_{i=0}^{p-k} x(W_i') \leq 2 + \sum_{i=1}^{p} (|W_i| - 2) + s(C')\\ &\leq s(C).\end{aligned}$$

Suppose finally that $v \in \bigcap_{j=1}^{p} W_j$ where $2 \leq p \leq k$ and $v \notin \bigcup_{i=p+1}^{k} W_i \cup W_0$. Define a new comb C' with $W_0' = \bigcup_{i=0}^{p} W_i$ and $W_{i-p}' = W_i$ for $i = p+1, \dots, k$. Note that $|W_0'| = |W_0| + \sum_{i=1}^{p} (|W_i| - 2) + 1$. Consequently it follows that

$$\sum_{e\in C} x_e \leq \sum_{i=0}^{k-p} x(W_i') \leq s(C).$$

It follows from Proposition 1.2 that our requirement (1.5) excludes only dominated Chvátal–combs.

Furthermore, the only undominated combs satisfying $k = 1$ yield subtour-elimination constraints and thus, in order to distinguish subtour-elimination constraints from comb inequalities, we will assume throughout that (1.6) holds with $k \geq 3$.

Proposition 1.3. *Every comb inequality* (1.7) *defines a proper face of Q_T^n.*

Proof. We first show that every comb inequality is a valid inequality for Q_T^n:

$$
\begin{aligned}
2a^C x &\le \sum_{w\in W_0} x(\omega(w)) + \sum_{i=1}^{k} \{x(W_i) + x(W_i - W_0) + x(W_i \cap W_0)\} \\
&\le 2|W_0| + \sum_{i=1}^{k} \{|W_i| - 1 + |W_i - W_0| - 1 + |W_i \cap W_0| - 1\} \\
&\le 2\Big\{|W_0| + \sum_{i=1}^{k} (|W_i| - 1)\Big\} - k.
\end{aligned}
$$

Inequality (1.7) follows by dividing by two and by integerization of the right-hand side. To prove that every comb inequality defines a proper face of Q_T^n, we construct a tour T satisfying (1.7) with equality and a tour T' satisfying (1.7) with (strict) inequality as follows: Choose any hamiltonian chain p_i' in $E(W_i - W_0)$ (of length $|W_i - W_0| - 1$) and any hamiltonian chain q_i in $E(W_i \cap W_0)$ (of length $|W_i \cap W_0| - 1$) for each $i = 1, \dots, k$. Connect each p_i' and q_i by an edge in $E(W_i)$ to form a hamiltonian chain p_i in $E(W_i)$ (of length $|W_i| - 1$) for $i = 1, \dots, k$. If $W_{k+1} = W_0 - \bigcup_{i=1}^{k} W_i$ is nonempty, choose a hamiltonian chain p_{k+1} in W_{k+1} (of length $|W_{k+1}| - 1$). To construct T, we connect the hamiltonian chain p_i to p_{i+1} by an edge in $E(W_0)$ for $i = 1, 3, \dots, k$ and complete the collection of hamiltonian chains to a tour T in K_n. It follows that

$$
\begin{aligned}
a^C x^T &= \sum_{i=1}^{k} (|W_i| - 1) + \sum_{i\in H} (|W_i \cup W_0| - 1) + \tfrac{1}{2}(k-1) + |W_{k+1}| \\
&= |W_0| + \sum_{i=1}^{k} (|W_i| - 1) - \tfrac{1}{2}(k+1) = s(C),
\end{aligned}
$$

where $H = \{i \in \{1, \dots, k\} \mid |W_i \cap W_0| \ge 2\}$. To construct a tour T' satisfying (1.7) with inequality, we proceed as before with the following differences: We connect the pending node in $W_1 \cap W_0$ of the chain p_1 to the pending node in $W_2 - W_0$ of the chain p_2, the pending node in $W \cap W_0$ to the pending node in $W_3 - W_0$ and complete the resulting collection of hamiltonian chains to a tour T' in K_n. Necessarily, none of the additional edges in T' is in the edge-set of the comb C and consequently, $x^{T'}$ does not satisfy (1.7) with equality.

