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ON THE SYMMETRIC TRAVELLING SALESMAN PROBLEM II: LIFTING THEOREMS AND FACETS

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Four lifting theorems are derived for the symmetric travelling salesman polytope. They provide constructions and state conditions under which a linear inequality which defines a facet of the n-city travelling salesman polytope retains its facetial property for the $(n+m)$-city travelling salesman polytope, where $m \geq 1$ is an arbitrary integer. In particular, they permit a proof that all subtour-elimination as well as comb inequalities define facets of the convex hull of tours of the n-city travelling salesman problem, where n is an arbitrary integer.

Key words: Linear Inequalities, Convex Polytopes, Facets, Lifting Theorems, Travelling Salesman Problem.

This is the second part of a two-part paper addressing itself to the facial structure of the symmetric travelling salesman problem. All definitions and notational conventions of the first paper ("On the symmetric travelling salesman I: Inequalities" [3]) apply unchanged. In this paper we prove four lifting theorems for Q_T^n, the convex hull of tours of the n-city travelling salesman problem, and obtain by application of these theorems the result that all subtour-elimination as well as comb inequalities (including the special Chvátal-combs, see [1]) define facets of Q_T^n. As we have demonstrated in Section 1 of the first paper, the number of linear inequalities that we thereby know are necessary to linearly describe Q_T^n is incredibly large and by far exceeds previous expectations based on the exponentially growing number of subtour-elimination constraints. Yet in at least two recent computational studies [2, 5] it has been found empirically that in order to prove optimality or near-optimality in large-scale travelling salesman problems a truely small number of inequalities of this type suffices.

4. Lifting theorem I for the travelling salesman polytope

While in the first paper we have used a "direct" method to prove that certain inequalities define facets of Q_T^n, a direct method of proof becomes rather impossible as combinatorially more complicated types of inequalities are considered. Since the complete graph K_{n+1} differs from K_n by "only" one node and n edges, it is, however, reasonable to ask for conditions under which an inequality $ax \le a_0$ which defines a facet for Q_T^n retains its facetial property if it is "somehow" lifted to a valid inequality $a^*x^* \le a_0^*$ for Q_T^{n+1}. We prove next several lifting theorems which spell out sufficient conditions under which such a proceeding is possible. These lifting theorems are not only powerful enough to prove that all subtour-elimination and comb-inequalities define facets of Q_T^n, but they also require relatively few assumptions about (a, a_0) and allow conclusions for combinatorially even more complicated types of inequalities. Since the proofs of these theorems require a number of auxiliary lemmata we first define several symbols that will be used consistently throughout the rest of the paper in addition to those defined in the introduction of the first paper [3].

Definition 4.0. (1) $ax \le a_0$ is an arbitrary, but fixed valid inequality that defines a *facet* of Q_T^n, satisfies $a_e \ge 0$ for all $e \in E$ and $a_e = \alpha > 0$ for some $e \in E$. (Note that $x \in Q_T^n$ implies $x(E) = n$ and thus $a \ge 0$ is not restrictive at all.)

(2) $G_a = (V_a, E_a)$ is the partial (sub-)graph induced by the nonzero components of a, i.e. $E_a = \{e \in E \mid a_e > 0\}$ and with node set $V_a = V$, i.e. isolated nodes are permitted.

(3) (W, C) denotes a *clique* in G_a, i.e. a maximal complete subgraph of G_a which is not a single node.

(4) $Z = \{w \in W \mid a_{wi} = 0 \; \forall \, i \in V - W\}$, i.e. Z is the subset of nodes of W whose nodes are not connected by a single edge of G_a to nodes in $V - W$.

(5) $Y = \{v \in V - W \mid \exists \, w \in W - Z$ such that $a_{vw} = 0\}$. If $W = Z$ we define $Y = V - W$.

(6) $E(a, X) = \{[i, j] \in E_a \mid \{i, j\} \cap X \neq \emptyset\}$ where X is a subset of W. Note that by definition and since W is the node-set of a clique in G_a, $E(X)$ is a proper subset of $E(a, X)$.

(7) $H(a) = \{x \in Q_T^n \mid ax = a_0\}$ and $H = \{T \in T_n \mid ax^T = a_0\}$. If $bx \le b_0$ is a valid inequality, then $H(b)$ is defined analogously to $H(a)$.

(8) All symbols with a star * pertain to the higher-dimensional polytope $Q_T^{n^*}$ under consideration where $n^* > n$. Thus, for instance for $n^* = n + 1$, A^* is the node-edge incidence matrix of K_{n+1} and has the general form

$$A^* = \begin{pmatrix} A & & I_n & \\ 0 & 1 & \cdots & 1 \end{pmatrix},$$

where A is node-edge incidence matrix of K_n and I_n the $n \times n$ identity matrix. Likewise $a \in \mathbf{R}^m$ is extended to $a^* \in \mathbf{R}^{m^*}$ in the natural way e.g. in the case $n^* = n + 1$ by adding components $a^*_{i,n+1}$ for $i = 1, \ldots, n$ at the "end" of a. If $n^* > n + 1$, then we proceed likewise by first adjoining $n + 1$, then $n + 2$, etc.

The next lemma is a useful characterization of a facet of Q_T^n. It is probably a classical result about polytopes, but we have been unable to locate an appropriate reference.

Lemma 4.1. *Let $cx \le c_0$ be a valid inequality for Q_T^n satisfying $H(c) \ne \emptyset$. The following two statements are equivalent:*

(i) *$H(c)$ is a facet of Q_T^n.*

(ii) *For every valid inequality $bx \le b_0$ satisfying $H(b) \supseteq H(c)$ and $H(b) \ne Q_T^n$, there exist $\lambda \in \mathbf{R}^n$ and $\pi \ne 0$ such that $\lambda A + \pi c = b$.*

Proof. If (i) holds, then $\lambda A = c$ is impossible. Furthermore, $H(b) = H(c)$ holds. If the statement (ii) is false, then the matrix obtained from A by adjoining the vectors b and c has row rank $n+2$ and hence

$$\dim H(c) \le \tfrac{1}{2}n(n-1) - (n+2) = \dim Q_T^n - 2,$$

is a contradiction to (i). If (ii) holds, choose $bx \le b_0$ such that $H(b)$ is a facet of Q_T^n. But then $\lambda A + \pi c = b$, $\pi \ne 0$ and $H(c) \ne \emptyset$ imply $H(c) = H(b)$.

The following remark which we will refer to as *Argument A* permits one to assign to certain components of a valid inequality $cx \le c_0$ arbitrarily chosen numerical values without affecting $H(c)$ and will be used repeatedly in the theorems to follow.

Remark 4.2 (Argument A). Let $u \in V$, let $f \in E - \omega(u)$ and define $F = \omega(u) \cup \{f\}$. If $cx \le c_0$ is a valid inequality for Q_T^n and α_e for all $e \in F$ are arbitrary real numbers, then there exist $b \in \mathbf{R}^m$ and a ***unique*** $\mu \in \mathbf{R}^n$ such that $b = c + \mu A$ and $b_e = \alpha_e$ for all $e \in F$. Consequently, setting $b_0 = c_0 + 2\sum_{j=1}^n \mu_j$, it follows that $H(b) = H(c)$ holds.

Proof. After a reordering of its rows and columns the $n \times n$ submatrix B of A with columns indices in F has the general form

$$B = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & & & \\ 1 & & 1 & 0 & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Clearly, $|B| = \pm 2$ and thus there is a unique $\bar{\mu}$ satisfying $\bar{\mu}B = \alpha_F - c_F$, where c_F is the vector with components c_e of c with $e \in F$ and α_F is the vector with components α_e, $e \in F$, (in the same ordering that is implied by B). Thus defining $b = c + \bar{\mu}A$, the assertion follows.

The following remark which we will refer to as *Argument B* permits one to "locally" solve the system of equations $b = \pi c + \lambda A$ for certain components of λ and will be used repeatedly in both the following lemmata and theorems.

Remark 4.3 (Argument B). Given any $b \in \mathbf{R}^m$ and $c \in \mathbf{R}^m$, consider a triple u, v and w of mutually distinct nodes in V. Then the system of equations $b = \pi c + \lambda A$ implies

$$\begin{aligned} \lambda_u + \lambda_v \qquad + \pi c_{uv} &= b_{uv}, \\ \lambda_u \qquad + \lambda_w + \pi c_{uw} &= b_{uw}, \\ \lambda_v + \lambda_w + \pi c_{vw} &= b_{vw} \end{aligned}$$

and consequently,

$$\begin{aligned} \lambda_u &= \tfrac{1}{2}(b_{uv} + b_{uw} - b_{vw} - \pi(c_{uv} + c_{uw} - c_{vw})), \\ \lambda_v &= \tfrac{1}{2}(b_{uv} - b_{uw} + b_{vw} - \pi(c_{uv} - c_{uw} + c_{vw})), \\ \lambda_w &= \tfrac{1}{2}(-b_{uv} + b_{uw} + b_{vw} - \pi(-c_{uv} + c_{uw} + c_{vw})). \end{aligned}$$

In the actual applications of Argument B, one always knows some or all of the numerical values of the quantities c_{uv} etc. and thus one can compute the vector λ "locally" for the respective components.

The following lemmata and their corollaries concern properties of the tours that are contained in a particular facet $H(a)$ of Q_T^n. These properties, such as the containment of a particular edge in a tour $T \in H$, permit us later to carry out the "inductive" step in the lifting theorems.

Lemma 4.4. *If $|W| \geq 3$, $|Z| \geq 2$ and $a_e = \alpha$ for all $e \in E(a, Z)$ hold, then for every $w \in Z$ and for every $i \in V - Z$ there exists a tour $T \in H$ such that $[i, w] \in T$.*

Proof. For any $w \in V$ and $i \in V$ there can be at most one edge $[w, i] \notin T$ for all $T \in H$, since otherwise $H(a)$ is contained in the intersection of two facets $x_e = 0$ and $x_f = 0$ of Q_T^n, contradicting the assumption that $H(a)$ is a facet of Q_T^n. Suppose now that for some $w \in Z$ and $i \in V - Z$, $[w, i] \notin T$ holds for all $T \in H$. Define $bx = x_{wi}$; then $H(a) = \{x \in Q_T^n \mid bx = 0\}$ holds and it follows from Lemma 4.1 that, for some $\lambda \in \mathbf{R}^n$ and $\pi \neq 0$, $\lambda A + \pi a = b$.

(a) Suppose that $i \in V - W$ holds. By assumption there exist $v \in Z - w$ and $u \in W - \{v, w\}$. Applying Argument B to the triple u, v and w we get $\lambda_u = \lambda_v = \lambda_w = -\frac{1}{2}\pi\alpha$. From $\lambda_w + \lambda_i = 1$ it follows that $\lambda_i = 1 + \frac{1}{2}\pi\alpha$. On the other hand, since $v \in Z$ it follows that $a_{vi} = 0$ and hence $\lambda_v + \lambda_i = 0$. Consequently, $\lambda_i = \frac{1}{2}\pi\alpha$. Contradiction.

