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CLIQUE TREE INEQUALITIES AND THE SYMMETRIC TRAVELLING SALESMAN PROBLEM*†

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The linear programming cutting plane approach for solving the travelling salesman problem has recently proven to be highly successful, cf. Crowder and Padberg (1980), Grötschel (1980a), Padberg and Hong (1980). One of the reasons for this success is certainly the fact that instead of ordinary cutting planes (Gomory-cuts etc.) problem-specific cutting planes could be used which define facets of the underlying integer programming polytopes.

In this paper we shall define a new class of inequalities (clique tree inequalities) valid for the travelling salesman polytope which properly contains many of the known classes of inequalities (like subtour elimination constraints, 2-matching constraints, comb inequalities), and we show that all these new inequalities induce facets of the travelling salesman polytope. Since the general structure of these new inequalities is quite simple we hope that it will be possible to use the inequalities efficiently in cutting plane procedures for the travelling salesman problem.

1. Introduction and notation. The linear programming cutting plane approach for solving the travelling salesman problem has recently proven to be highly successful, cf. Crowder and Padberg (1980), Grötschel (1980a), Padberg and Hong (1980). One of the reasons for this success is certainly the fact that instead of ordinary cutting planes (Gomory-cuts etc.) problem-specific cutting planes could be used which define facets of the underlying integer programming polytopes.

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Our purpose, therefore, is threefold. First we introduce this set of so-called clique tree inequalities and show that they are valid for both the travelling salesman polytope and its monotone extension. Second we show that these inequalities are facet-inducing for these polytopes and, almost always, induce distinct facets. Third, we show that in a sense made precise in §7, there is no facet-inducing generalization.

For our purposes, a *graph* $G = [V, E]$ consists of a finite set V of *nodes* and a finite set E of two-element subsets of V. The elements of E are called *edges*. If $e \in E$ is an edge then the two nodes, say i and j, contained in e are called the *endnodes* of e, e is said to *join* or *link* i and j, and i and j are called *adjacent*. For ease of notation we denote an edge e linking two nodes i and j by ij.

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Note that our definition implies that (our) graphs have no loops or multiple edges. The cardinality of the node set of a graph is called the *order* of this graph. A graph in which every two nodes are adjacent is called *complete*. The complete graph of order n is denoted by K_n.

If $G=[V,E]$ is a graph and $W \subseteq V$ then the set of edges in G which have both endnodes in W is denoted by $E(W)$. If $F \subseteq E$ then the sets of nodes of G which are contained in at least one edge $e \in F$ is denoted by $V(F)$. A graph $H=[W,F]$ is called a *subgraph* of $G=[V,E]$ if $W \subseteq V$ and $F \subseteq E(W)$. If $W \subseteq V$, $\emptyset \neq W \neq V$, then $\delta(W)$ is the set of edges with one endnode in W and the other endnode in $V \setminus W$. The edge set $\delta(W)$ is called a *cut*. We write $\delta(v)$ instead of $\delta(\{v\})$ for $v \in V$.

A *clique* in a graph $G=[V,E]$ is a set W of nodes such that any two nodes in W are adjacent. This definition implies that the graph $[W,E(W)]$ is a complete subgraph of G. In the sequel we shall use the word clique only for those cliques in a graph which are maximal with respect to set inclusion.

For any graph $G=[V,E]$, for any $S \subseteq V$, we let $G-S$ denote the subgraph of G obtained by deleting the nodes of S, plus all incident edges. If $G_1=[V_1,E_1]$ and $G_2=[V_2,E_2]$ are subgraphs of a graph G then the *union* of G_1 and G_2, denoted by $G_1 \cup G_2$, is the subgraph $[V_1 \cup V_2, E_1 \cup E_2]$ of G.

An *articulation set* of a graph $G=[V,E]$ is a minimal set S of nodes such that $[V\setminus S, E(V\setminus S)]$ has more connected components than G. If $|S|=1$, i.e. $S=\{v\}$ for some $v \in V$, then v is called a *cutnode* of G. Every subgraph $[W,E(W)]$ of G which contains no cutnode and which is maximal with respect to this property is called a *block* of G.

A *path* P in a graph $G=[V,E]$ is a sequence of edges $e_1,e_2,\ldots,e_{k-1},e_k$ such that $e_1=v_0v_1$, $e_2=v_1v_2,\ldots$, $e_{k-1}=v_{k-2}v_{k-1}$, $e_k=v_{k-1}v_k$ and such that $v_i \neq v_j$ holds for $0 \leqslant i<j \leqslant k$. The nodes v_0 and v_k are the *endnodes* of P. We say that P goes from v_0 to v_k or that P links v_0 and v_k. The number k of edges of P is called the *length* of P. Since a path is uniquely determined by the sequence of its nodes we also write $[v_0,v_1,\ldots,v_{k-1},v_k]$ to denote a path from v_0 to v_k. A graph G is called *connected* if any two nodes of G can be linked by a path in G. A path of length $|V|-1$ is called a *hamiltonian path* of G.

Sometimes it is convenient to consider a single node as a path of length zero. We call such a path *degenerate*. However, if we speak of a path, it will never be degenerate, unless otherwise specified.

If $P=[v_1,\ldots,v_k]$ is a path of length $k-1$ and $v_1v_k \in E$ then the sequence of edges $v_1v_2,v_2v_3,\ldots,v_{k-1}v_k,v_kv_1$ is called a *cycle* of length k. To abbreviate notation we denote such a cycle by $\langle v_1,v_2,\ldots,v_k\rangle$. A cycle of length $|V|$ is called a *hamiltonian cycle* or *tour*.

A *path system* is a set of paths $P_1,P_2,\ldots,P_s$ such that no two paths $P_i \neq P_j$ have a common node. In addition, the empty set is a path system, and we will consider each tour to be a path system. Note that any path system in K_n is either a tour or can be extended to a tour by adding appropriate edges. We frequently identify a path system with the set of edges belonging to its paths. Thus, for example, when we refer to the incidence vector of a path system we are actually referring to the incidence vector of the set of edges belonging to the paths of the path system.

We denote the set of all tours in K_n by $\mathcal{T}_n$ and the set of all path systems in K_n by $\tilde{\mathcal{T}}_n$ (i.e., the edges comprising any element of $\tilde{\mathcal{T}}_n$ are contained in some element of $\mathcal{T}_n$).

2. The symmetric travelling salesman problem. Given a complete graph $K_n=[V,E]$ and edge weights $c_{ij} \in \mathbb{R}$ for all $ij \in E$, the symmetric (n-city) travelling salesman problem (henceforth TSP) is to find a tour T such that $\sum_{e\in T} c_e$ is as small as possible. (In the sequel we abbreviate $\sum_{e\in T} c_e$ by $c(T)$.) More formally the TSP can be stated as $\min\{c(T) \mid T \in \mathcal{T}_n\}$.

Note that if we add a constant k to all edge weights the optimum tour does not change; only the value will change by $n \cdot k$. By setting $\tilde{c}_{ij} := -c_{ij}+k$ for all $ij \in E$ where k is a constant such that $\tilde{c}_{ij}>0$ for all $ij \in E$ we can therefore also solve the TSP by solving $\max\{\tilde{c}(T) \mid T \in \tilde{\mathcal{T}}_n\}$.

In the linear programming approach to the TSP, a polytope is associated with the TSP such that every vertex of this polytope corresponds to a tour and vice versa. More precisely, consider the n-city TSP and let $m := |E| = n(n-1)/2$. Every vector $x \in \mathbb{R}^m$ is considered as an m-tuple $(x_{12},x_{13},\ldots,x_{n-1,n})^{\mathrm{T}}$ where every component is indexed by an edge $e=ij$ of E. (Note that the pairs ij are unordered, i.e. the variables x_{ij} and x_{ji} are identical.) For every edge set $F \subseteq E$ the incidence vector $x^F \in \mathbb{R}^m$ is defined as follows

$$x_e^F = \begin{cases} 1 & \text{if } e \in F, \\ 0 & \text{if } e \notin F. \end{cases}$$

The (equality-constrained, symmetric) *travelling salesman polytope* Q_T^n is the convex hull of the incidence vectors of all tours in K_n, i.e.

$$Q_T^n := \operatorname{conv}\{x^T \mid T \in \mathcal{T}_n\}. \tag{2.1}$$

Similarly, the *monotone* (symmetric) *travelling salesman polytope* $\tilde{Q}_T^n$ is the convex hull of the incidence vectors of all path systems in K_n, i.e.

$$\tilde{Q}_T^n := \operatorname{conv}\{x^T \mid T \in \tilde{\mathcal{T}}_n\}. \tag{2.2}$$

It thus follows that every TSP can be solved via one of the linear programs

$$\min c^{\mathrm{T}}x, \quad x \in Q_T^n \quad \text{or} \quad \max \tilde{c}^{\mathrm{T}}x, \quad x \in \tilde{Q}_T^n. \tag{2.3}$$

In order to use linear programming techniques, however, the definitions of Q_T^n and $\tilde{Q}_T^n$ given above are not appropriate. What is needed is a description of Q_T^n and $\tilde{Q}_T^n$ by means of a system of linear inequalities and equations.

Recent results on polyhedra associated with combinatorial optimization problems indicate that it is quite unlikely that we will ever obtain a tractable linear description of Q_T^n or $\tilde{Q}_T^n$, cf. Grötschel (1980b), Grötschel, Lovász and Schrijver (1981), Karp and Papadimitriou (1980). Nevertheless, even partial knowledge of such inequality systems has shown to be very useful from the practical and algorithmic points of view.

The polyhedra Q_T^n and $\tilde{Q}_T^n$ have been intensively studied in the past, and we shall survey some of the results known to date.