Proposition 1.4. (i) *The comb inequalities* (1.7) *given by* $W_0, W_1, \dots, W_k$ *and by* $V - W_0, W_1, \dots, W_k$, *respectively, define the same face of* Q_T^n. (ii) *The number of comb inequalities defining distinct faces of* Q_T^n *is equal to*

$$
\nu^C(n) = \sum_{q=3}^{n-3} \tfrac{1}{2}\binom{n}{q} \sum_{j=3}^{n-q} \binom{n-q}{j} \sum_{\substack{k=3\\ k \text{ odd}}}^{\min(j,q)} A_k^j \sum_{p=k}^{q} \frac{1}{k!}\binom{q}{p} A_k^p, \tag{1.8}
$$

where

$$
A_k^p = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^p.
$$

Proof. To prove (i) we note that

$$
x(W_0) + \tfrac{1}{2}x(W_0 : V - W_0) = |W_0|, \qquad x(V - W_0) + \tfrac{1}{2}x(V - W_0 : W_0) = n - |W_0|
$$

imply $x(W_0) = 2|W_0| - n + x(V - W_0)$, since $x(W_0 : V - W_0) = x(V - W_0 : W_0)$. Consequently, a comb inequality (1.7) given by $W_0, W_1, \dots, W_k$ is satisfied with equality if and only if the comb-inequality (1.7) given by $V - W_0, W_1, \dots, W_k$ is satisfied with equality. Consequently, (i) follows. To prove (1.8), we note that every comb has at least three teeth and at least three nodes in its handle. Thus due to (i), we have $\frac{1}{2}\binom{n}{q}$ different possibilities for the choice of a handle where $3 \le q \le n-3$. For any such q, we can choose any j nodes from among the remaining $n-q$ nodes of K_n to form the part $W_i - W_0$ of the teeth of the comb, where $i = 1, \dots, k$. Thus we have $\binom{n-q}{j}$ possibilities to choose j nodes. For any such j nodes, we want to form an odd number $k \ge 3$ of teeth. By (1.3) and (1.4) we have $k \le \min(j, q)$ and by [25, §180] there are A_k^j possibilities to distribute j distinct elements into k distinct boxes such that no box is empty. Having made this choice, we must now select k nonempty subsets of W_0 which together with the already selected subsets of nodes form the teeth W_i for $i = 1, \dots, k$. To this end we can choose any p nodes of W_0 with $k \le p \le q$. Thus for any such p, we have $\binom{q}{p}$ possibilities. Again, we can distribute the p selected nodes in A_k^p different ways to k distinct boxes such that no box is empty. Since we have taken into account already the different permutations of the k boxes (in choosing the parts $W_i - W_0$ of the teeth), we must divide this number by the number of possible permutations $k!$. Thus (1.8) follows. By an argument closely following the proof of Proposition 1.3 one proves that any two combs whose teeth differ define distinct faces of Q_T^n. Then one shows by the same constructive argument that any two combs with identical teeth and handles W_0 and W_0' satisfying $W \cup W_0' \neq V$ define distinct faces of Q_T^n. (These proofs are quite elementary, but lengthy and therefore omitted.) Consequently, there are exactly $\nu^C(n)$ comb inequalities defining distinct faces of Q_T^n.

To get an idea about the comparative growth of the number $\nu^S(n)$ of subtour-elimination constraints and of the number $\nu^C(n)$ of comb constraints we have computed the respective numbers and tabulated them in Fig. 2. (For $n \ge 20$ we give only the order of magnitude.) As n gets large, $\nu^S(n)$ becomes marginal by comparison to $\nu^C(n)$.

While subtour-elimination constraints are intuitively readily understood, the logical implication of a comb inequality is more complicated. To illustrate the point, consider e.g. a comb inequality for $n = 8$ with $W_0 = \{1, 2, 3, 4\}$, $W_1 = \{1, 2, 5, 6\}$, $W_2 = \{3, 7\}$ and $W_3 = \{4, 8\}$, i.e.

$$2x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{23} + x_{24} + x_{25} + x_{26} + x_{34} + x_{37} + x_{48} + x_{56} \le 7. \tag{1.9}$$

Using the relations $\sum_{j=2}^{8} x_{1j} = 2$ and $x_{12} + \sum_{j=3}^{8} x_{2j} = 2$ to eliminate the variable x_{12} from (1.9) we obtain the equivalent constraint

$$x_{34} + x_{37} + x_{48} + x_{56} \le 3 + x_{17} + x_{18} + x_{27} + x_{28}. \tag{1.10}$$