(b) Suppose that $i \in W - Z$ holds. By assumption there exists $v \in Z - w$. Applying Argument B to the triple w, i and v we get $\lambda_i = \lambda_w = \frac{1}{2}(1 - \pi\alpha)$ and $\lambda_v = -\frac{1}{2}(1 + \pi\alpha)$. Since $W - Z \neq \emptyset$, it follows that $V - W \neq \emptyset$. Let $j \in V - W$, then $a_{wj} = a_{vj} = 0$. Consequently, $\lambda_j + \lambda_w = 0$ implies $\lambda_j = \frac{1}{2}(\pi\alpha - 1)$, but $\lambda_j + \lambda_v = 0$ implies $\lambda_j = \frac{1}{2}(1 + \pi\alpha)$. Contradiction.

Corollary 4.5. *If $|W| \geq 3$, $|Z| \geq 2$, $a_e = \alpha$ for all $e \in E(a, Z)$ and $a_e \geq \alpha$ for all*

$e \in E(W-Z)$ hold, then for every pair of nodes v, $w \in Z$ and for every node $i \in V-Z$ there exists a tour $T \in H$ containing the chain $[v, \ldots, w, i]$, such that $[v, \ldots, w]$ is a hamiltonian chain in $(Z, E(Z))$.

Proof. Let v, $w \in Z$ and $i \in V-Z$. By Lemma 4.4 there exists a tour $T' \in H$ such that $[w, i] \in T'$. Let $u \in Z - \{v, w\}$ and $u = v$ if $|Z| = 2$. If $[u, w, i] \notin T'$ holds, i.e. if T' has the form $\langle ju \cdots pkwi \cdots q\rangle$, we construct a new tour $T = \langle jkp \cdots uwi \cdots q\rangle$ by the indicated interchange. Then $ax^T - ax^{T'} = a_{uw} + a_{kj} - a_{wk} - a_{uj} \geq 0$ follows from our assumption about the coefficients, since $a_{wk} > 0$ implies $k \in Z$ and $a_{uj} > 0$ implies $j \in Z$. Consequently, $ax^T = a_0$, i.e. $T \in H$, and T contains the chain $[u, w, i]$. We repeat the argument working with node u and adjoining any node of $Z - \{v, w, u\}$ and finally the node v.

Remark 4.6. With the assumptions of Corollary 4.5 the following propositions follow from Corollary 4.5:

(i) *For every triple $u, v, w \in Z$ there exists a tour $T \in H$ which contains the chain $[u, v, w]$.*

(ii) *For every pair $v, w \in Z$ and for every $i \in V-Z$ there exists a tour $T \in H$ which contains the chain $[v, w, i]$.*

Lemma 4.7. *If $W = Z = \{v, w\}$, then for every $i \in V - W$ there exists a tour $T \in H$ which contains the chain $[v, w, i]$.*

Proof. Suppose there exists a tour $T' \in H$ such that $[v, w] \notin T'$. Let $T' = \langle jvk \cdots pw \cdots q\rangle$ and construct a new tour $T = \langle jp \cdots kvw \cdots q\rangle$ by the indicated interchange. By assumption $a_{vj} = a_{wp} = 0$ and consequently, $ax^T - ax^{T'} = a_{vw} + a_{jp} - a_{vj} - a_{wp} \geq \alpha$. Since $\alpha > 0$ and $T' \in H$, we have a contradiction. Consequently, $H(a) \subseteq \{x \in Q_T^n \mid x_{vw} = 1\}$. Since both $H(a)$ and $x_{vw} = 1$ are facets of Q_T^n, they are identical. For the facet $x_{vw} = 1$, the assertion of the lemma is trivially true.

Lemma 4.8. *If $W = \{v, w\}$, $Z = \{w\}$, $|Y| \geq 2$ and $a_{vj} = \alpha$ or $a_{vj} = 0$ for all $j \in V$ hold, then for every $i \in V - W$ there exists a tour $T \in H$ which contains $[w, i]$.*

Proof. Suppose that for some $i \in V - W$, $[w, i] \notin T$ holds for all $T \in H$. (There can be at most one such i, see the proof of Lemma 4.4.) Define $bx = x_{wi}$; then $H(a) = \{x \in Q_T^n \mid bx = 0\}$ and by Lemma 4.1 there exist $\lambda \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\lambda A + \pi a = b$ holds. Since $|Y| \geq 2$, there exists $j \in Y$, $j \neq i$, such that $a_{vj} = 0$, and since $w \in Z$, $a_{wj} = 0$ holds. Applying Argument B to the triple v, w and j we get $\lambda_v = \lambda_w = -\frac{1}{2}\pi\alpha$ and $\lambda_j = \frac{1}{2}\pi\alpha$. Since $\lambda_w + \lambda_i = 1$, it follows that $\lambda_i = 1 + \frac{1}{2}\pi\alpha$. If $a_{vi} = 0$ holds, then $\lambda_v + \lambda_i = -\frac{1}{2}\pi\alpha + 1 + \frac{1}{2}\pi\alpha = 1$ contradicts $\lambda_v + \lambda_i = 0$. If $a_{vi} = \alpha$ holds, i.e. $i \notin Y$, then $0 = \pi\alpha + \lambda_v + \lambda_i$ implies $\pi\alpha = -1$ and hence $\pi < 0$, since

$\alpha > 0$. Now choose $k \in Y$, $k \neq j$, and compute λ_k from $\lambda_w + \lambda_k = 0$ to be $\lambda_k = \frac{1}{2}\pi\alpha$. Since $\lambda_k + \lambda_j + \pi a_{ij} = 0$, it follows that $\pi a_{ij} = 1$, contradicting $a_{ij} \geq 0$ and $\pi < 0$.

Corollary 4.9. *With the assumptions of Lemma 4.8 the following propositions hold:*

(i) *For every $i \in V - W$ there exists a tour $T \in H$ which contains the chain $[v, w, i]$.*

(ii) *There exist $j \in Y$ and a tour $T \in H$ which contains the chain $[w, v, j]$.*

Proof. (i) Let $i \in V - W$. By Lemma 4.8 there exists a tour $T' \in H$ such that $[w, i] \in T'$. Suppose $[v, w] \notin T'$ and let $T' = \langle jwi \cdots kv \cdots q\rangle$. Construct a new tour $T = \langle jk \cdots iwv \cdots q\rangle$ by the indicated interchange. Since $a_{vk} \leq \alpha$ and $a_{jk} \geq 0$ it follows that $ax^T - ax^{T'} = a_{vw} + a_{jk} - a_{wj} - a_{vk} = \alpha + a_{jk} - a_{vk} \geq 0$ and hence $T \in H$.
(ii) Since $|Y| \geq 2$ holds, there exist $j \in Y$ and a tour $T' \in H$ which contains the edge $[v, j]$, since otherwise $H(a)$ were contained in the intersection of two facets $x_e = 0$ and $x_f = 0$, say. Suppose $[w, v] \notin T'$ and let $T' = \langle ivj \cdots kw \cdots q\rangle$. Construct a new tour $T = \langle ik \cdots jvw \cdots q\rangle$ by the indicated interchange. Like under (i) we conclude that $T \in H$.

Lemma 4.10. *If $|W| \geq 3$, $Z = \{w\}$, $|Y| \geq 2$ and $a_{jw} = \alpha$ for all $j \in W - Z$ hold, then for every $i \in V - w$ there exists a tour $T \in H$ which contains the edge $[w, i]$.*

Proof. Suppose that for some $i \in V - W$, $[w, i] \notin T$ holds for all $T \in H$. (There can be at most one such i, see the proof of Lemma 4.4.) Define $bx = x_{wi}$; then $H(a) = \{x \in Q_T^n \mid bx = 0\}$ and by Lemma 4.1 there exist $\lambda \in \mathbf{R}^n$ *and* $\pi \neq 0$ such that $\lambda A + \pi a = b$ holds.

(a) Suppose that $i \in V - W$ holds. Let $u, v \in W - w$, $a_{uv} = \beta$, and apply Argument B to the triple u, v and w. It follows that $\lambda_u = \lambda_v = -\frac{1}{2}\pi\beta$ and $\lambda_w = \frac{1}{2}\pi\beta - \pi\alpha$. Since u and v are arbitrary nodes in $W - w$ it follows also that $\lambda_u = -\frac{1}{2}\pi\beta$ holds for all $u \in W - w$. Furthermore since $\lambda_w + \lambda_i = 1$, it follows that $\lambda_i = 1 + \pi\alpha - \frac{1}{2}\pi\beta$. Choose now $j \in Y$, $j \neq i$, such that $a_{vj} = 0$ holds for some $v \in W - w$. Since $\lambda_w + \lambda_j = 0$ holds, it follows that $\lambda_j = \pi\alpha - \frac{1}{2}\pi\beta$. Consequently, since $\lambda_v + \lambda_j = 0$, it follows from $\pi \neq 0$ that $\alpha = \beta$. If $i \in Y$ holds, then $\lambda_u + \lambda_i = 0$ for some $u \in W - w$, which is impossible for the calculated values of λ_u and λ_i. If $i \notin Y$ holds, i.e. $a_{ui} > 0$ for all $u \in W - w$, then choose $k \in Y, k \neq j$, and compute $\lambda_k = \frac{1}{2}\pi\alpha$ from $\lambda_k + \lambda_w = 0$. But then $\lambda_k + \lambda_j + \pi a_{kj} = 0$ implies $\pi(\alpha + a_{kj}) = 0$ and hence, $\pi \neq 0$ contradicts $\alpha > 0$ and $a_{kj} \geq 0$.

(b) Suppose $i \in W - w$. Choose $j \in Y$ and $v \in W - w$ such that $a_{vj} = 0$. If $v \neq i$ holds, apply Argument B to the triple w, v and j. It follows that $\lambda_v = \lambda_w = -\frac{1}{2}\pi\alpha$ and $\lambda_j = \frac{1}{2}\pi\alpha$. If $v = i$ holds, applying Argument B to the triple w, $v = i$ and j yields $\lambda_v = \lambda_w = \frac{1}{2}(1 - \pi\alpha)$ and $\lambda_j = -\frac{1}{2}(2 - \pi\alpha)$. In any case, since $v \notin Z$ holds, it follows that there exists $k \in V - W$ such that $a_{vk} > 0$. From $\lambda_k + \lambda_w = 0$ it follows that $\lambda_k = -\lambda_v$. But then $\lambda_v + \lambda_k + \pi a_{vk} = 0$ implies $\pi a_{vk} = 0$, contradicting $\pi \neq 0$.

Corollary 4.11. *If* $|W| \geq 3$, $Z = \{w\}$, $|Y| \geq 2$ *and* $a_e = \alpha$ *for all* $e \in E(a, W)$ *hold, then the following propositions hold*:

(i) *For every node* $i \in V - w$ *there exist a node* $v \in W - w$, $v \neq i$, *and a tour* $T \in H$ *which contains the chain* $[v, w, i]$.