If $a \in \mathbb{R}^m\setminus\{0\}$, $a_0 \in \mathbb{R}$, then the inequality $a^{\mathrm{T}}x \leqslant a_0$ is said to be *valid* with respect to a polyhedron $P \subseteq \mathbb{R}^m$ if $P \subseteq \{x \in \mathbb{R}^m \mid a^{\mathrm{T}}x \leqslant a_0\}$. We say that $F \subseteq P$ is a *face* of P if $F=\emptyset$, $F=P$ or if there exists a valid inequality $a^{\mathrm{T}}x \leqslant a_0$ such that $F=P \cap \{x \in \mathbb{R}^m \mid a^{\mathrm{T}}x=a_0\}$. If $F \neq P$ then F is called a *proper face* of P. A *facet* is a maximal nonempty proper face, or equivalently, a proper face containing $\dim P$ affinely independent points. We say that a valid inequality $a^{\mathrm{T}}x \leqslant a_0$ *defines* or *induces* the face $P \cap \{x \in \mathbb{R}^m \mid a^{\mathrm{T}}x=a_0\}$. Two face-defining inequalities $a^{\mathrm{T}}x \leqslant a_0$, $b^{\mathrm{T}}x \leqslant b_0$ are called *equivalent* if $P \cap \{x \mid a^{\mathrm{T}}x=a_0\} = P \cap \{x \mid b^{\mathrm{T}}x=b_0\}$.

Since every vertex of Q_T^n is also a vertex of $\tilde{Q}_T^n$ we have $Q_T^n \subseteq \tilde{Q}_T^n$.

In fact, it is easy to see that Q_T^n is a face of $\tilde{Q}_T^n$. Thus, every inequality valid for $\tilde{Q}_T^n$ is also valid for Q_T^n, and every complete linear description of $\tilde{Q}_T^n$ contains a complete linear description of Q_T^n (by setting some of the inequalities to equations).

In order to obtain results about the facial structure of a polyhedron it is important to know its dimension. For the travelling salesman polytope we have

$$\dim \tilde{Q}_T^n = m = |E|, \tag{2.4}$$

$$\dim Q_T^n = m - n = |E| - |V|, \tag{2.5}$$

where (2.4) is obvious, and a proof of (2.5) can be found in Grötschel and Padberg (1979a). Thus, Q_T^n is a face of $\tilde{Q}_T^n$ whose dimension is n less than the dimension of $\tilde{Q}_T^n$.

By definition, $\tilde{Q}_T^n$ is contained in the unit hypercube which implies that the inequalities

$$0 \leq x_{ij} \leq 1 \qquad \text{for all} \quad ij \in E \tag{2.6}$$

are valid for $\tilde{Q}_T^n$. In fact, these inequalities induce facets for $\tilde{Q}_T^n$ for all $n \geq 3$. For the travelling salesman polytope Q_T^n we have that the inequalities $x_{ij} \geq 0$ induce facets of Q_T^n for all $n \geq 5$, while the inequalities $x_{ij} \leq 1$ induce facets of Q_T^n for all $n \geq 4$ (in case $n = 4$ the facets induced by $x_{ij} \leq 1$ and $x_{pq} \leq 1$, $i \neq p \neq j$, $i \neq q \neq j$, are equivalent), cf. Grötschel (1977).

Since every node of K_n lies on exactly two edges of every tour, Q_T^n is contained in the n hyperplanes defined by

$$x(\delta(v)) = \sum_{e \in \delta(v)} x_e = 2 \qquad \text{for all} \quad v \in V. \tag{2.7}$$

The intersection of the hyperplanes defined by (2.7) is exactly the affine space spanned by Q_T^n. Let us denote the node-edge incidence matrix of K_n by A and let $\mathbf{2}$ be a n-vector all of whose components are 2; then

$$Ax = \mathbf{2} \tag{2.8}$$

is a minimal system of equations whose solution set contains Q_T^n. This implies (using Farkas' Lemma) that any facet inducing inequality of Q_T^n is unique up to scaling and adding multiples of (2.8), i.e.

Lemma 2.9. *Let $a^T x \leq a_0$ induce a nonempty proper face of Q_T^n and suppose that $b^T x \leq b_0$ induces a facet of Q_T^n. Then $a^T x \leq a_0$ and $b^T x \leq b_0$ are equivalent, i.e., define the same facet of Q_T^n, if and only if there are $\lambda \in \mathbb{R}^n$ and $\rho > 0$ such that $a^T = \rho b^T + \lambda^T A$.*

Note that since both $a^T x \leq a_0$ and $b^T x \leq b_0$ define nonempty faces of Q_T^n, the condition $a^T = \rho b^T + \lambda^T A$ implies $a_0 = \rho b_0 + \lambda^T \mathbf{2}$. Moreover, we can also express b in terms of a in a similar fashion since $b^T = a^T/\rho - \lambda^T A/\rho$.

It is obvious that the inequalities

$$x(\delta(v)) \leq 2 \qquad \text{for all} \quad v \in V \tag{2.10}$$

are valid for $\tilde{Q}_T^n$, and it was shown in Grötschel (1977) that the inequalities (2.10) are facet-inducing for $\tilde{Q}_T^n$ for $n \geq 4$.

Dantzig, Fulkerson & Johnson (1954) introduced the so-called *subtour elimination constraints*

$$x(E(W)) \leq |W| - 1 \qquad \text{for all} \quad W \subseteq V, \quad \emptyset \neq W \neq V \tag{2.11}$$

which algebraically state that no tour can contain a subtour (a cycle of length $|W|$, $W \neq V$). It is clear that the inequalities (2.11) are valid for $\tilde{Q}_T^n$ and Q_T^n. The following result was shown by Grötschel (1977), resp. Grötschel and Padberg (1979b):

Theorem 2.12. (a) *The subtour elimination constraints $x(E(W)) \leq |W| - 1$ are facet inducing for $\tilde{Q}_T^n$ for all $W \subseteq V$ with $2 \leq |W| \leq n-1$ and all $n \geq 4$.*

(b) *The subtour elimination constraints are facet inducing for Q_T^n for all $W \subseteq V$ with $2 \leq |W| \leq n-2$ and all $n \geq 4$. Two distinct subtour elimination constraints $x(E(W)) \leq |W| - 1$ and $x(E(W')) \leq |W'| - 1$ are equivalent if and only if $W' = V \setminus W$.*

Note that the subtour elimination constraints for $|W| = 2$ are nothing but the trivial inequalities $x_{ij} \leq 1$, i.e. these are contained in the system (2.11).

A more general class of inequalities for $\tilde{Q}_T^n$ was introduced in Grötschel and Padberg (1979a). Let $K_n = [V, E]$ be the complete graph on n nodes, and let $H, T_1, \ldots, T_k \subseteq V$ satisfy

$$\begin{array}{lll} \text{(a)} & |H \cap T_i| \geq 1 & \text{for } i = 1, \ldots, k, \\ \text{(b)} & |T_i \setminus H| \geq 1 & \text{for } i = 1, \ldots, k, \\ \text{(c)} & T_i \cap T_j = \emptyset, & 1 \leq i < j \leq k, \\ \text{(d)} & k \geq 3 \text{ and odd.} & \end{array} \tag{2.13}$$

Then the subgraph $[H \cup \bigcup_{i=1}^k T_i, E(H) \cup \bigcup_{i=1}^k E(T_i)]$ is called a *comb* with *handle* H and *teeth* T_i. For every comb C the *comb inequality*

$$x(E(H)) + \sum_{i=1}^{k} x(E(T_i)) \leq |H| + \sum_{i=1}^{k} (|T_i| - 1) - \frac{k+1}{2} \tag{2.14}$$

is valid with respect to $\tilde{Q}_T^n$. Moreover, we have the following:

Theorem 2.15. (a) *All comb inequalities (2.14) define facets of $\tilde{Q}_T^n$, $n \geq 6$.*

(b) *All comb inequalities (2.14) define facets of Q_T^n, $n \geq 6$.*

(c) *Given two different combs defined by $H, T_1, \ldots, T_r$ and $H', T_1', \ldots, T_s'$, then the corresponding comb inequalities are equivalent with respect to Q_T^n if and only if $r = s$, $T_i = T_i'$ for $i = 1, \ldots, r$ and $H' = V \setminus H$. Comb inequalities (2.14) and subtour elimination constraints (2.11) are nonequivalent with respect to Q_T^n.*

A proof of 2.15(a) can be found in Grötschel (1977), proofs of (b) and (c) are in Grötschel and Padberg (1979b). Comb inequalities are generalizations of the class of comb inequalities introduced by Chvátal (1973a). The Chvátal combs (which define facets of Q_T^n and $\tilde{Q}_T^n$) are exactly those combs (2.13) where (a') $|H \cap T_i| = 1$ holds for $i = 1, \ldots, k$. We call such combs *simple*. Chvátal's comb inequalities in turn are generalizations of the 2-matching constraints introduced by Edmonds (1965) in connection with the perfect 2-matching problem. 2-matching inequalities are exactly those comb inequalities (2.14) where the corresponding combs (2.13) satisfy (a') $|H \cap T_i| = 1$ for $i = 1, \ldots, k$ and (b') $|T_i| = 2$ for $i = 1, \ldots, k$.

There are further classes of facets of Q_T^n resp. $\tilde{Q}_T^n$ known which are related to complicated classes of graphs like hypohamiltonian, hypotraceable, or nonhamiltonian and hypomatchable graphs, cf. Cornuéjols and Pulleyblank (1982), Grötschel (1980b), Maurras (1976). Our aim here is to generalize comb inequalities to a further very large class of facets.

Definition 2.16. A *clique tree* is a connected graph C for which the maximal cliques satisfy the following properties:

(1) The cliques are partitioned into two sets, the set of *handles* and the set of *teeth*.

(2) No two teeth intersect.

(3) No two handles intersect.

(4) Each tooth contains at least two nodes, at most $n-2$ nodes, and at least one node belonging to no handle.

(5) For each handle, the number of teeth intersecting it is odd and at least three.

(6) If a tooth T and a handle H have a nonempty intersection, then $H \cap T$ is an articulation set of the clique tree.

(In the sequel we always consider clique trees as subgraphs of K_n.)

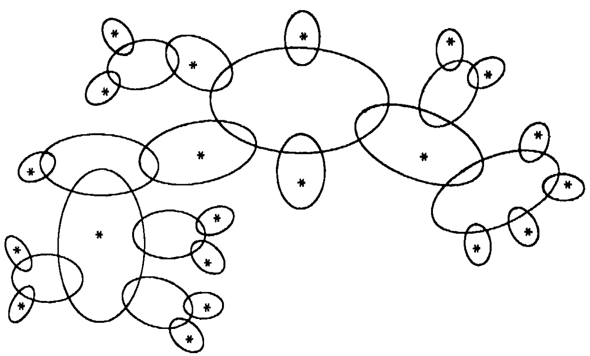


FIGURE 2.1

Figure 2.1 shows an example of a clique tree, where cliques are indicated by ellipse-shaped figures. Each ellipse containing a "*" is a tooth. The "*" indicates that there must be a node in the tooth which is not in any handle.