This constraint now expresses quite clearly the logical implication of the comb

n	$\nu^S(n)$	$\nu^C(n)$
6	25	60
7	56	2100
8	119	41420
9	246	667800
10	501	8841970
15	16368	1993711339620
20	$0.5 \cdot 10^6$	$1.5 \cdot 10^{18}$
30	$0.5 \cdot 10^9$	$1.5 \cdot 10^{31}$
40	$0.5 \cdot 10^{12}$	$1.5 \cdot 10^{45}$
50	$0.5 \cdot 10^{15}$	10^{60}
59	$0.3 \cdot 10^{18}$	10^{74}
120	$0.6 \cdot 10^{36}$	$2 \cdot 10^{179}$

Fig. 2. Comparative growth of $\nu^S(n)$ and $\nu^C(n)$.

inequality (1.9): If $x_{34} = x_{37} = x_{48} = x_{56} = 1$, i.e. if the travelling salesman travels on the chain [7, 3, 4, 8] and includes the link [5, 6] as well, then $x_{17} + x_{18} + x_{27} + x_{28} \geq 1$ must hold, i.e. then the travelling salesman must choose one of the links [1, 7], [1, 8], [2, 7] or [2, 8], since otherwise there exists no round trip for the eight cities. (See also Fig. 3; solid lines correspond to the variables on the lefthand side of the inequality (1.10), dashed lines to those on the right-hand side.) A similar interpretation of comb inequalities can be given in more general cases as well.

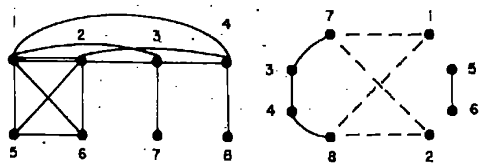

Fig. 3. Comb inequalities (1.9) and (1.10).

2. The dimension of Q_T^n

Since the node-edge incidence matrix A of the complete graph K_n has full row rank and since Q_T^n satisfies $Q_T^n \subseteq Q_A^n$, it follows that the dimension of Q_T^n satisfies

$$\dim Q_T^n \leq \dim Q_A^n \leq m - n = \tfrac{1}{2}n(n-3). \tag{2.1}$$

Furthermore, since the vector $\bar{x}$ with all positive components $\bar{x}_{ij} = 2/(n-1)$ for $1 \leq i < j \leq n$ is contained in Q_A^n, the polytope Q_A^n is not contained in any one of the subspaces $\{x \in \mathbf{R}^m \mid x_{ij} = 0\}$ or $\{x \in \mathbf{R}^m \mid x_{ij} = 1\}$ and hence, $\dim Q_A^n = m - n$. The proof that equality holds in the first inequality as well makes use of the following result from graph theory which can be found e.g. in [17, p. 89].

Lemma 2.1. *Let $K_n(V, E)$ be the complete graph on n nodes and k denote any integer.*

(i) *If $|V| = 2k+1$, then there exist k edge-disjoint tours T_l such that $E = \bigcup_{l=1}^{k} T_l$.*

(ii) *If $|V| = 2k$, then there exist $k-1$ edge-disjoint tours T_l and a perfect matching M edge-disjoint from any T_l such that $E = M \cup \bigcup_{l=1}^{k-1} T_l$.*

Theorem 2.2. *The dimension of Q_T^n equals $d_n = \frac{1}{2}n(n-3)$ for all $n \geq 3$.*

Proof. We show that Q_T^n contains $d_n + 1$ linearly independent tours. For $n = 3$ the claim is trivially true.

(a) Assume the $n = 2k+2$ with $k \geq 1$ and integer. The subgraph $K_{n-1} = (V', E')$ of K_n induced by the $n-1$ first nodes is again complete and by Lemma 2.1, its edge-set is the union of k edge-disjoint tours T_l of length $n-1$. From each $(n-1)$-tour T_l we construct $n-1$ tours T_{lj} of length n by replacing, one at a time, each edge $[u, v] \in T_l$ by the chain $[u, n, v]$. The incidence matrix of the tours T_{lj} for $j = 1, \dots, n-1$ (rows) versus the edges of K_n (columns) contains the submatrix $E - I$, where E is the $(n-1)\times(n-1)$ matrix of all ones and I is the $(n-1)\times(n-1)$ identity matrix. Furthermore, we obtain a total of $k(n-1) = d_n + 1$ tours. Since the tours T_l of length $n-1$ are edge-disjoint, the incidence matrix of all $d_n + 1$ tours T_{lj} contains a $(d_n+1)\times(d_n+1)$ submatrix N which is block-diagonal and whose diagonal blocks all equal to $E - I$ after a suitable arrangement of the rows and columns. Since $E - I$ is nonsingular, it follows that N is nonsingular and consequently, Theorem 2.2 holds if $n = 2k+2$.