(ii) *If for* $v \in W - w$ *there exist a node* $i \in V - W$ *and a tour* $S \in H$ *containing the chain* $[v, w, i]$, *then for every node* $u \in W - \{w, v\}$ *there exists a tour* $T \in H$ *which contains the chain* $[v, w, u]$.

Proof. (i) By Lemma 4.10 there exists for every $i \in V - w$ a tour $T' \in H$ containing $[w, i]$. If T' contains a chain $[k, w, i]$ with $k \notin W - w$, let $T' = \langle kwi \cdots jv \cdots p\rangle$ where $v \in W - w$. Then we construct a new tour $T = \langle kj \cdots iwv \cdots p\rangle$ by the indicated interchange. Since $a_{wv} = \alpha$, $a_{wk} = 0$, $a_{vj} \leq \alpha$ and $a_{kj} \geq 0$ hold, it follows that $ax^T - ax^{T'} \geq 0$ and thus $T \in H$.

(ii) Let $S = \langle vwi \cdots kuj \cdots p\rangle$. Then we construct the new tour $T = \langle vwuk \cdots ij \cdots p\rangle$ by the indicated interchange. Since $a_{wi} = 0$, $a_{uj} \leq \alpha$, $a_{ij} \geq 0$ and $a_{wu} = \alpha$ hold, it follows that $ax^T - ax^S \geq 0$ and thus $T \in H$.

With these preparations we prove now the first lifting theorem for the travelling salesman problem which relates facets of Q_T^n and Q_T^{n+1}.

Theorem 4.12. *Let* $ax \leq a_0$ *be a facet of* Q_T^n *satisfying* $a \geq 0$ *and let* (W, C) *be a clique in* $G_a = (V_a, E_a)$. *Suppose that one of the following two conditions is satisfied*:

(i) $|Z| \geq 2$; $a_e = \alpha$ *for all* $e \in E(a, Z)$ *and* $a_e \geq \alpha\ \forall e \in E(W - Z)$ *or*,

(ii) $|Z| = 1$, $|Y| \geq 2$, $a_e = \alpha$ *for all* $e \in E(a, W)$.

If (a^*, a_0^*) *is defined by* $a_0^* = a_0 + \alpha$, $a_e^* = a_e$ *for all* $e \in E$, $a_e^* = \alpha$ *for all* $e = [1, n+1]$ *with* $i \in W$ *and* $a_e^* = 0$ *otherwise, then* $a^*x^* \leq a_0^*$ *is a facet of* Q_T^{n+1}.

Proof. Let $H(a^*) = \{x^* \in Q_T^{n+1} \mid a^*x^* = a_0^*\}$ and $H^* = \{T \in T_{n+1} \mid x^T \in H(a^*)\}$. From the definition of (a^*, a_0^*) it follows immediately that $H(a^*)$ is a proper face of Q_T^{n+1}. Let $b^*x^* \leq b_0^*$ be any valid inequality defining a facet of Q_T^{n+1} and satisfying $H(b^*) \supseteq H(a^*)$, where $H(b^*)$ is defined analogously to $H(a^*)$. We want to prove that there exist $\lambda^* \in \mathbf{R}^{n+1}$ and $\pi \neq 0$ such that $\lambda^*A^* + \pi a^* = b^*$ holds, where A^* is the node-edge incidence matrix of K_{n+1}. Without restriction of generality we can assume by Argument A (applied with $u = n+1$) that

$$b^*_{i,n+1} = \alpha\ \forall i \in W, \qquad b^*_{i,n+1} = 0 \quad \forall i \in V - W \quad \text{and} \quad b^*_{vw} = \alpha \quad \text{for some } v, w \in V. \tag{4.1}$$

We proceed by calculating further components of b^* using the assumptions of the theorem.

(a) Suppose that $|Z| \geq 2$ and let v, w be two arbitrary nodes in Z. By Remark 4.6 and Lemma 4.7 there exists for every $i \in V - \{v, w\}$ a tour $T \in H$ which contains the chain $[v, w, i]$. Replacing this chain by the chain $[v, w, n+1, i]$ and $[v, n+1, w, i]$, respectively, we get two $(n+1)$-tours T^* and T^{**} which by construction are both contained in H^*. Consequently, $0 = b^*x^{T^*} - b^*x^{T^{**}} = b^*_{vw} + b^*_{i,n+1} - b^*_{v,n+1} - b^*_{wi}$ holds and we obtain from (4.1) that $b^*_{i,n+1} = b^*_{wi}$ where $i \in V - \{v, w\}$. It follows from (4.1) and from the fact that v and w can be interchanged that

$$b^*_{wi} = 0 \quad \forall i \in V - W, \qquad b^*_{wi} = \alpha \quad \forall i \in W - w \quad \text{and}$$
$$b^*_{vi} = 0 \quad \forall i \in V - W. \tag{4.2}$$

(b1) Suppose $Z = \{w\}$ and $W = \{v, w\}$. By Corollary 4.9(i) there exists for every $i \in V - W$ a tour $T \in H$ containing the chain $[v, w, i]$. Like in case(a) we conclude that $b^*_{wi} = 0$ for all $i \in V - W$. By Corollary 4.9(ii) there exists $j \in Y$ and a tour $T \in H$ containing the chain $[w, v, j]$. Since $j \in Y$ and $v \in W - Z$ it follows that $a_{vj} = 0$. Consequently, the two $(n+1)$-tours T^* and T^{**} obtained from T replacing the chain $[w, v, j]$ by the chains $[w, v, n+1, j]$ and $[w, n+1, v, j]$, respectively, are contained in H^*. Hence, $b^*x^{T^*} = b^*x^{T^{**}} = b^*_0$ and taking the difference we obtain $b^*_{vj} = 0$. Consequently, we have in this case

$$b^*_{wi} = 0 \quad \forall i \in V - W, \qquad b_{vw} = \alpha \quad \text{and}$$
$$\exists\, j \in V - W \quad \text{such that } a_{vj} = 0 \quad \text{and} \quad b^*_{vj} = 0. \tag{4.3}$$

(b2) Suppose $Z = \{w\}$ and $|W| \geq 3$. By Corollary 4.11(i) there exists for every $i \in V - W$ a node $v \in W - w$ and a tour $T \in H$ containing the chain $[v, w, i]$. We choose first an arbitrary, fixed $i \in V - W$ and let $v \in W - w$ be the associated node given by Corollary 4.11(i). The node v satisfies the assumptions of Corollary 4.11(ii), and thus for every node $u \in W - \{v, w\}$ there exists a tour $T \in H$ which contains the chain $[v, w, u]$. Constructing two tours $T^*, T^{**} \in H^*$ and taking differences as previously, it follows from (4.1) that $b^*_{wu} = \alpha$ for all $u \in W - w$. Let now i be any node in $V - W$. Then, by repeated application of Corollary 4.11(i), there exists a node $u \in W - w$ and a tour $T \in H$ containing the chain $[u, w, i]$. Since $b^*_{wu} = \alpha$, it follows from (4.1) by the usual procedure as in case (a) that $b^*_{wi} = 0$ for all $i \in V - W$. We want to show that a statement similar to the last part of (4.3) holds in this case as well. Since $|Y| \geq 2$ holds, there are least two edges $[p, i] \neq [u, j]$ with $p, u \in W - w$ and $i \neq j \in Y$ such that $a_{pi} = a_{uj} = 0$. There exists a tour $T \in H$ containing at least one of these two edges, $[u, j]$, say, since otherwise $H(a)$ is contained in the intersection of the two facets $x_{pi} = 0$ and $x_{uj} = 0$. From T we construct a new tour T^* by replacing the edge $[u, j]$ by the chain $[u, n+1, j]$ and, letting q be such that $[w, q] \in T$, a second tour T^{**} by replacing $[w, q]$ by the chain $[w, n+1, q]$. By construction, we have T^*, $T^{**} \in H$ and

$$0 = b^*x^{T^*} - b^*x^{T^{**}} = b^*_{u,n+1} + b^*_{j,n+1} + b^*_{wq} - b^*_{w,n+1} - b^*_{q,n+1} - b^*_{uj} = -b^*_{uj}.$$

(c) In all three cases (a), (b1) and (b2) we have thus proven the following two statements:

$$\exists\, w \in Z \quad \text{such that } b^*_{wi} = 0 \quad \forall i \in V - W \quad \text{and} \quad b^*_{wi} = \alpha \quad \forall i \in W - w. \tag{4.4}$$

$$\exists\, v \in W - w \quad \text{and} \quad j \in V - W \quad \text{such that } a_{vj} = 0 \quad \text{and} \quad b^*_{vj} = 0. \tag{4.5}$$

We define a hyperplane $bx = b_0$ in $\mathbf{R}^m$ by $b_e = b^*_e$ for all $e \in Z$ and $b_0 = b^*_0 - \alpha$. Let T be any tour in H. T contains the edge $[w, j]$ where $w \in Z$ satisfies (4.4) and $j \in V - w$ is arbitrary. We construct $T^* \in H^*$ by replacing the edge $[w, j]$ by the chain $[w, n+1, j]$. Then by (4.1) and (4.4) we have

$$b^* x^{T^*} - b x^T = b^*_{w,n+1} + b^*_{j,n+1} - b^*_{wj} = b^*_{w,n+1} = \alpha.$$

Consequently, $H(a) \subseteq H(b)$ and by Lemma 4.1 there exist $\gamma \in \mathbf{R}^n$ and $\pi \neq 0$ such that $b = \gamma A + \pi a$, if $H(b) \neq Q^n_T$. If $H(b) = Q^n_T$, the same relation holds with $\pi = 0$. In any case, applying Argument B to the triple w, v and j given by (4.4) and (4.5) we get $\gamma_w = \gamma_v = \frac{1}{2}\alpha(1 - \pi)$ and $\gamma_j = -\gamma_w$. Consequently, by (4.4) and the assumptions about a $\gamma_w + \gamma_i = 0$ holds for all $i \in V - W$ and thus $\gamma_i = -\gamma_w$ for all $i \in V - W$. Also, by the same argument, $\gamma_w + \gamma_i + \pi\alpha = \alpha$ holds for all $i \in W - w$ and thus $\gamma_i = \gamma_w$ for all $i \in W - w$. Defining $\lambda^*_j = \gamma_j$ for $j = 1, \ldots, n$ and $\lambda^*_{n+1} = \gamma_w$ it follows that by construction of a^* we have $\lambda^* A^* + \pi a^* = b^*$ and hence, if $\pi = 0$, we get $H(b^*) = Q^{n+1}_T$, which is a contradiction. Consequently, $\pi \neq 0$ and $H(a^*) = H(b^*)$ holds.