The graph of Figure 2.2 shows the smallest clique tree which is not a comb or a single clique. It is a graph on eleven nodes which has two handles (encircled) and five teeth, four of which have two nodes and one (the center) has three nodes.

We shall call those clique trees *simple* for which any handle and any tooth have at most one node in common. For example, the clique tree in Figure 2.2 is simple.

Suppose we have a clique tree C with handles $H_1, H_2, \ldots, H_r$ and teeth $T_1, T_2, \ldots, T_s$. We shall show in the sequel that the following *clique tree inequality* defines a facet of Q_T^n and $\tilde{Q}_T^n$

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} \qquad (2.17)$$

where for every tooth T_j, the integer t_j denotes the number of handles which intersect $T_j, j = 1, \ldots, s$. Note that in case there is a tooth T and a handle H with $|H \cap T| \geq 2$ then the coefficients on the left-hand side of (2.17) are 0, 1 and 2. The inequality (2.17) is a 0/1-inequality only if the clique tree is simple. If W is the set of all nodes of a clique tree, then for simple clique trees, inequality (2.17) can be written as

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq |W| - \frac{s+1}{2}. \qquad (2.18)$$

Note that the inequalities (2.11) and (2.14) are special cases of the clique tree inequalities (2.17). Any subtour elimination constraint can be considered as a clique

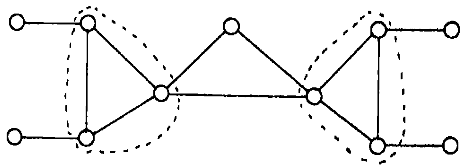


FIGURE 2.2

tree inequality where the clique tree consists of one tooth and no handle. The combs are the clique trees with exactly one handle, and for these the right-hand sides of (2.14) and (2.17) coincide.

To abbreviate our notation we shall call the right-hand side of a clique tree inequality (2.17) the *size* of the clique tree C, and we denote it by $s(C)$.

3. Construction of clique trees, validity. In this section we shall give constructive characterizations of clique trees ("gluing" and "splitting") and we shall prove that the clique tree inequalities (2.17) are valid with respect to $\tilde{Q}_T^n$ and Q_T^n.

The name clique tree reflects the fact that the structure of such a graph is "tree-like," namely, replacing every clique by a node and linking a pair of nodes whenever the corresponding cliques intersect we obtain a tree, the tree underlying the clique tree. It follows from Definition 2.16 that the partition of a clique tree into handles and teeth can be obtained with an analogue of "tree pruning" as follows.

Let us call a clique *pendent* if it meets at most one other clique. By (5) of (2.16) no handle can be pendent. Thus all the pendent cliques of a clique tree are teeth. If we remove these teeth from the clique tree the resulting graph will have new pendent cliques, call these cliques handles. Removing these handles we obtain a further graph whose pendent cliques are called teeth, etc. Obviously, Definition 2.16 implies that this pruning procedure ends in a nonambiguous way.

We shall now describe how one can glue and split clique trees. These methods will be important tools in subsequent inductive proofs.

3.1. *Gluing clique trees at a tooth.* Let C' and C'' be two clique trees with node sets V' and V'', and suppose $T := V' \cap V''$ is a tooth of C' and C'', that T contains a node not in any handle of C' and C'', and that the handles of C' and C'' do not intersect. Then $C' \cup C''$ is a clique tree called the clique tree obtained from C' and C'' by gluing at tooth T.

3.2. *Splitting a clique tree at a tooth and a handle.* Let C be a clique tree and T a tooth of C. Let H be a handle of C intersecting T. Delete the nodes $H \setminus T$ from C and let C'' be the component of $C - (H \setminus T)$ containing T. Delete the nodes from C which are in handles meeting T but not in T or H. Let C' be the component of this graph containing T. Then C' and C'' are clique trees called the clique trees obtained from C by splitting at T and H.

It is clear from Definition 2.16 and the operations defined in (3.1) and (3.2) that the graphs resulting from gluing and splitting are indeed clique trees. Figure 3.1 shows a gluing operation, and Figure 3.2 a splitting operation. In case T is a tooth intersecting one handle H only, we obtain the original clique tree and the complete graph

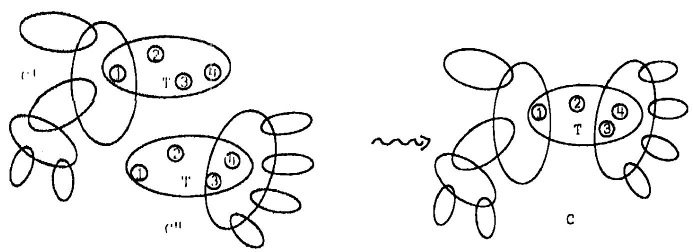


FIGURE 3.1

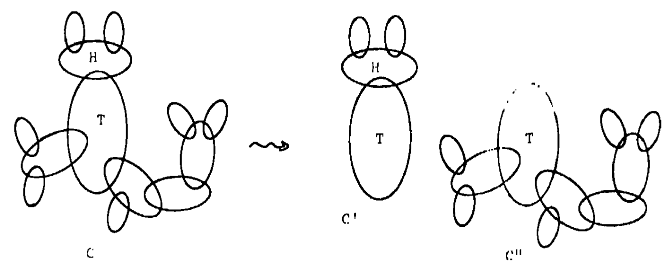


FIGURE 3.2

$[T, E(T)]$ as the result of splitting at T and H. If C is a clique tree which we split at a tooth T and a handle H to obtain two clique trees C' and C'' then, by gluing C' and C'' at T, the original clique tree C is reproduced. The gluing operation can be used to define clique trees inductively as follows:

DEFINITION 3.3. (a) Complete graphs with at least two and at most $n-2$ nodes are clique trees, and combs are clique trees.

(b) If C' and C'' are clique trees satisfying the assumptions of (3.1) then the graph resulting from gluing at tooth T is a clique tree.

(c) All graphs that can be generated by (a) and (b) are called clique trees.

LEMMA 3.4. *The clique tree Definitions* 3.3 *and* 2.16 *are equivalent.*

PROOF. For clique trees which are complete graphs there is nothing to prove. All clique trees in the sense of Definition 2.16 with one handle are combs, so they are clique trees in the sense of Definition 3.3. Suppose we have shown that the definitions are equivalent for all clique trees with $r \geq 1$ handles.

Assume that C is a clique tree in the sense of Definition 2.16 with $r+1$ handles. Then there is a tooth T meeting at least two handles. Let H be one of the handles and split at T and H. The resulting clique trees C' and C'' in the sense of Definition 2.16 are clique trees in the sense of Definition 3.3 by the induction hypothesis. By gluing C' and C'' at T we get back our original clique tree C. Thus by axiom (b) of Definition 3.3, C is a clique tree.

The gluing operation 3.1 shows that clique trees in the sense of Definition 3.3 are clique trees in the sense of Definition 2.16. ∎

It is easy to see that inductive definitions even more restrictive than Definition 3.3 would generate all clique trees. For example, requiring that one of the clique trees C' or C'' in axiom (b) be a comb would result in the same class of graphs. So all clique trees can be obtained by starting with a comb or a clique with k nodes, $2 \leq k \leq n-2$, and iteratively gluing combs onto the previously constructed clique tree.

Another splitting operation on clique trees is the following:

3.5. *Splitting a clique tree at a handle.* Let C be a clique tree and H a handle of C. Let $T_1, \ldots, T_k$ be the teeth of C which intersect H. For every tooth T_l, $l \in \{1, \ldots, k\}$, let C_l be the clique tree not containing H obtained from C by splitting at T_l and H. Then the clique trees $C_1, \ldots, C_k$ are called the clique trees obtained from C by splitting at H. Figure 3.3 shows the result of a splitting operation 3.5. It is easy to see how the sizes of the clique trees obtained by gluing or splitting relate to the sizes of the initial clique trees.

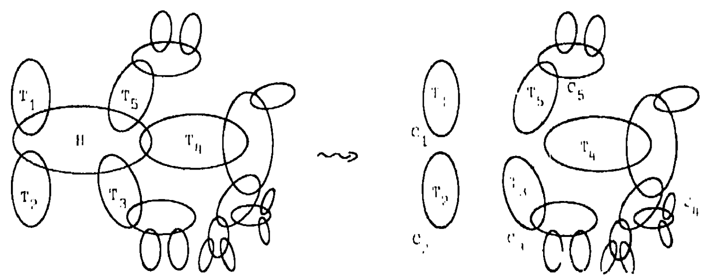


FIGURE 3.3

REMARK 3.6. (a) Let C be the clique tree obtained by gluing (3.1) clique trees C' and C'' at tooth T. Then

$$s(C') + s(C'') = s(C) + |T| - 1.$$

(b) Let C' and C'' be the clique trees obtained from C by splitting (3.2) at tooth T and handle H. Then

$$s(C') + s(C'') = s(C) + |T| - 1.$$

(c) Let C be a clique tree and H a handle of C intersecting k teeth. Let $C_1, \ldots, C_k$ be the clique trees obtained from C by splitting (3.5) at handle H. Then

$$\sum_{l=1}^{k} s(C_l) = s(C) - |H| + \frac{k+1}{2}.$$

THEOREM 3.7. *Let C be a clique tree in K_n with handles $H_1, \ldots, H_r$ and teeth $T_1, \ldots, T_s$. Then the clique tree inequality*

$$\sum_{i=1}^{r} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) \leq \sum_{i=1}^{r} |H_i| + \sum_{j=1}^{s} (|T_j| - t_j) - \frac{s+1}{2} = s(C)$$

is valid with respect to $\tilde{Q}_T^n$ (and hence with respect to Q_T^n).

PROOF. We prove by induction on the number of handles. If C has no handle then the clique tree inequality is a subtour elimination constraint and there is nothing to prove.