(b) Assume that $n = 2k+1$ with $k \geq 2$ and integer. We proceed as in the case (a) and construct $(k-1)(n-1)$ linearly independent tours from the $k-1$ tours of length $n-1$. The perfect matching in K_{n-1} is completed abritrarily to a $(n-1)$-tour in K_{n-1} and subsequently, used to construct k tours of length n by replacing, one at a time, each edge $[u, v] \in M$ by the chain $[u, n, v]$. This way we obtain a total of $d_n + 1$ tours whose incidence matrix contains a $(d_n+1)\times(d_n+1)$ block-triangular matrix N with $k-1$ blocks $E - I$ of size $(n-1)\times(n-1)$ and an additional block $E' - I'$ of size $k \times k$.

Remark 2.3. Since every tour of the symmetric travelling salesman problem defines exactly two edge-disjoint tours for the *asymmetric* travelling salesman problem, the proof of Theorem 2.2 can be used to prove the dimensionality result for the asymmetric travelling salesman problem, see [13] for a different argument.

Remark 2.4. A further implication of Theorem 2.2 is that for complete graphs the hamiltonian cycles form a *cycle-basis*, see [1, p. 15], which, however, is not true for arbitrary finite graphs.

3. Trivial facets of Q_T^n and special combs

Since the dimension d_n of Q_T^n satisfies $d_n < \frac{1}{2}n(n-1)$, it is to be expected that it is rather difficult to prove that a particular inequality is a facet of Q_T^n. The proof that the inequalities $x_e \geq 0$ and $x_e \leq 1$ for all $e \in E$ constitute the "trivial" facets of Q_T^n is by no means obvious. Also included in this section is the proof that a very special sublcass of the comb inequalities (1.7) define facets of Q_T^n. The results of this section are used in Section 6 [16] in connection with the lifting theorems to prove that subtour-elimination constraints as well as comb inequalities define facets of Q_T^n. The general proof-procedure used in this section consists in showing that the incidence matrix of all n-tours satisfying a given inequality with equality versus the edges of K_n contains a $d_n \times d_n$ nonsingular submatrix. We leave it to the reader to verify that each inequality is satisfied with inequality by at least one tour as well as to verify the assertions for "small" n by enumeration.

We start by proving some general properties of facets $ax \leq a_0$ of Q_T^n. Note that $Q_T^n \subseteq Q_A^n$ implies that for every (valid or facetial) inequality $ax \leq a_0$ there exists an *equivalent* (valid or facetial) inequality $\bar{a}x \leq \bar{a}_0$ satisfying $\bar{a} \geq 0$. This follows because $x \in Q_T^n$ implies $\sum_{e\in E} x_e = n$ and thus $\bar{a}_e = a_e + \theta$ for all $e \in E$ and $\bar{a}_0 = a_0 + n\theta$ has the required property for $\theta \geq -\min\{a_e \mid a_e < 0, e \in E\}$.

Proposition 3.0. *If $ax \leq a_0$ is a facet of Q_T^n satisfying $a \geq 0$, then either (a, a_0) can be written as $a = b + c$ and $a_0 = b_0 + c_0$ where $b \neq 0 \neq c$ are not multiples of a and $ax \leq a_0$, $bx \leq b_0$ and $cx \leq c_0$ are all equivalent with respect to Q_T^n or else, the graph $G_a = (N_a, E_a)$ is connected, where $E_a = \{e \in E \mid a_e > 0\}$ and $N_a = V(E_a)$.*