The assumptions of Theorem 4.12 require a modicum in terms of knowledge about the coefficients of the facetial inequality $ax \leq a_0$ and thus Theorem 4.12 has a considerably broader spectrum of potential application than, prima facie, its technical assumptions seem to imply.

5. Lifting Theorems II, III and IV for Q^n_T

While the first lifting theorem deals with a single clique of the graph G_a and gives two conditions under which it is possible to substitute in an appropriate way a single node of that clique by two nodes joined by an edge (and thus by repeated application, by a complete subgraph on an arbitrary number of $q \geq 2$ nodes), the following lifting theorems deal with the situation where some or all of the nodes of G_a under consideration are contained in exactly two different cliques of G_a. The second and third lifting theorem require this to be the case for every node in some chosen clique (W, C) of G_a and due to Proposition 3.0, the facets considered in these two lifting theorems are in fact equivalent to certain comb-inequalities. Several of the lemmata preceeding the theorems, however, require fewer assumptions and thus the entire development is carried out at a more general level, especially as we need some of the lemmata later on. The fourth lifting theorem, very much like the first one, requires very little information about the facet to be lifted and shows that certain nodes of G_a can be replaced by a complete graph in quite a different way than in Theorem 4.12 while ensuring that the resulting inequality retains its facetial property. We first define again a few symbols that we will use in addition to those of the Introduction and Definition 4.0 and prove several auxiliary lemmata.

Definition 5.0. (1) (W', C') denotes a clique in G_a different from (W, C). Z', Y', $E(a, X')$ for $X' \subseteq W'$ are defined exactly like the quantities without a prime (see Definition 4.0).

(2) $W \Delta W' = (W - W') \cup (W' - W)$ is the symmetric difference of W and W'.

(3) $w \in W$ satisfies *condition* $S(w)$, if w is contained in exactly one additional 2-element clique (W^w, E^w) of G_a, $W^w = \{w, p\}$, say, and $a_{pj} = 0$ holds for all $j \in V - w$. E^w, of course, is the edge $[w, p]$ and $a_{wp} > 0$.

Lemma 5.1. *Suppose that $w \in V$ is contained in exactly two different cliques W and W' of G_a and that $a_{wi} = \alpha$ for all $i \in W \Delta W'$, $a_{wi} = 0$ for all $i \in V - (W \cup W')$ hold. If there exist two distinct $u, v \in W - W'$ and two distinct $i, j \in V - (W \cup W')$ such that $a_{uj} = a_{vi} = 0$ holds, then there exists for every node $h \in V - (W \cap W')$ a tour $T \in H$ which contains $[w, h]$.*

Proof. Suppose that there exists $h \in V - (W \cap W')$ such that $[w, h] \notin T$ for all $T \in H$. Define $bx = x_{wh}$ and $b_0 = 0$. Then $H(a) = H(b)$ and by Lemma 4.1 there exist $\lambda \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\lambda A + \pi a = b$. By assumption there exist $u \in W - W'$, $u \neq h$, and $j \in V - (W \cup W')$, $j \neq h$, such that $a_{uj} = 0$. Applying Argument B to the triple u, w and j we find $\lambda_w = \lambda_u = \frac{1}{2}\pi\alpha$ and $\lambda_j = \frac{1}{2}\pi\alpha$. Since $W \neq W'$ are cliques in G_a, there exist $v \in W - W'$ and $k \in W' - W$ such that $a_{vk} = 0$. If $k \neq h \neq v$ holds, then from $\lambda_v + \lambda_w + \pi\alpha = 0$ and $\lambda_k + \lambda_w + \pi\alpha = 0$ we get $\lambda_v = \lambda_k = -\frac{1}{2}\pi\alpha$, contradicting $\lambda_v + \lambda_k = 0$. Thus $h = k$ or $h = v$ holds, i.e. $h \in W \Delta W'$. Let $i \in V - (W \cup W')$, $i \neq j$. Since $h \neq i$, $\lambda_w + \lambda_i = 0$ holds and thus $\lambda_i = \frac{1}{2}\pi\alpha$. But then $\lambda_i + \lambda_j + \pi a_{ij} = 0$ yields $\pi(\alpha + a_{ij}) = 0$, a contradiction to $\alpha > 0$ and $a_{ij} \geq 0$.

Lemma 5.2. *Suppose that $|W| \geq 3$ holds and that there exist two distinct $v, w \in W$ satisfying condition $S(v)$, $S(w)$ respectively. If $a_e = \alpha$ for all $e \in E(a, \{v, w\})$ holds, then every tour $T \in H$ contains an edge $[i, j] \in C$ such that $\{i, j\} \cap \{v, w\} \neq \emptyset$.*

Proof. If $T \in H$ contains $[v, w]$ we are done. Else, suppose $T = \langle ivj \cdots hwku \cdots t\rangle$ and $W^w = \{w, p\}$, $W^v = \{v, q\}$.

(a) If only one of the four edges $[i, v]$, $[v, j]$, $[h, w]$ or $[w, k]$ is contained in E_a, say $[i, v] \in E_a$, then construct $T' = \langle ivwh \cdots jku \cdots t\rangle$ by the indicated interchange. Then $ax^{T'} - ax^T = a_{vw} + a_{jk} - a_{vj} - a_{wk} \geq \alpha > 0$, is a contradiction to $T \in H$.

(b) If at least three of the respective four edges are in E_a, then by the assumptions of Lemma 5.2 there must be one of them in C.

(c) Suppose $[v, j]$, $[w, h] \notin E_a$ and $[v, i]$, $[w, k] \in E_a$. If $i \in W$ or $k \in W$ holds, we are done. Otherwise, $i = q$ and $k = p$ and thus $a_{ku} = a_{kj} = 0$. Construct $T' = \langle ivwkj \cdots hu \cdots t \rangle$. Then $ax^{T'} - ax^T = a_{vw} + a_{hu} - a_{vj} - a_{wk} = \alpha + a_{hu} > 0$, is a contradiction to $T \in H$. The case $[v, i]$, $[w, k] \notin E_a$, $[v, j]$, $[w, h] \in E_a$ follows by symmetry.

(d) If either $[v, i]$, $[w, h] \in E_a$, $[v, j]$, $[w, k] \notin E_a$ or $[v, i]$, $[w, h] \notin E_a$, $[v, j]$, $[w, k] \in E_a$ hold, a construction of T' like in case (a) yields the desired contradiction to $T \in H$.

Lemma 5.3. *Suppose that $|W| \geq 3$ and $a_e = \alpha$ for all $e \in C$. Let $T \in H$ contain an edge $[i, j]$ satisfying $a_{ij} = 0$ for some $j \in W$. Then the following propositions hold:*

(i) *For every $v \in W - j$, which satisfies condition $S(v)$, the chain $[u, v, w]$ of T satisfies $u \notin W$ or $w \notin W$.*

(ii) *For every $v \in W - j$, which satisfies condition $S(v)$, the chain $[u, v, w]$ of T satisfies $a_{uv} > 0$ and $a_{vw} > 0$.*

(iii) *If j satisfies condition $S(j)$, then letting $W^j = \{j, s\}$ it follows that $[j, s] \in T$.*

Proof. (i) Suppose that for some $v \in W - j$ the chain $[u, v, w]$ of T satisfies $u, w \in W$. Let $W^v = \{v, q\}$. Consequently, the chain $[k, q, h]$ of T satisfies by assumption $a_{kq} = a_{qh} = 0$. If $u \neq j \neq w$, let w.r.o.g. $T = \langle uvw \cdots ij \cdots hqk \cdots t \rangle$ and construct $T' = \langle uw \cdots ih \cdots jvqk \cdots t \rangle$ by the indicated interchanges. Then $ax^{T'} - ax^T = a_{vq} + a_{hi} > 0$ yields a contradiction to $T \in H$. Suppose next w.r.o.g. that $w = j$ and let $T = \langle uvwi \cdots hqk \cdots t \rangle$ and construct $T' = \langle uwvqh \cdots ik \cdots t \rangle$ by the indicated interchanges. Again, $ax^{T'} - ax^T = a_{vq} + a_{ki} > 0$ contradicts $T \in H$.

(ii) Suppose that for some $v \in W - j$ the chain $[u, v, w]$ violates (ii). Let $W^v = \{v, q\}$ and suppose w.r.o.g that $T = \langle ruvw \cdots ij \cdots t \rangle$ and that $a_{uv} = 0$. Construct $T' = \langle rui \cdots wvj \cdots t \rangle$ by the indicated interchange. Then $ax^{T'} - ax^T = \alpha + a_{ui} > 0$ contradicts $T \in H$. Suppose next that $a_{vw} = 0$. If $u = q$ holds, then $a_{ru} = 0$ and construct $T' = \langle rw \cdots iuvj \cdots t \rangle$. Then $ax^{T'} - ax^T = \alpha + a_{rw} > 0$ contradicts $T \in H$. If $u \neq q$ holds, then the chain $[k, q, h]$ of T satisfies $a_{kq} = a_{qh} = 0$. Let w.r.o.g. $T = \langle ruvw \cdots ij \cdots kqh \cdots t \rangle$ and construct $T' = \langle ruvqk \cdots ji \cdots wh \cdots t \rangle$. Then $ax^{T'} - ax^T = a_{vq} + a_{wh} > 0$ contradicts $T \in H$.

(iii) If (iii) is false, then the chain $[k, s, h]$ of T satisfies by assumption $a_{ks} = a_{sh} = 0$. Let $T = \langle ji \cdots ksh \cdots t \rangle$ and $T' = \langle jsk \cdots ih \cdots t \rangle$. Then $ax^{T'} - ax^T = a_{js} + a_{ih} > 0$ contradicts $T \in H$.

Corollary 5.4. *Suppose that $|W| \geq 3$ and $a_e = \alpha$ for all $e \in E(a, W)$ hold and that every node $w \in W$ satisfies condition $S(w)$. Then the following propositions hold:*

(i) *For every node $u \in W$, for every node $i \in V - (W \cup W^u)$ and for every pair* *of nodes $v, w \in W - u$ there exists a tour $T \in H$ which contains the edges $[u, i]$ and $[v, w]$.*

(ii) *For every triple of nodes $u, v, w \in W$ there exists a tour $T \in H$ which contains the chain $[u, v, w]$.*

Proof. (i) The node $u \in W$ satisfies the assumptions of Lemma 5.1. Consequently, there exists a tour $T' \in H$ containing $[u, i]$. Suppose that $[v, w] \notin T'$ for some pair $v, w \in W - u$. It follows from Lemma 5.3 with $j = u$ that the two chains $[hvk]$ and $[rwt]$ of T' satisfy w.r.o.g. $r \in W$, $t \notin W$ and either $h \in W$, $k \notin W$ or $h \notin W$, $k \in W$. Let $T' = \langle ui \cdots hvk \cdots rwt \cdots s\rangle$ and suppose that $h \in W$, $k \notin W$ hold. Then $T = \langle ui \cdots hr \cdots kvwt \cdots s\rangle$ satisfies $ax^T - ax^{T'} = a_{hr} + a_{vw} - a_{hv} - a_{rw} = 0$, i.e. $T \in H$ and we are done. If $h \notin W$, $k \in W$ hold, then $k = r$ is impossible since otherwise Lemma 5.3(i) is violated for the chain $[vkw]$. Consequently, $T' = \langle ui \cdots hvk \cdots pqrwtl \cdots s\rangle$. The node p exists, because r is contained in a 2-element clique $W^r = \{r, q\}$ and $k \in W$ satisfies $a_{qk} = 0$. The node l exists, because $t \notin W$ and $a_{ui} = a_{ut} = 0$ hold. Consequently, $T = \langle ui \cdots hvwtqrk \cdots pl \cdots s\rangle$ satisfies $ax^T - ax^{T'} = a_{vw} + a_{tq} + a_{rk} + a_{pl} - a_{vk} - a_{pq} - a_{rw} - a_{tl} = a_{pl} \geq 0$. Consequently, $T \in H$ and (i) follows.