Suppose the claim is true for all clique trees with r handles, and assume C is a clique tree with $r+1$ handles. Pick any handle H of C and denote the other handles of C by $H_1, \ldots, H_r$. Let $T_1, \ldots, T_k$ be the teeth of C intersecting H, and let $C_1, \ldots, C_k$ be the clique trees obtained from C by splitting at H. Every such clique tree has at most r handles. We assume that C_l contains T_l, $l = 1, \ldots, k$. Let $a_l^T x \leq s(C_l)$ be the corresponding clique tree inequalities.

For every clique tree C_l, $l \in \{1, \ldots, k\}$, let $\overline{C}_l$ be the clique tree obtained from C_l by replacing T_l with $T_l \setminus H$, and let $\bar{a}_l^T x \leq s(\overline{C}_l)$ be the corresponding clique tree inequality.

By Remark 3.6 we have

$$\sum_{l=1}^{k} s(C_l) = s(C) - |H| + \frac{k+1}{2}$$

which implies

$$\sum_{i=1}^{k} s(\overline{C}_i) = s(C) - |H| - \sum_{i=1}^{k} |H \cap T_i| + \frac{k+1}{2}.$$

From this we obtain (setting $H_{r+1} := H$)

$$2\left(\sum_{i=1}^{r+1} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j))\right) \leq \sum_{i=1}^{k} \left(a_i^{\mathrm{T}}x + \bar{a}_i^{\mathrm{T}}x + x(E(H \cap T_i))\right) + \sum_{v \in H} x(\delta(v))$$

$$\leq \sum_{i=1}^{k} \left(s(C_i) + s(\overline{C}_i) + |H \cap T_i| - 1\right) + 2|H|$$

$$= 2s(C) + 1.$$

For every incidence vector of a path system the left-hand side above is an even integer. So, dividing by two and rounding down the right-hand side we get the desired result. ∎

4. Clique trees induce facets. In this section we prove that clique trees define facets of Q_T^n. The proof of the main theorem, Theorem 4.7, proceeds by induction on the number of handles and is rather long. Much of the proof relies on the existence of path systems satisfying several properties. We have separated the proofs of the existence of these path systems into several technical lemmas which precede Theorem 4.7.

In order to simplify the following we first introduce some new notions.

DEFINITION 4.1. Let C be a clique tree in K_n.

(a) Let T be a tooth of C and $H_1, \ldots, H_k$ be the handles of C that intersect T. We say that a path π *saturates* T, if the following holds

$$|\pi \cap E(T)| = |T| - 1,$$

$$|\pi \cap E(H_i \cap T)| = |H_i \cap T| - 1 \quad \text{for} \quad i = 1, \ldots, k.$$

(b) A path system P is said to *saturate* a clique tree C if its incidence vector satisfies the clique tree inequality (2.17) for C with equality. If in addition all edges in the path system are edges of C, then we call P a *path cover* of C.

We shall need the following construction several times in the sequel.

CONSTRUCTION 4.2. Suppose C is a comb with handle H and teeth $T_1, \ldots, T_k$. Pick any tooth, say T_1, and any (possibly empty) subset S of $H \setminus (T_2 \cup \cdots \cup T_k)$. Remove $S \cup (T_1 \setminus H)$ from C.

Now pick any pair of remaining teeth, say T_2 and T_3, and construct the following path. Start in some node of $T_2 \setminus H$, go through all nodes of $T_2 \setminus H$, go to a node in $T_2 \cap H$, go through all nodes of $T_2 \cap H$, if there is any node in $\overline{H} := H \setminus (T_2 \cup \cdots \cup T_k \cup S)$ then go to a node in $\overline{H}$, go through all nodes of $\overline{H}$, go to a node in $T_3 \cap H$, go through all nodes in $T_3 \cap H$, go to a node in $T_3 \setminus H$, go through all nodes in $T_3 \setminus H$ and end in some node of $T_3 \setminus H$.

Continue this pairing of teeth with T_l and T_{l+1} $l \in \{4, 6, \ldots, k-1\}$ just as described for T_2 and T_3 the only difference being that we go directly from $T_l \cap H$ to $T_{l+1} \cap H$ without running through $\overline{H}$. Call this construction *pairing of an even number of teeth* of a comb.

LEMMA 4.3. *Let C be a clique tree, let T be a pendent tooth of C and let H be the handle intersecting T. Let π be a path saturating T.*

(a) *For every node i of C, $i \notin T \cup H$, there is a path cover P of C which contains π as a path and which contains a path, say P_i, ending at i.*

(b) *Let u and z be the endnodes of π and suppose $z \in H$. Then for every $i \in H \setminus T$ there exists a path cover P of C which contains a path P_i ending at i and a path P_u ending at u and containing π as a subpath such that P_i and P_u are disjoint.*

PROOF. (1) Suppose C is a comb. If $i \notin H \cup T$, then remove T from C and pair the remaining teeth of C as described in Construction 4.2 making sure that one of the paths of the system ends in the specified node $i \notin T \cup H$.

If one endnode of π is in H and $i \in H \setminus T$, then proceed as follows. If i is in some other tooth, say T_1, then let P_i be a path saturating T_1 and ending at i. Suppose $i \in H$ and i is in no tooth of C, then pick any tooth, say T_1, construct a path saturating T_1 and ending at some node $j \in T_1 \cap H$. Add edge ij to this path and call it P_i. Now remove the nodes of the path P_i from C and pair the remaining teeth, making sure that π is a subpath of the path system constructed by pairing.

It is obvious that the path systems constructed above are covers of C, extend π and have the properties specified in (a) and (b).

(2) Now we assume that the lemma is true for all clique trees with $r \geq 1$ handles and suppose that C is a clique tree with $r + 1$ handles.

We remove all pendent teeth from C and call this new graph $\overline{C}$. The pendent cliques of $\overline{C}$ (there are at least two) are handles of C. Choose one of the pendent cliques of $\overline{C}$ which—as a handle of C—does not intersect T. Let H' be this handle of C. Clearly H' intersects one tooth, say T', of C which is not pendent in C. Let C' and C'' be the two clique trees obtained by splitting C at T' and H' (3.2). The clique tree C' is a comb with handle H' by construction, and C'' is a clique tree with r handles containing the initial tooth T and handle H.

We have to consider various cases.

Let D be the clique tree obtained from C'' by replacing T' by $T' \setminus H'$. Assume first that node i is in D. Then by our induction hypothesis there exists a path cover of D satisfying (a) resp. (b). Remove $T' \setminus H'$ from C' and pair the remaining even number of teeth of C' using Construction 4.2. Here we have to make sure that one of the paths obtained contains $|H' \cap T'| - 1$ edges of $E(H' \cap T')$ which obviously can be achieved. The union of these two path systems is a path cover of C with the required properties.

If i is not in D, then $i \notin H$ and we must thus consider only case (a). Assume that i is in C' but not in $T' \setminus H'$. Then let j be a node in T' which is not in any handle of C. Apply the induction hypothesis (a) to clique tree D with node j which gives us a path cover of D containing π and containing a path P_j ending at j.

We shall extend this path P_j in several ways depending on the location of i.

If $i \in H' \cap T'$ then we go from j to one node of $H' \cap T'$ and then through all nodes of $H' \cap T'$ and end at i. Call this path P_i. Remove all nodes of T' from the comb C' and pair the remaining even number of teeth of C' using Construction 4.2.

If $i \in H'$ but i is in no tooth of C' then we extend P_j by going from j to a node in $H' \cap T'$, then through all nodes in $H' \cap T'$, and finally to the node i. We remove all nodes in T' and node i from C' and pair the remaining teeth of C' using Construction 4.2.

If i is in some tooth, say $\overline{T}$, of C', then we proceed as follows. First we construct a path P_i saturating $\overline{T}$ and ending at i. We pick a tooth T'' different from T' and $\overline{T}$. Then we extend P_j by going from j to a node in $H' \cap T'$, through all nodes in $H' \cap T'$, then to a node in H' which is in no tooth (if there is such a node), we continue through all nodes in H' which are in no tooth, go to a node in $H' \cap T''$, continue through all nodes in $H' \cap T''$, go to a node in $T'' \setminus H'$ and run through all nodes in $T''\backslash H'$ ending in a node of $T''\backslash H'$. We remove all nodes from C' which are in P_i or in the extension of P_j just described before. Then we pair the remaining even number (possibly 0) of teeth of C' using Construction 4.2.

In all three cases we have constructed a path cover of C' containing a path P_i ending at i. It is easy to see that the union of the path cover of D and any of the three path covers of C (including the edge from j to a node in $T' \cap H'$) gives a path cover of C with the required properties. ∎

LEMMA 4.4. *Let C be a clique tree, T be a tooth of C and π be a path saturating T.*

(a) *For every node i of C which is not in T and not in any handle intersecting T there is a path cover P of C containing π as a path and containing a path P_i ending at i.*

(b) *Let u and z be the endnodes of π and suppose z is in a handle H. Then for every $i \in H\backslash T$ there exists a path cover P of C which contains a path P_i ending at i and a path P_u ending at u and containing π such that P_i and P_u are disjoint.*

PROOF. If T is a pendent tooth then the claim follows from Lemma 4.3. So suppose T is not pendent and the node i satisfies the requirements of (a) (or (b)).

Let $H_1, \dots, H_k$ be the handles intersecting T. Denote by C_j', C_j'' the clique trees obtained from splitting (3.2) at T and H_j, $j = 1, \dots, k$. By construction, the tooth T is pendent in every clique tree C_j', $j = 1, \dots, k$.

The given node i is in one of these clique trees C_j', say in C_1'. If i is in $H_1\backslash T$ then the endnode z of π must be in H_1. Applying Lemma 4.3(a) (or (b) if $i \in H_1\backslash T$) to C_1', T, H_1 and i we obtain a path cover P_1' of C_1' containing a path P_i ending at i and containing π as a path (resp. containing a path P_u containing π and ending at u such that P_i and P_u are disjoint).

For every other clique tree C_j', $j = 2, \dots, k$, we choose a node i_j which is in a tooth of C_j' different from T and not in a handle. Then we apply Lemma 4.3(a) to C_j', T, H_j and i_j to obtain a path cover P_j' of C_j' which contains π as a path.