Proof. Suppose G_a is not connected. Let $V_1 \subseteq N_a$ be the node-set of any connected component of G_a. Clearly, $|V_1| \geq 2$ holds, and we define b by $b_e = a_e$ for all $e \in E_1$, where $E_1 = E(V_1)$, $b_e = 0$ otherwise. Define $b_0 = \max\{bx \mid x \in Q_T^n\}$. Since $b \geq 0$ and $b \neq 0$ hold, b_0 is some finite number. Let $V_2 = V - V_1$ and define c by $c_e = a_e$ for all $e \in E_2$, where $E_2 = E(V_2)$, $c_e = 0$ otherwise. Clearly $c \geq 0$ and $c = a - b$. Define c_0 analogously to b_0. Then both $bx \leq b_0$ and $cx \leq c_0$ are valid inequalities for Q_T^n and the respective constants satisfy $a_0 \leq b_0 + c_0$. By construction, there exists tours $T1$ and $T2$ such that $bx^{T1} = b_0$ and $cx^{T2} = c_0$ hold respectively. Due to the nonnegativity of b and c we can assume without loss of generality that $T1$ and $T2$, respectively induce hamiltonian chains in the graphs (V_1, E_1) and (V_2, E_2), respectively. We can connect the respective hamiltonian chains to form a tour $T3$ satisfying $bx^{T3} = b_0$ and $cx^{T3} = c_0$. Consequently, $b_0 + c_0 = ax^{T3} \leq a_0$ and thus $a_0 = b_0 + c_0$ follows. Suppose now that for some tour T $ax^T = a_0$, but $bx^T < b_0$ or $cx^T < c_0$. Then $a_0 = ax^T = bx^T + cx^T < b_0 + c_0 = a_0$ is a contradiction and the proposition follows.

In fact, as we have not invoked any property of the nonnegative valid inequality $ax \le a_0$ other than the existence of a tour satisfying $ax \le a_0$ with equality, the proposition is true for any nonnegative valid inequality defining a nonempty face of Q_T^n.

Theorem 3.1. *The inequalities $x_{ij} \le 1$ for $1 \le i < j \le n$ define facets of Q_T^n for all $n \ge 4$.*

Proof. Without restriction of generality we prove the assertion for the inequality $x_{n-1,n} \le 1$.

(a) For $n = 4$ and $n = 5$ one verifies the assertion by enumerating all tours that satisfy $x_{n-1,n} = 1$ and proving that there are $d_4 = 2$ and $d_5 = 5$, respectively, linearly independent tours among them.

(b) Let $n \ge 6$ and $n = 2k + 2$ with $k \ge 2$ and integer. By Lemma 2.1 we know that the complete graph K_{n-2} on the $n-2$ first nodes is the union of $\frac{1}{2}(n-4)$ edge-disjoint $(n-2)$-tours and one perfect matching. Using a similar construction as in Theorem 2.2 we construct n-tours from these $(n-2)$-tours and the perfect matching.

(b1) We start by choosing a particular $(n-2)$-tour $\langle 1 \cdots n-2 \rangle$, say, and construct $n-3$ n-tours by replacing the edge $[j, j+1]$ alternatingly by the chain $[j, n, n-1, j+1]$ or by the chain $[j, n-1, n, j+1]$ starting with the first chain. We remember for later reference the $n-3$ edges $[n-1, 2]$, $[n, 3]$, $[n-1, 4], \dots, [n-1, n-2]$ that are contained in the $n-3$ n-tours obtained this way.

(b2) From the same $(n-2)$-tour used in (b1) we obtain $n-3$ different n-tours by interchanging n and $n-1$. Furthermore, we add the tour $\langle 1, \dots, n-2, n, n-1 \rangle$. Note that the edges marked under (b1) do not occur in these $n-2$ tours.

(b3) From the remaining $\frac{1}{2}(n-4)-1$ tours of length $n-2$ and the one matching that has been completed arbitrarily to a $(n-2)$-tour we construct n-tours by replacing each edge $[i, j]$ by the chain $[i, n-1, n, j]$ like in the proof of Theorem 2.2. (In case of the tour obtained from the perfect matching, we use only the edges of the perfect matching.)

This way we have obtained $n-3+n-2+(\frac{1}{2}(n-4)-1)(n-2)+\frac{1}{2}(n-2) = \frac{1}{2}n(n-3)$ tours in K_n which all contain the edge $[n-1, n]$. Let D be the $d_n \times m$ matrix whose rows correspond to these n-tours in the order as listed. We order the columns of D as follows: The $n-3$ first columns are those corresponding to the edges listed under (b1). All other columns are ordered like in the proof of Theorem 2.2. We next delete all columns of D given by the edges

$$\begin{array}{lll} [1, n] \ \text{ and } \ [n-1, n], & & \\ [k, n], & 2 \le k \le n-2 & \text{for even } k, \\ [k, n-1], & 1 \le k \le n-2 & \text{for odd } k. \end{array} \tag{3.1}$$