(ii) Let $u, v, w \in W$ be distinct nodes and $i \in V - (W \cup W^u)$. By part (i) there exists a tour $T' \in H$ containing both $[u, i]$ and $[v, w]$. If $T' = \langle ui \cdots wvp \cdots s\rangle$, then $T = \langle uvw \cdots ip \cdots s\rangle$ satisfies $T \in H$, since $ax^T - ax^{T'} = a_{uv} + a_{ip} - a_{ui} - a_{vp} \geq a_{ip} \geq 0$. If $T' = \langle ui \cdots qvwpr \cdots s\rangle$, then by Lemma 5.3(i) applied to node w it follows that $P \notin W$. Hence, by Lemma 5.3(ii), $p \in W^w - W$ and thus $a_{pr} = 0$. Likewise, we get $q \in W^v - W$. Consequently, $T = \langle uvwpi \cdots qr \cdots s\rangle$ satisfies $T \in H$, since $ax^T - ax^{T'} = a_{uv} + a_{pi} + a_{qr} - a_{ui} - a_{qv} - a_{pr} = a_{pi} + a_{qr} \geq 0$.

The next theorem is the second lifting theorem for facets of Q_T^n. Like in the case of the first theorem it involves an inductive step from Q_T^n to Q_T^{n+1}. The lifting procedure adjoins the node $n+1$ to the clique (W, C) in G_a in such a fashion that $W \cup \{n+1\}$ becomes the node set of clique in G_{a^*} and node $n+1$ is joined in G_{a^*} only to the nodes in W.

Theorem 5.5. *Let $ax \leq a_0$ be a facet of Q_T^n satisfying $a \geq 0$ and let (W, C) be a clique in $G_a = (V_a, E_a)$ with $|W| \geq 3$. Suppose that every node $w \in W$ satisfies condition $S(w)$ and that $a_e = \alpha$ for all $e \in E(a, W)$. If (a^*, a_0^*) is defined by $a_0^* = a_0 + \alpha$, $a_e^* = a_e$ for all $e \in E$, $a_e^* = \alpha$ for all $e = [i, n+1]$ with $i \in W$, $a_e^* = 0$ otherwise, then $a^*x^* \leq a_0^*$ is a facet of Q_T^{n+1}.*

Proof. Let $H(a^*) = \{x^* \in Q_T^{n+1} \mid a^*x^* = a_0^*\}$ and $H^* = \{T \in T_{n+1} \mid x^T \in H(a^*)\}$. From the definition of (a^*, a_0^*) it follows immediately that $H(a^*)$ is a proper face of Q_T^{n+1}. Let $b^*x \leq b_0^*$ be any valid inequality defining a facet of Q_T^{n+1} and satisfying $H(b^*) \supseteq H(a^*)$, where $H(b^*)$ is defined analogously to $H(a^*)$. We want to prove that there exist $\lambda^* \in \mathbf{R}^{n+1}$ and $\pi \neq 0$ such that $\lambda^*A^* + \pi a^* = b^*$

holds where A^* is the node-edge incidence matrix of K_{n+1}. To simplify the notation we assume that $W = \{1, \dots, k\}$ and $W^i = \{i, k+i\}$ for $i = 1, \dots, k$ where $k \geq 3$. Without restriction of generality we can assume further (by applying Argument A with $u = n+1$ and $f = [1, 2]$) that

$$b^*_{i,n+1} = \alpha \quad \forall i \in W, \qquad b^*_{i,n+1} = 0 \quad \forall i \in V - W \quad \text{and} \quad b^*_{12} = \alpha. \tag{5.1}$$

We compute next further components of b^* using the assumptions of the theorem.

(a) For every $w \in W - \{1, 2\}$ there exists by Corollary 5.4(ii) a tour $T \in H$ containing the chain $[1, 2, w]$. Replacing this chain by $[1, n+1, 2, w]$ and $[1, 2, n+1, w]$, respectively, we get two tours $T^*, T^{**} \in H^*$. Consequently by (5.1) $0 = b^*x^{T^*} - b^*x^{T^{**}} = b^*_{1,n+1} + b^*_{2,w} - b^*_{12} - b^*_{w,n+1} = b^*_{2w} - \alpha$ follows and thus $b^*_{2w} = \alpha$. Since nodes 1 and 2 can be interchanged, it follows that $b^*_{1w} = \alpha$ for all $w \in W$. For every pair $v, w \in W - \{1, 2\}$ there exists by Corollary 5.4(ii) a tour T containing the chain $[1, v, w]$. Replacing this chain by $[1, n+1, v, w]$ and $[1, v, n+1, w]$, respectively, we get by the same argument and using $b^*_{1v} = \alpha$

$$b^*_{ij} = \alpha \quad \forall i, j \in W \cup \{n+1\}. \tag{5.2}$$

(b) For every $u \in W$, $i \in V - (W \cup W^u)$ and $v, w \in W - u$ there exists by Corollary 5.4(i) a tour $T \in H$ which contains the edges $[u, i]$ and $[v, w]$. We construct a first new tour $T^* \in H^*$ by replacing $[u, i]$ by the chain $[u, n+1, i]$ and a second tour $T^{**} \in H^*$ by replacing $[v, w]$ by $[v, n+1, w]$. With the same argument used under (a) it follows from (5.1) and (5.2) that

$$b^*_{ui} = 0 \quad \forall u \in W \text{ and } i \in V - (W \cup W^u). \tag{5.3}$$

(c) We define a hyperplane $bx = b_0$ in $\mathbf{R}^m$ by $b_e = b^*_e$ for all $e \in E$ and $b_0 = b^*_0 - \alpha$. By Lemma 5.2 every $T \in H$ contains an edge $[v, w]$ with $v, w \in W$. If we construct $T^* \in H^*$ by replacing this edge by $[v, n+1, w]$, then

$$b^*x^{T^*} - bx^T = b^*_{w,n+1} + b^*_{v,n+1} - b^*_{wv} = \alpha.$$

Consequently, $H(a) \subseteq H(b)$ and by Lemma 4.1 there exist $\gamma \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\gamma A + \pi a = b$, if $H(b) \neq Q^n_T$. If $H(b) = Q^n_T$, the same relation holds with $\pi = 0$. In any case, applying Argument B to any triple u, v and $w \in W$ we get $\gamma_u = \gamma_v = \gamma_w = \frac{1}{2}\alpha(1-\pi)$ using (5.2) and thus $\gamma_i = \frac{1}{2}\alpha(1-\pi)$ for all $i \in W$. Since by our assumptions for every $i \in V - W$ there exists a $u \in W$ such that $i \notin W^u$, we get from (5.3) $\gamma_i + \gamma_u = 0$ and thus $\gamma_i = -\frac{1}{2}\alpha(1-\pi)$ for all $i \in V - W$. Defining $\lambda^*_j = \gamma_j$ for $j = 1, \dots, n$ and $\lambda^*_{n+1} = \frac{1}{2}\alpha(1-\pi)$ it follows that by construction of a^* and by (5.1) and (5.2) we have $\lambda^* A^* + \pi a^* = b^*$ and hence, since $H(b^*) \neq Q^n_T$, $\pi \neq 0$ follows. Consequently, $H(a^*) = H(b^*)$ holds.

The third lifting theorem shows that under the assumptions of Theorem 5.5 a facet $ax \leq a_0$ of Q^n_T can be lifted in a different way to a facet $a^*x^* \leq a^*_0$ of Q^{n+4}_T. This lifting procedure involves adjoining two nodes $n+1$ and $n+2$ to the clique

(W, C) in G_a and two additional nodes $n+3$ and $n+4$ such that $n+1$ and $n+2$, respectively, satisfy condition $S(n+1)$, $S(n+2)$ respectively, where $W^{n+1} = \{n+1, n+3\}$ and $W^{n+2} = \{n+2, n+4\}$. According to our conventions, all symbols with a star pertain now to K_{n+4} *and* Q_T^{n+4}, respectively.

Lemma 5.6. *Suppose that* $|W| \geq 3$ *and* $a_e = \alpha$ *for all* $e \in E(a, W)$ *hold and that* $v, w \in W$ *satisfy condition* $S(v)$, $S(w)$ *respectively. Let* $W^v = \{v, q\}$ *and* $W^w = \{w, p\}$. *Then there exists a tour* $T \in H$ *which contains* $[p, q]$.

Proof. Suppose the assertion is false. Then $[p, q] \notin T$ for all $T \in H$ and $H(a) = H(b)$ where $bx = x_{pq}$ and $b_0 = 0$. Consequently, by Lemma 4.1 there exist $\lambda \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\lambda A + \pi a = b$. Applying Argument B to the triple v, p and q we get $\lambda_v = -\frac{1}{2}(1+\pi\alpha)$, $\lambda_p = \frac{1}{2}(1+\pi\alpha)$ and $\lambda_q = \frac{1}{2}(1-\pi\alpha)$, while applying Argument B to the triple w, p and q we get $\lambda_w = -\frac{1}{2}(1+\pi\alpha)$, $\lambda_p = \frac{1}{2}(1-\pi\alpha)$ and $\lambda_q = \frac{1}{2}(1+\pi\alpha)$. Consequently, $\pi\alpha = 0$ is a contradiction to $\pi \neq 0$ and $\alpha > 0$.

Theorem 5.7. *Let* $ax \leq a_0$ *be a facet of* Q_T^n *satisfying* $a \geq 0$ *and let* (W, C) *be a clique in* $G_a = (V_a, E_a)$ *with* $|W| \geq 3$. *Suppose that every node* $w \in W$ *satisfies condition* $S(w)$ *and that* $a_e = \alpha$ *for all* $e \in E(a, W)$. *If* (a^*, a_0^*) *is defined by* $a_0^* = a_0 + 3\alpha$, $a_e^* = a_e$ *for all* $e \in E$, $a_e^* = \alpha$ *for all* $e = [i, j]$ *with* $i, j \in W \cup \{n+1, n+2\}$, $a_e^* = \alpha$ *for* $e \in \{[n+1, n+3], [n+2, n+4]\}$, $a_e^* = 0$ *otherwise, then* $a^*x^* \leq a_0^*$ *is a facet of* Q_T^{n+4}.