The union of the path covers $P_1', \dots, P_k'$ of $C_1', \dots, C_k'$ is a path cover of C (this is easy to see by comparing the sizes of $C_1', \dots, C_k'$ and C) which has the desired properties. ∎

LEMMA 4.5. *Let C be a clique tree, H a handle of C and T a tooth intersecting H. Let p, q be two nodes in H which are in different teeth intersecting H but not in T. Let π be a path whose endnodes are p and q and which contains all nodes of H which are not in teeth, all nodes in $H \cap T$ but no other nodes and which satisfies $|\pi \cap E(H \cap T)| = |H \cap T| - 1$. Then there exists a path cover of C containing π as a subpath and which contains two different paths each having at least one endnode in a (pendent) tooth of C. (In case $s = 3$ one of these two paths may be degenerate.)*

PROOF. Let $T_1, \dots, T_k$ be the teeth intersecting H. Let us assume without loss of generality that $T = T_1$, $p \in T_2$ and $q \in T_3$. For every tooth T_i, $i = 2, \dots, k$ construct a path π_i that saturates T_i and has one endnode in $H \cap T_i$. In case $i = 2$ or 3 this endnode should be p resp. q.

Let $C_1, C_2, \dots, C_k$ be the clique trees obtained by splitting (3.5) at H where T_i is in C_i, $i = 1, \dots, k$. For $i = 2, \dots, k$ use Lemma 4.4 and a node j_i in a (pendent) tooth of C_i different from T_i to obtain a path cover P_i of C_i containing π_i as a path. Note that if $C_i = T_i$, we set $P_i = \{\pi_i\}$. Let C_1' be the clique tree obtained from C_1 replacing T_1 by $T_1\backslash H$ and let P_1 be a path cover of C_1' in which one path ends at a node in a (pendent) tooth. Note that in case T_1 is pendent in C and $|T_1| = 2$, then P_1 is degenerate. Let P be the union of the path systems $P_1, \dots, P_k$, and π and the edges e_i, $i = 4, 6, \dots, k-1$, whose endnodes are the endnodes of π_i and π_{i+1} in $H \cap T_i$ resp. $H \cap T_{i+1}$.

By construction P is a path system, contains π as a subpath and contains at least two different paths ending at a node in a (pendent) tooth of C.

Let $a^{\mathrm{T}}x \le s(C)$ and $a_i^{\mathrm{T}}x \le s(C_i)$ be the clique tree inequalities for C and C_i, $i = 1, 2, \dots, k$. If F is the set of edges in $E(H)$ which do not have both endnodes in the same tooth and $\mathring{H}$ is the set of nodes in H which are in no tooth, then obviously $|F \cap P| = |\mathring{H}| + (k+1)/2$. Note also that $a_1^{\mathrm{T}}x^P = s(C_1) - 1$, then using Remark 3.6(c)

$$\begin{aligned} a^{\mathrm{T}}x^P &= \sum_{i=1}^{k} a_i^{\mathrm{T}}x^P + \sum_{i=1}^{k} |P \cap E(H \cap T_i)| + |F \cap P| \\ &= \sum_{i=1}^{k} s(C_i) - 1 + \sum_{i=1}^{k} (|H \cap T_i| - 1) + |\mathring{H}| + \frac{k+1}{2} \\ &= s(C) - |H| + \frac{k+1}{2} - 1 + |H| - k + \frac{k+1}{2} \\ &= s(C). \end{aligned}$$

Hence P saturates C. ∎

LEMMA 4.6. *Let C be a clique tree in K_n, T a tooth of C, and V' the node set of C. Let $H_1, \dots, H_k$ be the handles intersecting T, $\mathring{T} := T\backslash(\bigcup_{i=1}^{k} H_i)$, and let $a^{\mathrm{T}}x \le s(C)$ be the clique tree inequality for C. Then there are two nodes $u, u' \in V'\backslash T$ which are in different teeth of C (these teeth may be chosen to be pendent) such that there is a path P with endnodes u and u', containing all nodes of $(V'\backslash\mathring{T})$ and satisfying*

(a) $|P \cap E(H_j \cap T)| = |H_j \cap T| - 1$, $j = 1, \dots, k$,

(b) $a^{\mathrm{T}}x^P = s(C) - |\mathring{T}| + 1$.

PROOF. Let C_j be the clique tree obtained from C by splitting (3.2) at T and H_j which contains T and H_j, for $j = 1, \dots, k$.

For every $j \in \{1, \dots, k\}$ pick two nodes p_j, q_j in H_j and a path π_j such that all requirements of Lemma 4.5 are satisfied with respect to the tooth T of C_j. Then for every $j = 1, \dots, k$ there exists a path cover P_j' of C_j with the properties specified in Lemma 4.5.

Clearly, P_j' contains a hamiltonian path in $T\backslash H_j$. Remove this path from P_j' and call the remaining path system P_j. Let P' be the union of the path systems P_j, $j = 1, \dots, k$, then it is obvious from the construction that $|P' \cap E(H_j \cap T)| = |H_j \cap T| - 1$, $j = 1, \dots, k$ and $a^{\mathrm{T}}x^{P'} = s(C) - |\mathring{T}| + 1$. By adding appropriate edges (from K_n, not in $E(C)$) we can easily turn P' into a path P with the required properties. ∎

Using Lemmas 4.3, ..., 4.6 we shall now prove our main result.

THEOREM 4.7. *Let C be a clique tree in $K_n = [V, E]$. Then the corresponding clique tree inequality (2.17) defines a facet of Q_T^n.*

(Note that for $n \le 3$, K_n contains no clique trees. For $n = 4, 5$, the only clique trees consist of single cliques and their corresponding inequalities are equal to or equivalent to (see Theorem 2.12) the trivial inequalities $x_{ij} \le 1$ which we have already remarked are facet inducing. So although Theorem 4.7 is true for all values of n, it is of most interest for $n \ge 6$ and, in particular, for $n \ge 11$ when clique trees which are neither single cliques nor combs exist.)

PROOF. 1. We prove the theorem by induction on the number of handles. If C has no handle then C is a complete graph and (2.17) is a subtour elimination constraint. If

C has one handle then C is a comb. It was shown in Grötschel and Padberg (1979b) that subtour elimination constraints and comb inequalities define facets of Q_T^n.

2. Now we assume that the theorem is true for all clique trees with $r \geq 1$ handles and assume that C is a clique tree with $r+1$ handles. The corresponding clique tree inequality, denoted by $a^Tx \leq \alpha$ in the sequel, is valid with respect to Q_T^n by Theorem 3.5. Moreover, it is easy to find a tour in K_n whose incidence vector does not satisfy $a^Tx \leq \alpha$ with equality. Thus, $F_C := Q_T^n \cap \{x \in \mathbb{R}^m \mid a^Tx = \alpha\}$ is a proper facet of Q_T^n.

Suppose now that $d^Tx \leq \delta$ is a facet defining inequality for Q_T^n which satisfies $F_C \subseteq F_d := Q_T^n \cap \{x \mid d^Tx = \delta\}$. We shall show that there exist $\bar{\rho} > 0$ and a vector $\bar{\lambda} \in \mathbb{R}^n$ with $a^T = \bar{\rho} d^T + \bar{\lambda}^T A$, where A is the node-edge incidence matrix of K_n, cf. (2.8). By Lemma 2.9 this will prove our claim.

3. As can be expected our proof will be rather technical. We have to perform various splits, tour constructions and coefficient calculations to obtain the desired result. We now introduce the various clique trees, teeth, handles and node sets we are going to need in the sequel. In part 5 of this proof we introduce an inequality $b^Tx \leq \beta$ which is equivalent to $d^Tx \leq \delta$ by construction. The purpose of the remaining parts $6, \ldots, 14$ is to show that $b = \rho a$ holds for some $\rho > 0$. The best way to follow the stages of the proof is to keep track by drawing the parts of the clique tree C presently under consideration and recording the coefficients of b already calculated.

By assumption, C has at least two handles. Thus there is a tooth, say T, which intersects at least two handles. Choose one of these handles and call it H_1. Denote by $\mathring{T}$ the set of nodes of T which are in no handle of C. Note that $\mathring{T}$ is nonempty by Definition 2.16 (4), and let w be a fixed node in $\mathring{T}$.

Split (3.2) clique tree C at tooth T and handle H_1 to obtain clique trees C' and C''. By definition, C' contains T and H_1.

We now modify C' and C'' slightly. First denote by $\overline{H}$ the set of nodes in C which are in handles intersecting T but not in H_1. Now replace in C' the tooth T by the tooth $T_1 := T \setminus \overline{H}$ and call the new clique tree C_1. In C'' the tooth T is replaced by $T_2 := T \setminus H_1$, and the new clique tree is called C_2.

The set of nodes of K_n is as usual denoted by V. The set of nodes which are not contained in C is denoted by $\overline{V}$, and the node set of C_1 resp. C_2 is denoted by V_1 resp. V_2. Let K^1 be the complete graph on the node set V_1 and K^2 be the complete graph on the node set $V_2 \cup \overline{V}$.

4. By Lemma 4.6 there are two nodes, say u and u', which are in two different teeth ($\neq T_1$) of C_1 but in no handle of C_1 such that there is a hamiltonian path P_1 in $K^1 - \mathring{T}$ with endnodes u and u' satisfying $|P_1 \cap E(H_1 \cap T_1)| = |H_1 \cap T_1| - 1$ and $a^Tx^{P_1} = s(C_1) - |\mathring{T}| + 1$.

Similarly, by Lemma 4.6 there are two nodes v and v'' which are in two different teeth ($\neq T_2$) of C_2 but in no handle of C_2 such that there is a hamiltonian path P_2' in $V_2 - \mathring{T}$ with endnodes v and v'' satisfying $|P_2' \cap E(H_j \cap T_2)| = |H_j \cap T_2| - 1$ for all handles H_j of C_2 intersecting T_2 and such that $a^Tx^{P_2'} = s(C_2) - |\mathring{T}| + 1$. If $\overline{V} = \emptyset$, set $v' := v''$ and $P_2 := P_2'$. Otherwise choose any node $v' \in \overline{V}$ and extend P_2' to a hamiltonian path P_2 of K^2 by going from v'' through all nodes of $\overline{V}$ (in any order) ending at v'.