The matrix D' that results is of size $d_n \times d_n$ and has the form

$$D' = \begin{pmatrix} P & 0 \\ R & S \end{pmatrix}, \tag{3.2}$$

where S is the square, block-triangular matrix of the same general form of the matrix N that was found in the proof of Theorem 2.2 and P is given by

$$P = \begin{pmatrix} I' & N' \\ 0 & E-I \end{pmatrix}. \tag{3.3}$$

Here the matrix $E - I$ is of size $(n-2) \times (n-2)$ and I' is the band-matrix of size $(n-3) \times (n-3)$ given by

$$I' = \begin{pmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & \cdot & \cdot \\ & & & 1 & 1 \end{pmatrix}. \tag{3.4}$$

Consequently, since both P and S are nonsingular, the matrix D has full row rank.

(c) The case $n = 2k + 1$ for $k \geq 3$ and integer is entirely analogous to part (b) except that its proof does not require separate consideration of a perfect matching in K_{n-2}.

Theorem 3.2. *The inequalities $x_{ij} \geq 0$ for $1 \leq i < j \leq n$ define facets of Q_T^n for all $n \geq 5$.*

Proof. We show without restriction of generality that $x_{n-2,n-1} \geq 0$ defines a facet of Q_T^n for all $n \geq 7$. For $n = 5$ and $n = 6$, the assertion follows by enumeration.

(a) Using Theorem 3.1 with n replaced by $n-1$ we know that in K_{n-1} there exist $d_{n-1} = \frac{1}{2}(n-1)(n-4)$ linearly independent $(n-1)$-tours which all contain the edge $[n-2, n-1]$. In every one of these d_{n-1} tours we replace this edge by the chain $[n-2, n, n-1]$. Labelling the nodes $n-1$ and n of the proof of Theorem 3.1 by $n-2$ and $n-1$, respectively, it follows from (3.1) that these n-tours are linearly independent. Furthermore, it follows that the columns corresponding to the edges $[1, n-1]$, $[1, n-2]$ and $[2, n-1]$ are not contained in the nonsingular square matrix D' denoted by (3.2).

(b) We choose now an arbitrary $(n-1)$-tour containing the chain $[n-2, 1, n-1, 2]$ and replace this chain by $[n-2, 1, n-1, n, 2]$, $[n-2, 1, n, n-1, 2]$ and $[n-2, n, 1, n-1, 2]$, respectively. From the matrix D' given by (3.2) we construct the following $(d_{n-1}+3) \times (d_{n-1}+3)$-matrix N by adjoining the three columns corresponding to the edges $[2, n]$, $[n-1, n]$ and $[1, n]$ and the corresponding part of the incidence vectors of the three tours as rows:

$$N=\begin{pmatrix} & \vdots & \vdots & \vdots \\ D' & 0 & 1 & 0 \\ & \vdots & \vdots & \vdots \\ q & 1 & 1 & 0 \\ q & 0 & 1 & 1 \\ q & 0 & 0 & 1 \end{pmatrix} =: \begin{pmatrix} D' & R \\ Q & P \end{pmatrix}. \tag{3.5}$$

Due to the last sentence under (a) the three last rows of N are identical in their d_{n-1} first components. We note that

$$|N| = |P|\,|D' - RP^{-1}Q| = |D'|, \tag{3.6}$$

since $|P| = 1$, R consists of two zero vectors and a vector of all ones and the matrix Q consists of three identical rows. We have now $d_n - (n-5)$ linearly independent n-tours satisfying $x_{n-2,n-1} = 0$.

(c) We choose any $n-5$ n-tours which contain exactly one of the edges $[i, n]$ where $3 \le i \le n-3$. This can be done in K_n such that none of the $n-5$ n-tours contains the edge $[n-2, n-1]$. Consequently, the matrix N given by (3.5) can be completed to a nonsingular matrix of size $d_n \times d_n$ by adjoining these $n-5$ tours and the columns corresponding to the edges $[i, n]$ for $i = 3, \ldots, n-3$.

Remark 3.3. For $n = 4$, none of the inequalities $x_{ij} \ge 0$ defines a facet and for a complete, irredundant description of Q_T^4 only e.g. the three inequalities $x_{12} \le 1$, $x_{13} \le 1$ and $x_{14} \le 1$ of Q_A^4 are needed, i.e. the other inequalities of Q_A^4 are superfluous. If $n = 5$, all trivial facets are needed and $Q_T^5 = Q_A^5$, see also [26].