Proof. Let $H(a^*) = \{x^* \in Q_T^{n+4} \mid a^*x^* = a_0^*\}$ *and* $H^* = \{T \in T_{n+4} \mid x^T \in H(a^*)\}$. From the definition of (a^*, a_0^*) it follows immediately that $H(a^*)$ is a proper face of Q_T^{n+4}. Let $b^*x^* \leq b_0^*$ be any valid inequality defining a facet of Q_T^{n+4} and satisfying $H(b^*) \supseteq H(a^*)$, where $H(b^*)$ is defined analogously to $H(a^*)$. We want to prove that there exist $\lambda^* \in \mathbf{R}^{n+4}$ and $\pi \neq 0$ such that $\lambda^* A^* + \pi a^* = b^*$ holds where A^* is the node-edge incidence matrix of K_{n+4} (see Definition 4.0, point 8). To simplify the notation we assume that $W = \{1, \dots, k\}$ and $W^i = \{i, k+i\}$ for $i = 1, \dots, k$ where $k \geq 3$. Without restriction of generality we can assume further (by applying Argument A with $u = n+4$ and $f = [n+1, n+2]$) that

$$b_{i,n+4}^* = 0 \quad \forall\, i \in \{1, \dots, n+1, n+3\}, \qquad b_{n+2,n+4}^* = b_{n+1,n+2}^* = \alpha. \tag{5.4}$$

We compute next in steps (a)–(d) further components of b^* using the assumptions of the theorem.

(a) For every $i \in V - W$ there exists by assumption a node $w \in W$ such that $a_{wi} = 0$ holds. Furthermore, w is contained in exactly one additional clique W^w and $i \notin W^w$. Consequently, since $|W| \geq 3$ and every node w of W satisfies condition $S(w)$ Lemma 5.1 applies and there exists a tour $S \in H$ which contains the edge $[w, i]$.

(a1) From S we construct two tours S1 and S2, respectively, by replacing the edge $[w, i]$ by the chains $[w, n+2, n+4, n+1, n+3, i]$ and $[w, n+2, n+1, n+3, n+4, i]$. By construction, $S1 \in H^*$, $S2 \in H^*$ and consequently, from (5.4) we get $0 = b^*x^{S1} - b^*x^{S2} = b^*_{i,n+3}$.

(a2) From S we construct a third tour $S3$ by replacing the edge $[w, i]$ by the chain $[w, n+2, n+4, n+3, n+1, i]$. By construction $S3 \in H^*$ and consequently, from (5.4) we get $0 = b^*x^{S3} - b^*x^{S2} = b^*_{i,n+1}$.

Summarizing (a1) and (a2) we have obtained

$$b^*_{i,n+1} = b^*_{i,n+3} = 0 \quad \forall\, i \in V - W. \tag{5.5}$$

(b) For every $w \in W$ there exists by assumption a node $i \in V - W$ such that $a_{wi} = 0$. Applying Lemma 5.1 as before, there exists a tour $T \in H$ which contains $[w, i]$.

(b1) From T we construct two tours $T1 \in H^*$ and $T2 \in H^*$ by replacing $[w, i]$ by the chain $[w, n+4, n+2, n+1, n+3, i]$ and $[w, n+3, n+1, n+2; n+4, i]$, respectively. Consequently, from (5.4) and (5.5) we get $0 = b^*x^{T1} - b^*x^{T2} = -b^*_{w,n+3}$.

(b2) From T we construct a third tour $T3 \in H^*$ by replacing $[w, i]$ by the chain $[w, n+2, n+1, n+3, n+4, i]$. Consequently, from (5.4) and $b^*_{w,n+3} = 0$ we get

$$0 = b^*x^{T3} - b^*x^{T2} = b^*_{w,n+2} - \alpha, \quad \text{i.e. } b^*_{w,n+2} = \alpha.$$

(b3) From T we construct $T4 \in H^*$ by replacing $[w, i]$ by the chain $[w, n+1, n+2, n+4, n+3, i]$. Setting $b^*_{n+1,n+3} = \beta$, then from (5.4), (5.5) and from $b^*_{w,n+2} = \alpha$ we get

$$0 = b^*x^{T4} - b^*x^{T3} = b^*_{w,n+1} - \beta, \quad \text{i.e. } b^*_{w,n+1} = \beta.$$

Summarizing (b1), (b2) and (b3) we have obtained

$$\begin{aligned} &b^*_{w,n+3} = 0 \quad \forall\, w \in W, \qquad b^*_{w,n+2} = \alpha \quad \forall\, w \in W, \\ &b^*_{n+1,n+3} = \beta \quad \text{and} \quad b^*_{w,n+1} = \beta \quad \forall w \in W, \end{aligned} \tag{5.6}$$

where β is some scalar.

(a3) We take next the tour $S \in H$ found in (a) and construct a tour $S4 \in H^*$ by replacing $[w, i] \in S$ by the chain $[w, n+1, n+3, n+4, n+2, i]$. Consequently, from (5.5) and (5.6) we obtain $0 = b^*x^{S4} - b^*x^{S3} = b^*_{i,n+2} + \beta - \alpha$, i.e. $b^*_{i,n+2} = \alpha - \beta$.

(a4) From S we construct a further tour $S5 \in H^*$ by replacing $[w, i] \in S$ by the chain $[w, n+1, n+3, n+2, n+4, i]$. Consequently, from (5.4) and from (a3) it follows that $0 = b^*x^{S5} - b^*x^{S4} = b^*_{n+2,n+3} - \alpha + \beta$, i.e. $b^*_{n+2,n+3} = \alpha - \beta$.

Summarizing (a3) and (a4) we have obtained

$$b^*_{i,n+2} = \alpha - \beta \quad \forall\, i \in V - W, \qquad b^*_{n+2,n+3} = \alpha - \beta. \tag{5.7}$$

(c) By Corollary 5.4(i) there exists for every $u \in W$, for every $i \in V - (W \cup W^u)$ and for every pair $v, w \in W - u$ a tour $R \in .H$ containing both $[u, i]$ and $[v, w]$. Let w.r.o.g. $R = \langle ui \cdots wv \cdots s \rangle$.

(c1) From R we construct $R1 \in H^*$ by replacing $[u, i]$ by the chain $[u, n+2, n+4, n+3, n+1, i]$ and a tour $R2 \in H^*$ by $R2 = \langle u(n+3)(n+1)w \cdots i(n+4)(n+2)v \cdots s\rangle$. Using (5.4), (5.5), and (5.6) we get $0 = b^*x^{R1} - b^*x^{R2} = b^*_{vw} - \beta$, i.e. $b^*_{vw} = \beta$.

(c2) From R we construct $R3 \in H^*$ by replacing $[v, w]$ by the chain $[v, n+2, n+4, n+3, n+1, w]$. Consequently, from (5.5), (5.6) and (c1) we get $0 = b^*x^{R3} - b^*x^{R1} = b^*_{ui}$.

Summarizing (c1) and (c2) we have obtained

$$b^*_{vw} = \beta \quad \forall\, v, w \in W, \qquad b^*_{ui} = 0 \quad \forall\, u \in W \text{ and } i \in V - (W \cup W^u). \tag{5.8}$$

(d) Let $[i, j]$ be any edge such that $a_{ij} = 0$ and $P \in H$ be any tour which contains $[i, j]$. By Lemma 5.2, P contains an edge $[v, w] \in C$. We construct $P1 \in H^*$ and $P2 \in H^*$, respectively, by replacing $[i, j]$ by the chain $[i, n+4, n+2, n+1, n+3, j]$ and by replacing $[v, w]$ by the chain $[v, n+2, n+4, n+3, n+1, w]$, respectively. Using (5.4), (5.5), (5.6) and (5.8), we get $0 = b^*x^{P2} - b^*x^{P1} = b^*_{ij}$, i.e.

$$b^*_{ij} = 0 \quad \text{for all } i, j \in V \text{ such that } a_{ij} = 0 \text{ and } [i, j] \in T \text{ for some } T \in H. \tag{5.9}$$

(e) We define a hyperplane $bx = b_0$ in $\mathbf{R}^m$ by $b_e = b^*_e$ for all $e \in E$ and $b_0 = b^*_0 - 2\alpha - \beta$. By Lemma 5.2 every $T \in H$ contains an edge $[v, w] \in C$. If we construct $T^* \in H^*$ by replacing $[v, w]$ by the chain $[v, n+2, n+4, n+3, n+1, w]$, then by (5.4), (5.6) and (5.8) $b^*x^{T^*} - bx^T = 2\alpha + \beta$. Consequently, $H(a) \subseteq H(b)$ and by Lemma 4.1 there exist $\gamma \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\lambda A + \pi a = b$ holds, if $H(b) \neq Q^n_T$. If $H(b) = Q^n_T$, the same relation holds, with $\pi = 0$. In any case, applying Argument B to any triple u, v, and $w \in W$ we get $\gamma_w = \frac{1}{2}(\beta - \pi\alpha)$ for all $w \in W$. Since for every $i \in V - W$ there exists $w \in W$ with $a_{wi} = 0$, it follows from (5.8) that $\gamma_i + \gamma_w = 0$, i.e. $\gamma_i = -\frac{1}{2}(\beta - \pi\alpha)$ for all $i \in V - W$. By Lemma 5.6 there exists a tour $T \in H$ containing the edge $[k+1, k+2]$ and since $a_{k+1,k+2} = 0$ it follows from (5.9) and $\gamma_{k+1} + \gamma_{k+2} = 0$ that $\beta = \pi\alpha$. Consequently, $\gamma_i = 0$ for all $i \in V$ and $b = \pi a$. Defining $\lambda^*_j = 0$ for $j = 1, \dots, n+1, n+3, n+4$ and $\lambda^*_{n+2} = \alpha - \beta$, it follows by the construction of a^* and by (5.4)–(5.7) that $\lambda^* A^* + \pi a^* = b^*$ holds and hence, since $H(b^*) \neq Q^n_T$, $\pi \neq 0$ follows. Consequently, $H(a^*) = H(b^*)$ holds.

The fourth lifting theorem, like the first one, shows that under very mild conditions a facet $ax \le a_0$ of Q^n_T can be lifted to a facet $a^*x^* \le a^*_0$ of Q^{n+m}_T where $m \ge 1$ is an arbitrary integer. (Note that we depart here from the earlier convention regarding the use of the letter m.) This lifting procedure involves the substitution of a complete graph on $m+1$ nodes into a node that is the unique intersection of two different cliques of G_a. Furthermore, the coefficients of

$a^*x^* \le a_0^*$ deserve special attention. According to our conventions, all symbols with a star pertain now to K_{n+m} and Q_T^{n+m}. We first prove two lemmata.