Note that the following holds by construction and Remark 3.6(a) for the incidence vectors x^{P_1}, x^{P_2} of P_1 and P_2:

$$s(C_1) + a^Tx^{P_2} = s(C),$$

$$s(C_2) + a^Tx^{P_1} = s(C),$$

$$a^Tx^{P_1} + a^Tx^{P_2} = s(C) - |\mathring{T}| + 1.$$

This also implies that every path cover of C_1 resp. C_2 can be extended to a path system saturating C by adding P_2 resp. P_1, and that $P_1 \cup P_2$ can be extended to a path system saturating C by adding a hamiltonian path of $[\mathring{T}, E(\mathring{T})]$.

5. We shall now construct a vector b such that $b^Tx \leq \beta$ is equivalent to $d^Tx \leq \delta$ and such that some of the components of b are equal to the corresponding components of a.

It is easy to see that the columns of the node-edge incidence matrix A corresponding to the edges

$$\begin{aligned} &ui, && i \in V_2 \cup \overline{V}, \\ &vi, && i \in V_1 \setminus \mathring{T}, \\ &vw, && \end{aligned}$$

are linearly independent, where the nodes u, v were defined in 4 and $w \in \mathring{T}$ was fixed in 3. (Note that $V_2 \cup \overline{V} \cup (V_1 \setminus \mathring{T}) = V$.) Thus, there exists a unique vector $\lambda \in \mathbb{R}^n$ such that $b^T := d^T + \lambda^T A$ satisfies $b_{ui} = a_{ui}$, $b_{vi} = a_{vi}$, $b_{vw} = a_{vw}$, i.e.

$$b_{ui} = a_{ui} = 0 \qquad \text{for all} \quad i \in V_2 \cup \overline{V}, \tag{1}$$

$$b_{vi} = a_{vi} = 0 \qquad \text{for all} \quad i \in V_1 \setminus \mathring{T}, \tag{2}$$

$$b_{vw} = a_{vw} = 0, \tag{3}$$

and we shall show that there exists $\rho > 0$ such that $b = \rho a$ holds, which will prove our theorem. (Note that we also have $\beta = \delta + \lambda^T 2$.)

6. Using (1), (2), (3) we shall calculate further components of the vector $b \in \mathbb{R}^m$.

Let P_1 and P_2 be the paths introduced in 4. Let P_3 be a path containing all nodes of $\mathring{T}$ ending at w. If $|\mathring{T}| > 1$, then let w' be the other end of P_3. If $\mathring{T} = \{w\}$, then P_3 is degenerate and we set $w' := w$. Adding to $P_1 \cup P_2 \cup P_3$ the edges $u'w'$, vw and uv' we obtain a tour S in K_n which by construction saturates C, cf. 4. The tour $S' := (S \setminus \{uv', vw\}) \cup \{v'w, uv\}$ also saturates C. Thus the incidence vectors x^S and $x^{S'}$ are in F_b. Therefore we get from (1) and (3)

$$0 = b^Tx^S - b^Tx^{S'} = b_{uv'} + b_{vw} - b_{v'w} - b_{uv} = -b_{v'w}. \tag{4}$$

We construct two further tours from P_1, P_2, P_3. Adding to $P_1 \cup P_2 \cup P_3$ the edges uw', vw and $u'v'$ we obtain a tour S in K_n which saturates C. The clique tree C is also saturated by the tour $S' := (S \setminus \{u'v', vw\}) \cup \{v'w, vu'\}$. Using (3), (4) and (2) we obtain

$$0 = b^Tx^S - b^Tx^{S'} = b_{u'v'} + b_{vw} - b_{v'w} - b_{vu'} = b_{u'v'}. \tag{5}$$

Next we shall calculate further values $b_{u'i}$ and $b_{v'i}$.

Let T' be the tooth of C, resp. C_2, containing v'' and let i be any node in $V_2 \setminus T'$. Let Π' be a path saturating T' and starting at v''. In case i is in a handle intersecting T' we require that the other endnode of π' be in this handle. Then by Lemma 4.4 there exists a path cover of C_2 containing paths P_i and $P_{v''}$ which end at i and v'' respectively and have no node in common. By extending $P_{v''}$ to $P_{v'}$ ending at v' and adding appropriate edges, this path cover of C_2 can be extended to a hamiltonian path P of K^2 with endnodes i and v'. Adding the hamiltonian path P_1 of $K^1 - \mathring{T}$, cf. 4, and the edges $u'v'$, ui we obtain a tour S in K_n saturating C. Clearly, the tour $S' := (S - \{ui, u'v'\}) \cup \{u'i, uv'\}$ also saturates C. Thus, by construction, we have $a^Tx^S = a^Tx^{S'} = \alpha$ which by our assumption $F_C \subseteq F_b$ (5) and (1) implies

$$0 = b^Tx^S - b^Tx^{S'} = b_{u'v'} + b_{ui} - b_{u'i} - b_{uv'} = -b_{u'i}.$$

If i is a node in T' we use the node v instead of v'' in the construction above to obtain $b_{u'i} = 0$. If i is a node in $\bar{V}$ then a similar (but simpler) construction gives the same result. Therefore we get

$$b_{u'i} = a_{u'i} = 0 \qquad \text{for all} \quad i \in V_2 \cup \bar{V}. \tag{6}$$

Exchanging the roles of the clique trees C_1 and C_2, the nodes u, u' and v, v', and the paths P_1 and P_2 we obtain by symmetry

$$b_{v'i} = a_{v'i} = 0 \qquad \text{for all} \quad i \in V_1 \setminus \mathring{T}. \tag{7}$$

7. We shall now apply the induction hypothesis to C_2 to calculate the coefficients b_{ij} for $i, j \in V_2 \cup \bar{V}$.

Consider the clique tree C_2 as a clique tree in the complete graph, say $\bar{K}^2$, on the node set $V_2 \cup \bar{V} \cup \{u\}$, and let $Q_T^{n_2}$ be the corresponding travelling salesman polytope, $n_2 = |V_2| + |\bar{V}| + 1$. Let $\bar{a}^{\mathrm{T}}x \leq s(C_2)$ be the clique tree inequality for C_2 with respect to $Q_T^{n_2}$, i.e. $\bar{a}$ is a vector in $\mathbb{R}^{m_2}$ with $m_2 = n_2(n_2-1)/2$. Moreover, let $\bar{b}$ be the vector obtained from b by removing all components which do not correspond to an edge in $\bar{K}^2$.

Suppose S_2 is any hamiltonian cycle in $\bar{K}^2$ saturating C_2. Since u is in $\bar{K}^2$, S_2 contains an edge ui, $i \in V_2 \cup \bar{V}$. Removing this edge from S_2, adding the edge $u'i$ and the path P_1, cf. 4, we obtain a tour in K_n which by construction saturates C. From (1) and (6) we have $b_{ui} = b_{u'i} = 0$ for all $i \in V_2 \cup \bar{V}$. Thus we obtain that every tour S_2 in $\bar{K}^2$ saturating C_2 satisfies

$$\bar{b}^{\mathrm{T}}x^{S_2} = \beta - b^{\mathrm{T}}x^{P_1} := \bar{\beta}. \tag{8}$$

If $\bar{b}^{\mathrm{T}}x \leq \bar{\beta}$ were not valid with respect to $Q_T^{n_2}$, then we could pick a tour in $\bar{K}^2$ violating this inequality, extend it to a tour in K_n as above and obtain a tour in K_n whose incidence vector violated $b^{\mathrm{T}}x \leq \beta$. This implies that $F_{\bar{b}} := \{x \in Q_T^{n_2} \mid \bar{b}^{\mathrm{T}}x = \bar{\beta}\}$ is a face of $Q_T^{n_2}$. (Using the same extension argument it is easy to see that $F_{\bar{b}} \neq Q_T^{n_2}$.)

Now by our induction hypothesis, the clique tree inequality for C_2 defines a facet $F_{\bar{a}} := \{x \in Q_T^{n_2} \mid \bar{a}^{\mathrm{T}}x = s(C_2)\}$ of $Q_T^{n_2}$. (8) implies that $F_{\bar{a}} \subseteq F_{\bar{b}}$. Since $F_{\bar{b}} \neq Q_T^{n_2}$ we necessarily have $F_{\bar{a}} = F_{\bar{b}}$, i.e. $\bar{a}^{\mathrm{T}}x \leq s(C_2)$ and $\bar{b}^{\mathrm{T}}x \leq \bar{\beta}$ are equivalent with respect to $Q_T^{n_2}$. Thus Lemma 2.9 implies that there are λ_i, $i \in V_2 \cup \bar{V} \cup \{u\}$, $\rho_2 > 0$ such that $\bar{b}_{ij} = \rho_2 \bar{a}_{ij} + \lambda_i + \lambda_j$ for all edges ij in $\bar{K}^2$ which by construction gives

$$b_{ij} = \rho_2 a_{ij} + \lambda_i + \lambda_j \qquad \text{for all} \quad i, j \in V_2 \cup \bar{V} \cup \{u\}. \tag{9}$$

From (1) we know that $b_{ui} = a_{ui} = 0$ for all $i \in V_2 \cup \bar{V}$, therefore (9) implies $\lambda_i = -\lambda_u$ for all $i \in V_2 \cup \bar{V}$. By (3) we have $b_{vw} = a_{vw} = 0$. Since $v, w \in V_2$, we get from (9) $\lambda_v = -\lambda_w$, and thus $-\lambda_u = \lambda_v = -\lambda_w = \lambda_u$. Hence $\lambda_u = 0$, and therefore $\lambda_i = 0$ for all $i \in V_2 \cup \bar{V} \cup \{u\}$, and (9) implies

$$b_{ij} = \rho_2 a_{ij} \qquad \text{for all} \quad i, j \in V_2 \cup \bar{V} \cup \{u\}. \tag{10}$$

8. The induction hypothesis is now applied to the "other side" of C. Let $\bar{K}^1$ be the complete graph on $V_1 \cup \{v\}$, where v is defined in 4, and let $Q_T^{n_1}$ be the corresponding travelling salesman polytope. We consider C_1 as a clique tree in $\bar{K}^1$ and denote the clique tree inequality for C_1 with respect to $Q_T^{n_1}$ by $\bar{a}^{\mathrm{T}}x \leq s(C_1)$. Similarly let $\bar{b}$ be the vector obtained from b by removing all components not corresponding to edges in $\bar{K}^1$.