Theorem 3.4. *Let $n \ge 6$ and $\{u, v, w, u_1, v_1, w_1\} \subseteq V$. Let $W_0 = \{u, v, w\}$, $W_1 = \{u, u_1\}$, $W_2 = \{v, v_1\}$, and $W_3 = \{w, w_1\}$. Then the comb inequality*

$$x_{uv} + x_{uw} + x_{vw} + x_{uu_1} + x_{vv_1} + x_{ww_1} \le 4$$

defines a facet of Q_T^n.

Proof. Without restriction of generality let $u = n$, $v = n-1$, $w = n-2$, $u_1 = n-4$, $v_1 = n-3$ and $w_1 = 1$. We will prove that this comb defines a facet of Q_T^n for all $n \ge 9$. For $n = 6$, $n = 7$ and $n = 8$ the assertion follows by enumeration.

(a) Using Theorem 3.1 with n replaced by $n-3$ we know that in K_{n-3} there exist $d_{n-3} = d_n - 3(n-3)$ linearly independent $(n-3)$-tours which all contain the edge $[n-4, n-3]$. In every one of these d_{n-3} tours we replace this edge by the chain $[n-4, n, n-2, n-1, n-3]$. Labelling the nodes $n-1$ and n of the proof of Theorem 3.1 by $n-4$ and $n-3$, respectively, it follows from (3.1) that these n-tours are linearly independent. Furthermore, it follows from (3.1) that the columns corresponding to the edges $[1, n-3]$ and $[1, n-4]$ are not contained in the nonsingular square submatrix D' denoted by (3.2).

(b) We choose now any $(n-3)$-tour containing the chain $[n-4, 1, n-3]$ and replace this chain by $[n-4, 1, n-2, n, n-1, n-3]$ and $[n-4, n, n-1, n-2, 1, n-3]$, respectively. From the matrix D' given by (3.2) we construct the following $(d_{n-3}+2)\times(d_{n-3}+2)$-matrix N' by adjoining the two columns corresponding to the edges $[n-4, n]$ and $[1, n-2]$ and the corresponding part of the incidence vectors of the two tours as rows:

$$N' = \begin{pmatrix} & \vdots & \vdots \\ D' & 1 & 0 \\ & \vdots & \vdots \\ q & 1 & 1 \\ q & 0 & 1 \end{pmatrix}. \tag{3.7}$$

Due to the last sentence under (a) the two last rows of N' are identical in their d_{n-3} first components. From the nonsingularity of D' it follows readily that N' is nonsingular. Note that the column corresponding to the edge $[n-4, n-3]$ is not contained in the matrix N'.

(c) We choose now any $(n-3)$-tour containing the chain $[n-4, n-3, 1]$, replace it by $[n-4, n-3, n-1, n, n-2, 1]$ and adjoin the corresponding part of the incidence vector of this tour as row and the column corresponding to the edge $[n-4, n-3]$ as column to N'. The resulting matrix is denoted N. N is of size $(d_{n-3}+3)\times(d_{n-3}+3)$ and nonsingular.

(d) We construct next $3(n-3)-3$ n-tours containing all three edges contained in $E' = \{[n-4, n], [n-3, n-1], [1, n-2]\}$, exactly one of the edges $[n-1, n]$, $[n-2, n-1]$ or $[n-2, n]$ and exactly one of the edges $[i, j]$ not already counted with $i \in \{1, \dots, n-3\}$ and $j \in \{n-2, n-1, n\}$. We append the corresponding part of the incidence vectors of these tours to the matrix N as rows and the new columns corresponding to the edges $[i, j] \in E''$, where $E'' = \{[i, j] \mid i \in \{1, \dots, n-3\}, j \in \{n-2, n-1, n\}\} - E'$. The resulting matrix is of size $d_n \times d_n$, nonsingular and a submatrix of all incidence vectors of tours satisfying the comb-inequality with equality.

Remark 3.5. For a complete and irredundant description of Q_T^6 one has to intersect Q_A^6 with the 10 subtour-elimination constraints on all node-sets of cardinality 3 and the 60 comb-inequalities that are possible in K_6. This is stated in [26] as well as the following: Q_T^7 is defined by a total of 2177 inequalities in addition to the seven equations. Among these there are 840 comb-inequalities as defined in Theorem 3.4 as well as all remaining 1260 comb-inequalities that are possible in K_7. For $n = 8$ the number of inequalities that we know already is astronomical and no explicit characterization of Q_T^8 is known to date.

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