Lemma 5.8. *If (W', C') is a clique in G_a satisfying $|Z'| = |W'| - 1$ and $a_e = \alpha$ for all $e \in C'$, then every tour $T \in H$ contains a hamiltonian chain in W' or in Z'.*

Proof. Let $|W'| \ge 3$ (otherwise there is nothing to be proven) and suppose that $T \in H$ does not have a hamiltonian chain in either W' or Z'. Then T has the general form $T = \langle pi_1 \cdots i_s q \cdots uj_1 \cdots j_r v \cdots s\rangle$ where $i_k \in Z'$ for $k = 1, \ldots, s$, $j_k \in Z'$ for $k = 1, \ldots, r$, and $p, q, t, s \notin Z'$. W.r.o.g. let $p, q, u \notin W'$. Then $T' = \langle pi_1 \cdots i_s j_r \cdots j_1 u \cdots qv \cdots s\rangle$ satisfies $ax^{T'} - ax^T = \alpha + a_{qv} > 0$ since $a_{qi_s} = a_{vj_r} = 0$. Consequently, $T \notin H$ holds, contradicting our assumption.

Lemma 5.9. *Let $w \in V$ be contained in exactly two different cliques (W, C) and (W', C') of G_a which satisfy $|W \cap W'| = 1$ and $Z' = W' - w$. If there exist two distinct $p, q \in W - W'$ and two distinct $i, j \in V - (W \cup W')$ such that $a_{pj} = a_{qi} = 0$ holds and if $a_e = \alpha$ holds for all $e \in E(a, W')$, then for every node $v \in Z'$ and every node $i \in V - \{v, w\}$ there exists a tour $T \in H$ which contains the chain $[v, w, i]$.*

Proof. Let $v \in Z'$. Then, by Lemma 5.1, for every $i \in V - \{v, w\}$ there exists $T \in H$ such that $[w, i] \in T$. By Lemma 5.8, T contains a hamiltonian chain in Z' or W'. If the hamiltonian chain is in W', i.e. $T = \langle pv_1 \cdots v_j \cdots v_k wi \cdots s\rangle$ where $p \notin W'$ and $v_j \in W'$ for $j = 1, \ldots, k$, and if $v_j = v$ with $1 < j < k$ holds, then $T' = \langle pv_1 \cdots v_{j-1} v_k \cdots v_j wi \cdots s\rangle$ satisfies $T' \in H$ and $[v, w, i] \in T'$. If the hamiltonian chain is in Z', i.e. $T = \langle pv_1 \cdots v_j \cdots v_k q \cdots iws \cdots t\rangle$ where $p, q \notin W'$ and $v_j \in W'$ for $j = 1, \ldots, k$, and if $v_j = v$ holds with $1 < j < k$, $T' = \langle pv_1 \cdots v_{j-1} v_k \cdots v_j wi \cdots qs \cdots t\rangle$ satisfies $T' \in H$ and $[v, w, i] \in T'$, since $a_{qv_k} = 0$, $a_{ws} \le \alpha$, $a_{v_j v_{j-1}} = a_{v_k v_{j-1}}$, $a_{qs} \ge 0$ and $a_{wv_j} = \alpha$. A similar construction holds in both cases if $v = v_1$.

Theorem 5.10. *Let $ax \le a_0$ be a facet of Q_T^n satisfying $a \ge 0$ and let $w \in V$ be contained in exactly two different cliques (W, C) and (W', C') of G_a satisfying $|W \cap W'| = 1$ and $Z' = W' - w$. Suppose that there exist two distinct nodes $p, q \in W - W'$, and two distinct nodes $i, j \in V - (W \cup W')$ such that $a_{pj} = a_{qi} = 0$ holds and that $a_e = \alpha$ for all $e \in E(a, W')$ and $a_e \ge \alpha$ for all $e \in C$ hold. If (a^*, a_0^*) is defined by $a_0^* = a_0 + 2m\alpha$, $a_e^* = a_e$ for all $e \in E$, $a_e^* = \alpha$ for all $e = [i, j]$ with $i \in W \Delta W'$, $j \in \{n+1, \ldots, n+m\}$, $a_e^* = 2\alpha$ for all $e = [i, j]$ with $i, j \in \{w, n+1, \ldots, n+m\}$ and $a_e^* = 0$ otherwise, then $a^*x^* \le a_0^*$ is a facet of Q_T^{n+m}, where $m \ge 1$ is any integer.*

Proof. Let $H(a^*) = \{x^* \in Q_T^{n+m} \mid a^*x^* = a_0^*\}$ and $H^* = \{T \in T_{n+m} \mid x^T \in H(a^*)\}$. From the definition of (a^*, a_0^*) it follows immediately that $H(a^*)$ is a proper face of Q_T^{n+m}. Let $b^*x^* \le b_0^*$ be any valid inequality defining a facet of Q_T^{n+m} and satisfying $H(b^*) \supseteq H(a^*)$, where $H(b^*)$ is defined analogously to $H(a^*)$. We want to show that there exist $\lambda^* \in \mathbf{R}^{n+m}$ and $\pi \neq 0$ such that $\lambda^* A^* + \pi a^* = b^*$ holds where A^* is the node-edge incidence matrix of K_{n+m} (see Definition 4.0, point 8). To simplify the notation we assume that $w = n$ and $n-1 \in W' - W$. Without restriction of generality we can assume further (by applying Argument A with $u = n+m$ and $f = [n-1, n]$) that

$$b^*_{i,n+m} = 2\alpha \quad \forall\, i \in \{n, \dots, n+m-1\}, \qquad b^*_{i,n+m} = \alpha \quad \forall\, i \in W \Delta W',$$
$$b^*_{i,n+m} = 0 \quad \forall\, i \in V - (W \cup W'), \qquad b^*_{n-1,n} = \alpha. \quad (5.10)$$

We compute next in step (a)–(b) further components of b^* using the assumptions of the theorem.

(a) By Lemma 5.9 there exists for $n-1 \in Z'$ and every $i \in V - \{n, n-1\}$ a tour $T \in H$ which contains the chain $[n-1, n, i]$.

(a0) From T we construct two tours $T1 \in H^*$ and $T2 \in H^*$ by replacing the edge $[n, i]$ by the chain $[n, n+1, \dots, n+m, i]$ and the edge $[n-1, n]$ by the chain $[n-1, n+m, n+m-1, \dots, n]$. Consequently, from (5.10) we get

$$0 = b^* x^{T1} - b^* x^{T2} = b^*_{i,n+m} - b^*_{i,n}$$

and hence by (5.10)

$$b^*_{i,n} = \alpha \quad \forall\, i \in W \Delta W', \qquad b^*_{i,n} = 0 \quad \forall\, i \in V - (W \cup W'). \qquad (5.11)$$

We claim next that

$$b^*_{i,n+k} = \alpha \quad \forall\, i \in W \Delta W', \qquad b^*_{i,n+k} = 0 \quad \forall\, i \in V - (W \cup W'),$$
$$b_{n+j,n+k} = 2\alpha \quad \text{for } 0 \le j < k \le m \quad (5.12)$$

holds for all $m \ge 1$. For $m = 1$ this follows from (5.10). In order to get a backward induction started if $m \ge 2$ is an arbitrary integer we have to first carry out calculations for $k = m-1$.

(a1) From T we construct a tour $T3 \in H^*$ by replacing the chain $[n-1, n, i]$ by the chain $[n-1, n+m-1, \dots, n+1, n, n+m, i]$. Consequently, from (5.10) and (5.11) we get $0 = b^* x^{T3} - b^* x^{T2} = b^*_{n-1,n+m-1} - \alpha$.

(a2) From T we construct a tour $T4 \in H^*$ by replacing the chain $[n-1, n, i]$ by the chain $[n-1, n+m-1, \dots, n+1, n+m, n, i]$ and get from (5.10) and (5.11) $0 = b^* x^{T4} - b^* x^{T3} = 2\alpha - b^*_{n,n+1}$.

(a3) From T we construct a tour $T5 \in H^*$ by replacing $[n-1, n, i]$ by $[n-1, n+m, n, n+1, \dots, n+m-1, i]$ and get $0 = b^* x^{T5} - b^* x^{T1} = b^*_{i,n+m-1} - b^*_{i,n+m}$ for all $i \in V - \{n-1, n\}$.

(a4) From T we construct a tour $T6 \in H^*$ by replacing $[n-1, n, i]$ by the chain $[n-1, n+m, n+1, \dots, n+m-1, n, i]$ and get $0 = b^* x^{T6} - b^* x^{T2} = b^*_{n,n+m-1} - 2\alpha$.

One verifies now that (5.12) is true for $j = 0$ and $k \ge m-1$. Assuming that the first two statements of (5.12) hold for $k \le m-1$, we will show that they hold for

$k-1$ as well. Furthermore, assuming that the last statement of (5.12) holds for $j=0$, for all h with $k<h\le m$ and $k\le m-1$, we will show that the statement is true if k is replaced by $k-1$.

(a5) From T we construct $T1k\in H^*$ by replacing $[n-1, n, i]$ by the chain $[n-1,\ n+k,\dots,n+m,\ n+k-1,\dots,n+1,\ n,\ i]$ and $T2k\in H^*$ by replacing $[n-1,\ n,\ i]$ by $[n-1,\ n+k-1,\dots,n+1,\ n,\ n+k,\dots,n+m,\ i]$. Taking the difference we get that by induction hypothesis, (5.10) and (5.11) $0=\alpha-b^*_{n-1,n+k-1}$ holds.

(a6) From T we construct $T3k\in H^*$ by replacing $[n-1, n, i]$ by the chain $[n-1,\ n+k,\dots,n+m,\ n+1,\dots,n+k-1,\ n,\ i]$ and using $T2k$ we get that $0=2\alpha-b^*_{n,n+k-1}$ holds by (5.10), (5.11), (a2), (a3) and induction hypothesis.

(a7) From $T1k$ and $T2$ we obtain $0=2\alpha-b_{n+k-1,n+k}$ by (5.10) and the induction hypothesis. From T we construct $Tkh\in H^*$ by replacing $[n-1, n, i]$ by $[n-1,\ n+m,\dots,n+h+1,\ n+k,\dots,n+h,\ n+k-1,\dots,n+1,\ n,\ i]$ where $k<h\le m$ and calculate by taking the difference with $T2$ that $0=b^*_{n+k-1,n+h}-2\alpha$ holds by induction hypothesis.

(a8) From T we construct finally $T4k\in H^*$ by replacing $[n-1, n, i]$ by $[n-1,\ n+m,\dots,n+k,\ n,\ n+1,\dots,n+k-1,\ i]$ and obtain using $T2$ that $0=b^*_{i,n+k-1}-b^*_{i,n}$ holds for all $i\in V-\{n-1,n\}$.

Consequently, (5.12) follows.