Let S_1 be any hamiltonian cycle in $\bar{K}^1$ saturating C_1. Node v has two neighbours in S_1. At most one of these two nodes is in T_1, for otherwise it would be easy to construct a tour S' in K^1 with $\bar{a}^{\mathrm{T}}x^{S'} > \bar{a}^{\mathrm{T}}x^{S_1} = s(C_1)$, a contradiction. Suppose i is a neighbour of v in S_1 which is not in T_1. Removing edge vi from S_1, adding edge $v'i$ and the path P_2, cf. 4, we obtain a tour in K_n saturating C. Since by (2) and (7) $b_{vi} = b_{v'i} = 0$ for all $i \in V_1 \setminus \mathring{T}$ we obtain

$$\bar{b}^{\mathrm{T}}x^{S_1} = \beta - b^{\mathrm{T}}x^{P_2}. \tag{11}$$

By our induction hypothesis, $\bar{a}^{\mathrm{T}}x \leq s(C_1)$ defines a facet; thus, using the same argument as in 7, we obtain

$$b_{ij} = \rho_1 a_{ij} + \mu_i + \mu_j \qquad \text{for all } i, \quad j \in V_1 \cup \{v\} \tag{12}$$

and some $\mu_i \in \mathbb{R}$, $i \in V_1 \cup \{v\}$, $\rho_1 > 0$.

By (2) $b_{vi} = a_{vi} = 0$ for all $i \in V_1 \setminus \mathring{T}$ and hence (12) gives $\mu_i = -\mu_v$ for all $i \in V_1 \setminus \mathring{T}$. By (1) $b_{ui} = a_{ui} = 0$ for all $i \in \mathring{T}$; thus we get from (12) $\mu_i = -\mu_u$ for all $i \in \mathring{T}$. Moreover (3) implies $\mu_w = -\mu_v$. Since $w \in \mathring{T}$ we get $-\mu_v = \mu_w = -\mu_u$, so $\mu_v = \mu_u$. On the other hand we have, from (1), $b_{uv} = a_{uv} = 0$ and so (12) implies $\mu_v = -\mu_u$. Hence $\mu_u = \mu_v = -\mu_u$ which implies $\mu_u = 0$ and therefore $\mu_i = 0$ for all $i \in V_1 \cup \{v\}$. Thus we get from (12)

$$b_{ij} = \rho_1 a_{ij} \qquad \text{for all} \quad i, j \in V_1 \cup \{v\}. \tag{13}$$

9. Since $\mathring{T} \subseteq V_1 \cap V_2$, we immediately have $\rho_1 = \rho_2$ in case $|\mathring{T}| \geq 2$.

Suppose $\mathring{T} = \{w\}$. Let $j \in H_1 \cap T$ and $k \in \bar{H} \cap T$ and let π be a hamiltonian path in T starting at w, going to j and ending at k which saturates T with respect to C. This implies that π saturates T also with respect to the clique trees C' and C'', see 3.

Apply Lemma 4.4(a) to C' so as to obtain a path cover containing π as a path and having a path ending at u. The paths different from π can be linked up to give a single path Q_1 in C' ending at u and some other node, say $\bar{u}$. Similarly, we apply Lemma 4.4(a) to C'' and thereby obtain a path Q_2, having one end equal to v and the other to $\bar{v}$ say, such that Q_2 contains all nodes not in $V_1 \cup T$ and such that $Q_2 \cup \pi$ saturates C''. Note that by the construction used in the proof of Lemma 4.4(a) we may assume that $\bar{u} \notin H_1$ and $\bar{v}$ does not belong to the handle in C'' containing k.

We now combine Q_1, Q_2 and π to form a tour S_1 by adding the edges $uw, k\bar{v}, v\bar{u}$. By construction, S_1 saturates C. Note that $b_{uw} = b_{k\bar{v}} = b_{v\bar{u}} = 0$ and $b_{jw} = \rho_1$. The tour $S_2 = S_1 \setminus \{jw, uw, \bar{u}v, \bar{v}k\} \cup \{wk, vw, j\bar{u}, u\bar{v}\}$ also saturates C and we have $b_{wk} = \rho_2$, $b_{vw} = b_{j\bar{u}} = b_{u\bar{v}} = 0$. Thus we get $0 = b^{\mathrm{T}}x^{S_1} - b^{\mathrm{T}}x^{S_2} = \rho_1 - \rho_2$, i.e.

$$\rho := \rho_1 = \rho_2. \tag{14}$$

Note that if S is a tour in K_n constructed in 7 or 8 by removing an edge from a tour on $\bar{K}^2$ or $\bar{K}^1$ and adding the path P_1 or P_2 and an edge $u'i$ or $v'i$, then the incidence vector x^S of S satisfies $a^{\mathrm{T}}x^S = \alpha$ and $b^{\mathrm{T}}x^S = \beta$, and, moreover, (1), (2), ..., (14) imply that $b_e = \rho a_e$ for all $e \in S$. Thus $b^{\mathrm{T}}x^S = \rho a^{\mathrm{T}}x^S = \rho\alpha$ which gives

$$\beta = \rho\alpha. \tag{15}$$

10. We still have to calculate the values b_e for "crossing" edges e. The principal idea is to construct a tour S saturating C such that all coefficients b_e, $e \in S$, are known to equal ρa_e from the discussions before except for one such coefficient, say b_f, whose value is not yet known. Since S saturates C, $\rho\alpha = \beta$ by (15) and by the assumption $F_C \subseteq F_b$ we have $0 = \rho\alpha - \beta = \rho a^{\mathrm{T}}x^S - b^{\mathrm{T}}x^S = \rho a_f - b_f$ which implies $b_f = \rho a_f$.

It follows from (1), ..., (15) that the so far undetermined coefficients b_{ij} are the following:

$$i \in V_1 \setminus (\mathring{T} \cup \{u, u'\}), \qquad j \in (\bar{V} \cup V_2) \setminus (\mathring{T} \cup \{v, v'\}) \tag{16}$$

cf. 3 for the definitions.

11. We first consider the case $i \in V_1 \setminus T$, $j \in (\bar{V} \cup V_2) \setminus T$.

three cliques of C_2 each meeting exactly one node of U. This contradicts $b_{uv} = 2$ for all $u, v \in U$. Thus, the first case cannot occur.

Case 2. We claim that $V \setminus (N \cup U) = \emptyset$. For if not, choose any node p in this set. Since C_1 is a clique tree there must be a node $u \in U$ with $a_{up} = 0$. Since $\lambda_u = \frac{1}{2}$ and $\lambda_p \geq 0$ we must have $b_{up} = 1$ or $b_{up} = 2$, and so $\lambda_p = \frac{1}{2}$ or $\lambda_p = \frac{3}{2}$.

If $\lambda_p = \frac{3}{2}$, then for any $v \in U$ we have $b_{vp} = 2$, hence $b_{vw} = 2$ for any $v, w \in U$. But this means that $a_{vw} = 1$ for all $v, w \in U$. Thus U is a clique of C_1 which is impossible.

Therefore we have $\lambda_p = \frac{1}{2}$. By definition, in C_1 node p is adjacent to every node $w \in N$ which implies $a_{pw} \geq 1$ and thus $b_{pw} \geq 1$. Moreover, since $\lambda_p = \frac{1}{2}$ we have $b_{pu} \geq 1$ for all $u \notin N$. But then C_2 contains the star $\delta(p)$ which is impossible. Therefore we conclude that $V \setminus (N \cup U)$ is empty and that $N = V \setminus U$ is a maximal clique of C_1.

Suppose that N is a tooth of C_1. Then U contains two teeth T_1 and T_2 of C_1 which meet the same handle H. Let $u \in T_1 \cap H$ and $v \in T_2 \cap H$ and let $w \in T_1 \setminus H$, then $b_{uv} = b_{uw} = 2$ and $b_{vw} = 1$ which is impossible.

Suppose that N is a handle. If C_1 is a comb then we have alternative (b) of our theorem. Otherwise, in C_1 a tooth T joins N to a handle H which is completely contained in U. By Definition 2.16 there are nodes $u \in T \setminus (N \cup H)$, $v \in T \cap H$ and $w \in H \setminus T$ and we can conclude that $b_{uv} = b_{vw} = 2$ and $b_{uw} = 1$ which is impossible in a clique tree. This finishes the second case and therefore the proof of Theorem 5.1. ∎

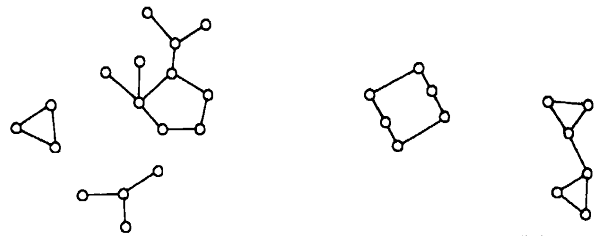
6. The relationship of Q_T^n and $\tilde{Q}_T^n$. We noted in the introduction that the travelling salesman polytope Q_T^n is a face of the monotone travelling salesman polytope $\tilde{Q}_T^n$, namely the face obtained by requiring $Ax = 2$ where A is the node-edge incidence matrix of K_n. Therefore, for each facet F of Q_T^n there exist one or more facets F' of $\tilde{Q}_T^n$ such that $F = F' \cap Q_T^n$. In this section we consider the question of when a facet inducing inequality $a^T x \leq a_0$ of Q_T^n is also facet inducing for $\tilde{Q}_T^n$. We prove a sufficient condition for this to be the case and show that this condition is indeed the case for clique tree inequalities. We also describe how any facet inducing inequality can be transformed into an equivalent facet inducing inequality which will also be facet inducing for $\tilde{Q}_T^n$.