(b) Since $|Y'|=|V-(W\cup W')|\ge 2$ it follows by Lemma 4.4 (Lemma 4.8, respectively) when applied to W' that for every node $v\in W-W'$ and every node $i\in Z'$ there exists a tour $S\in H$ which contains the edge $[i, v]$. By Lemma 5.8 we know that S contains a hamiltonian chain in W' or Z'. Since by assumption $n\in W\cap W'$ is contained in exactly two cliques and $W'-n=Z'$ holds, we can assume w.r.o.g. that the hamiltonian chain terminates in node n if W' contains such a chain. If Z' contains the hamiltonian chain, let $S=\langle jw_1\cdots w_kiv\cdots pnq\cdots s\rangle$ and construct a new tour $R=\langle jnw_1\cdots w_kiv\cdots pq\cdots s\rangle$ where $[w_1,\dots,i]$ is the hamiltonian chain in Z'. Consequently, $ax^R-ax^S=\alpha+a_{jn}+a_{pq}-a_{pn}-a_{qn}\ge 0$, since $a_{jn}\ge 0$ and $a_{pn}=a_{qn}=\alpha$ implies $a_{pq}\ge\alpha$. Consequently, in both cases, there exists a tour $S\in H$ which contains the chain $[n, w_1,\dots,w_k, i, v]$ satisfying $v\in W-W'$ and $w_j\in Z'$ for $j=1,\dots,k$. We construct now $S1\in H^*$ and $S2\in H^*$ by replacing this chain by the chains $[n,\ n+1,\dots,n+m,\ w_1,\dots,w_k,\ i,\ v]$ and $[n,\ w_1,\dots,w_k,\ i,\ n+1,\dots,n+m,\ v]$, respectively. Taking differences as usual, we obtain from (5.10), (5.11) and (5.12) that $b^*_{v,i}=0$ holds. Consequently we have proven

$$b^*_{v,i}=0\quad\forall\, i\in W'-W,\quad\forall\, v\in W-W'. \tag{5.13}$$

(c) We define a hyperplane $bx=b_0$ in $\mathbf{R}^m$ by $b_e=b^*_e$ for all $e\in E$ and $b_0=b^*_0-2m\alpha$. Every tour $T\in H$ contains an edge $[i, n]$ where $i\in V-n$. If we construct $T^*\in H^*$ by replacing this edge by the chain $[n,\ n+1,\dots,n+m,\ i]$, then

$$b^*x^{T^*}-bx^T=\sum_{i=0}^{m-1} b^*_{n+i,n+i+1}+b^*_{i,n+m}-b^*_{in}=2m\alpha.$$

Consequently, $H(a) \subseteq H(b)$ and by Lemma 4.1 there exist $\gamma \in \mathbf{R}^n$ and $\pi \neq 0$ such that $\gamma A + \pi a = b$ holds, if $H(b) \neq Q_T^n$. If $H(b) = Q_T^n$, the same relation holds with $\pi = 0$. In any case, applying Argument B to the triple $n-1$, n and $w \in W - W'$ we get $\gamma_n = \alpha - \pi\alpha$ and $\gamma_w = \gamma_{n-1} = 0$. For $i \in W \Delta W'$ we have $\gamma_i + \gamma_n + \pi\alpha = \alpha$ and thus $\gamma_i = 0$. For $i \in V - (W \cup W')$ we have $\gamma_i + \gamma_n = 0$ and thus $\gamma_i = \pi\alpha - \alpha$. Defining $\lambda_i^* = \gamma_i$ for $i = 1, \dots, n$ and $\lambda_{n+i}^* = \alpha - \pi\alpha$ for $i = 1, \dots, m$ it follows by the construction of a^* and by (5.10) and (5.12) that $\lambda^* A^* + \pi a^* = b^*$ holds, and hence, since $H(b^*) \neq Q_T^n$, $\pi \neq 0$ follows. Consequently, $H(a^*) = H(b^*)$ holds.

6. Facets of the travelling salesman polytope

We prove next that subtour-elimination as well as comb inequalities define facets of Q_T^n. As it will be seen, the lifting Theorems I–IV apply to these inequalities and in connection with the facts established in Section 3 of the first paper [3], the result now follows quite easily. It should be noted, however, that the lifting theorems, especially Theorem 4.12 and Theorem 5.10, are by no means limited to the class of inequalities considered here.

Theorem 6.1. *For every $n \geq 4$ and $U \subseteq V$ satisfying $2 \leq |U| \leq [\frac{1}{2}n]$ the subtour-elimination constraint*

$$x(U) \leq |U| - 1 \tag{6.1}$$

defines a facet of Q_T^n.

Proof. In Theorem 3.1 (see [3]) we have proven that the assertion is correct for $|U| = 2$. Hence we can assume that $2 < |U| \leq [\frac{1}{2}n]$ holds and that w.r.o.g. $U = \{1, 2, n-k, \dots, n\}$ where $k = |U| - 3$. Now let $W = \{1, 2\}$, $ax = x_{12}$ with $a_0 = 1$ and consider the graph K_{n-k-1} and its associated travelling salesman polytope Q_T^{n-k-1}. W is the node-set of a clique in this graph, by Theorem 3.1 $ax \leq a_0$ is a facet of this polytope and $Z = W$ satisfies $|Z| \geq 2$. Consequently, Theorem 4.12 part (i) applies and $a^* x^* \leq a_0^*$ as defined there is a facet of Q_T^{n-k}. But $a^* x^* = x_{12} + x_{1,n-k} + x_{2,n-k}$ and $a_0^* = 2$ is again a subtour-elimination constraint which clearly satisfies again the assumptions of Theorem 4.12 part (i). Consequently, the assertion follows after k successive applications of Theorem 4.12.

As a subtour-elimination constraint on the node-set U is equivalent to a sub-tour-elimination constraint on the node set $V - U$, in fact all subtour-elimination constraints define facets, though it is only half the number of all possible constraints that matter in describing Q_T^n linearly. Furthermore, since a subtour-elimination constraint on the node-set U of V is equivalent to the cut-set constraint $x(U : V - U) \geq 2$, these inequalities provide yet another linear description of the same facet as is defined by (6.1). We prove next that all comb-inequalities define facets of Q_T^n.

Theorem 6.2. *Every comb inequality defines a facet of Q_T^n. More precisely, let $n \geq 6$ and $W_0, W_1, \dots, W_k \subseteq V$ satisfy* (i) $|W_0 \cap W_i| \geq 1$ *and* $|W_i - W_0| \geq 1$ *for* $i = 1, \dots, k$ *and* (ii) $W_i \cap W_j = \emptyset$ *for* $1 \leq i < j \leq k$, *where* $k \geq 3$ *is an odd integer. Then the comb inequality*

$$\sum_{i=0}^{k} x(W_i) \leq |W_0| + \sum_{i=1}^{k} (|W_i| - 1) - \langle \tfrac{1}{2}k \rangle \tag{6.2}$$

is a facet of Q_T^n.

Proof. In order to prove the theorem we first "shrink" the comb to a comb having a 3-element handle and three 2-element teeth so as to be able to use Theorem 3.4 (see [3]) which states that such combs define facets of Q_T^n for all $n \geq 6$. We start by removing all nodes *except one* from $W_i \cap W_0$ for $i = 1, \dots, k$ (which gives us a Chvátal-comb [1]). Then we remove all but one of the nodes in $W_i - W_0$ for $i = 1, \dots, k$ (which gives a 2-matching constraint [3]). Next we remove all nodes in $W_0 - \bigcup_{i=1}^{k} W_i$. The resulting comb has k nodes in the handle and k 2-element teeth U_i, say, for $i = 1, \dots, k$. We remove all but three of the k teeth. The resulting comb has the desired form and defines by Theorem 3.4 a facet of Q_T^l where $l = n - \sum_{i=0}^{k} |W_i| + 6 \geq 6$. Furthermore, it satisfies the conditions of Theorem 5.7 and thus by adjoining the teeth U_4 and U_5 we get a new comb defining a facet of Q_T^{l+4}. (Note that we added four of the original nodes.) Furthermore, this new comb satisfies again the assumptions of Theorem 5.7 and thus by successive application of Theorem 5.7 we retrieve the comb having k nodes in its handle U_0, say, and k 2-element teeth. Furthermore, it defines a facet for the polytope Q_T^{l+4i} where i counts the number of times Theorem 5.7 was applied. If $|W_0 - \bigcup_{i=1}^{k} W_i| = 0$ and $|W_i| = 2$ for $i = 1, \dots, k$, we are done. Else, by Theorem 5.5 we can adjoin any node in $W_0 - \bigcup_{i=1}^{k} W_i$ to the current handle U_0. The resulting new handle W_0 is the node-set of a clique in the associated graph induced by the non-zero components of the current comb inequality and satisfies $|Z_0'| = 1$ (see Definition 4.0, point 4). Furthermore, because of Proposition 1.4(i) (see [3]) we can assume w.r.o.g. that $|W_0| \leq [\frac{1}{2}n]$. Consequently, if $|W_0 - \bigcup_{i=1}^{k} W_i| \geq 2$ it follows that $|Y_0'| \geq 2$ (see Definition 4.0, point 5) and by Theorem 4.12 part (ii) we can adjoin a further node to the current handle W_0'. Call the resulting handle again W_0'. If $|W_0 - \bigcup_{i=1}^{k} W_i| \geq 3$, then, since now $|Z_0'| \geq 2$, Theorem 4.12(i) applies and can be reapplied to adjoin all remaining such nodes.

The inequality that results defines a facet for Q_T^p where $p = |W_0 - \bigcup_{i=1}^{k} W_i| + 2k$ and has k 2-element teeth. For each i with $|W_i - W_0| \geq 2$ we adjoin one node to the corresponding tooth in the current comb using Theorem 4.12 part (ii) as above and the remaining nodes by an application of Theorem 4.12 part (i). The inequality that results defines a facet of the corresponding polytope. For each i with $|W_i \cap W_0| \geq 2$ we can now apply Theorem 5.10 once and adjoin the remaining nodes. Theorem 5.10 proves that the resulting inequality is a facet of Q_T^n. That inequality, however, is the comb inequality (6.2) that we started out with and thus the theorem is proven.

Theorems 6.1 and 6.2 provide a rather large class of facets for the symmetric travelling salesman problem and hence, a "fiarly good" linear approximation to the travelling salesman polytope. For larger n, however, these linear inequalities do not fully describe the polytope; in fact, we do not even know at present whether or not this class of facets does the job for Q_T^8. In fact, Maurras [4] has shown that for $n = 10$ subgraphs isomorphic to the (non-hamiltonian) Petersen graph give rise to facets of Q_T^n, and the more general question of hypo-hamiltonian graphs and their relationship to facets of the closely related monotone travelling salesman problem has been investigated by Grötschel [2]. In brief, his results are that "almost all" hypo-hamiltonian subgraphs of the complete graph K_n give rise to facets of the monotone travelling salesman polytope.

A complete list of references appears at the end of [3].

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