Let $a^T x \leq a_0$ be facet inducing for Q_T^n. If $a \geq 0$ then the inequality is valid for $\tilde{Q}_T^n$. If not, by Lemma 2.9 we can let $\bar{a}^T := a^T + \lambda^T A$ and $\bar{a}_0 := a_0 + \lambda^T \mathbf{2}$ for λ sufficiently large and obtain an equivalent inequality $\bar{a}^T x \leq \bar{a}_0$ with $\bar{a} \geq 0$.

Let $\mathscr{J}$ be the family of all subsets of the edges for which the corresponding subsets of the columns of A are linearly independent. Then $(E, \mathscr{J})$ is a matroid defined on E, which we denote by $\mathscr{M}(G)$. This matroid is sometimes called the *real matroid* of G. It is well known, and easily verified, that $J \subseteq E$ is independent in $\mathscr{M}(G)$ (i.e. the corresponding columns of A are linearly independent) if and only if each component of the graph $[V, J]$ contains no even cycle and at most one odd cycle. The circuits or minimal dependent sets of $\mathscr{M}(G)$ are the sets $C \subseteq E$ such that either C is the edge set of an even cycle of G or else C is the edge set of two edge disjoint odd cycles of G joined by a path. See Figure 6.1.

It is easy to see that $J \subseteq E$ is a basis of $\mathscr{M}(K_n)$ for $n \geq 3$ if and only if every component of $[V, J]$ has exactly one odd cycle and no even cycle. For example, if T is the edge set of a spanning tree of G and $j \in E \setminus T$ creates an odd cycle when added to T, then $T \cup \{j\}$ is a basis.

For $j \in E$, let A_j denote the corresponding column of A. For $S \subseteq E$, let A_S denote the $(n \times |S|)$-submatrix of A consisting of those columns corresponding to members of S. The set S contains a basis of $\mathscr{M}(K_n)$ if and only if the rows of A_S are linearly independent, i.e., for any $\lambda \in \mathbb{R}^n$ satisfying $\lambda^T A_S = 0$ we must have $\lambda = 0$. Since the rows of A are linearly independent, $\lambda \in \mathbb{R}^n$ satisfies $\lambda^T A = 0$ if and only if $\lambda = 0$. Thus



An independent set of edges of $\mathscr{M}(G)$. Two circuits of $\mathscr{M}(G)$.

FIGURE 6.1

we have the following:

LEMMA 6.1. *The set $S \subseteq E$ contains no basis of $\mathscr{M}(K_n)$ if and only if there exists $\lambda \in \mathbb{R}^n$ such that $\lambda^T A \neq 0$ but $\lambda^T A_j = 0$ for all $j \in S$.*

(In fact this is a specialization of the theory of chain groups developed by Tutte (1965) as an alternative representation for matroids defined by matrices. The original form, considering linear independence of columns as we did here, appeared in Whitney (1935), the original paper on matroids.)

Let $a \geq 0$ be such that $a^T x \leq a_0$ induces a facet of Q_T^n. We let $E^0(a) := \{j \in E : a_j = 0\}$. ($E \setminus E^0(a)$ is usually called the *support* of a.) If $E^0(a)$ contains no basis of $\mathscr{M}(K_n)$ then by Lemma 6.1 there exists λ such that $\lambda^T A_j = 0$ for all $j \in E^0(a)$ but $\lambda^T A$ is not identically zero. By subtracting an appropriate multiple of $\lambda^T A$ from a and $\lambda^T \mathbf{2}$ from a_0 we obtain (using Lemma 2.9) an inequality $\bar{a}^T x \leq \bar{a}_0$ that induces the same facet of Q_T^n, that satisfies $\bar{a} \geq 0$ and $E^0(\bar{a}) \supsetneq E^0(a)$. We call this procedure *reducing* the inequality $a^T x \leq a_0$. Clearly we need only reduce an inequality at most $|E|$ times before obtaining an equivalent inequality $\hat{a}^T x \leq \hat{a}_0$ with $\hat{a} \geq 0$ and such that $E^0(\hat{a})$ contains a basis of $\mathscr{M}(K_n)$. Finally we scale the inequality so that the smallest nonzero coefficient of a has value one. We call such an inequality *support reduced.*

We call a facet of Q_T^n *trivial* if it is induced by an inequality $x_k \geq 0$ for some $k \in E$ and *nontrivial* otherwise. First we show that it is easy to recognize when a support reduced inequality induces a trivial facet.

LEMMA 6.2. *Let $a^T x \leq a_0$ be a support reduced facet inducing inequality for Q_T^n. Then $a^T x \leq a_0$ induces a trivial facet of Q_T^n if and only if*

(a) *$E \setminus E^0(a) = \delta(v) \setminus \{k\}$ for some $v \in V$ and $k \in \delta(v)$, and in this case $a_j = 1$ for all $j \in \delta(v) \setminus \{k\}$ and $a_0 = 2$; or*

(b) *$E^0(a) = \delta(v) \cup \{k\}$ for some $v \in V$ and $k \in E(V \setminus \{v\})$, and in this case $a_j = 1$ for all $j \in E \setminus E^0(a)$ and $a_0 = n - 2$.*

PROOF. By Lemma 2.9 $a^T x \leq a_0$ induces a trivial facet if and only if for some $k \in E$ there exist $\lambda \in \mathbb{R}^n$ and $\rho > 0$ such that $a_j = \lambda^T A_j$ for $j \in E \setminus \{k\}$ and $a_k = \lambda^T A_k - \rho$. The sufficiency of (a) follows by defining $\lambda_v := 1$, $\lambda_w := 0$ for $w \in V \setminus \{v\}$ and $\rho = 1$. The sufficiency of (b) follows by defining $\lambda_v := -\frac{1}{2}$, $\lambda_w := \frac{1}{2}$ for $w \in V \setminus \{v\}$ and $\rho = 1$.

We now prove the necessity. Suppose $a^T x \leq a_0$ is equivalent to $-x_k \leq 0$. Set $E^0 = E^0(a)$. If there exist distinct $u, v \in V$ such that $\lambda_u < 0$ and $\lambda_v < 0$ then $a_{uv} < 0$, a contradiction. So there is at most one $v \in V$ with $\lambda_v < 0$.

Suppose that no v satisfies $\lambda_v < 0$. Then if there exist distinct $w, v \in V$ with $\lambda_v > 0$ and $\lambda_w > 0$, we would have $\delta(v) \cup \delta(w) \setminus \{k\}$ contained in $E \setminus E^0$. Therefore at least one of $\{v, w\}$ is contained in a component of $[V, E^0]$ containing no odd cycle, so

First suppose $j \in V_2 \setminus T$ and let H'' be the handle intersecting T for which removal of $H'' \cap T$ disconnects j from $\mathring{T}$. Let π be a path saturating T ending in H_1 and H''. Using Lemma 4.4 extend π to a path cover P' of C' containing a path P_i ending at i and extend π to a path cover P'' of C'' containing a path P_j ending at j. It is obvious that $P' \cup P''$ can be extended to a hamiltonian path in K_n with endnodes i and j such that the edges added to $P' \cup P''$ do not belong to those given in (16). Thus adding edge ij we obtain a tour S saturating C such that all values b_e, $e \in S$, are known to equal ρa_e except for $e = ij$. The case $j \in \bar{V}$ is treated similarly. By the argument of 10, we obtain

$$b_{ij} = \rho a_{ij} \qquad \text{for all} \quad i \in V_1 \setminus T, \quad j \in (\bar{V} \cup V_2) \setminus T. \tag{17}$$

The other three cases one has to consider are

12. $i \in V_1 \setminus T, j \in V_2 \cap T$.
13. $i \in V_1 \cap T, j \in (\bar{V} \cup V_2) \setminus T$.
14. $i \in V_1 \cap T, j \in V_2 \cap T$.

Cases 12 and 13 are symmetric. The construction of the desired tour is essentially the same as in 11. In case 14 one starts with a path π from i to j saturating T and extends it to a tour S in K_n saturating C. Then the edges in S from i and j to nodes not in T, say i' and j', are removed as well as an edge from w to some other node in T, say w'. The edges ij, wi' and $w'j'$ are added to obtain a tour S' saturating C. In order to verify that S' saturates C we want to use $a_{ii'} = a_{jj'} = a_{w'j'} = a_{wi'} = 0$. It follows from the fact that $i' \notin T$ and (13) that $a_{wi'} = 0$. The construction of S indicated above ensures this for the edges ii', jj', $w'j'$ as well. Then b_{ij} can be shown to equal ρa_{ij} using the argument of 10.

Altogether we have shown that $a^{\mathrm{T}} = \bar{\rho} d^{\mathrm{T}} + \bar{\lambda}^{\mathrm{T}} A$ and $\alpha = \bar{\rho}\delta + \bar{\lambda}^{\mathrm{T}} \mathbf{2}$ where $\bar{\rho} = 1/\rho > 0$ and $\bar{\lambda} = \lambda/\rho$. This completes the proof of our theorem. ∎

5. Equivalence of clique trees. We say that clique trees C_1 and C_2 are *equivalent* if their respective inequalities $a^{\mathrm{T}}x \leq a_0$ and $b^{\mathrm{T}}x \leq b_0$ induce the same facet of Q_T^n. In this section we show that there are only two cases of equivalent clique trees. Otherwise, every distinct clique tree inequality induces a distinct facet of Q_T^n. In particular, the facets induced by all clique tree inequalities which are neither comb inequalities nor subtour elimination inequalities are all distinct.

The importance of this result is that it shows that from the point of view of cutting planes, the set of clique tree inequalities is essentially minimal. That is, there is no small subset of the clique trees that suffices to provide all the cuts provided by the whole set.

Theorem 2.12(b) asserted that two distinct subtour elimination constraints are equivalent if and only if they are of the form $x(E(W)) \leq |W| - 1$ and $x(E(V \setminus W)) \leq |V \setminus W| - 1$. Theorem 2.15(c) asserted that two distinct facet-inducing comb inequalities are equivalent if and only if the two combs have the same sets of teeth and complementary handles. Moreover, no comb inequality is equivalent to a subtour elimination inequality. The results of this section not only provide proofs of these results, but also show that these are the only cases of equivalence of clique trees.

Let $a^{\mathrm{T}}x \leq a_0$ and $b^{\mathrm{T}}x \leq b_0$ be two facet inducing inequalities for Q_T^n. By Lemma 2.9 these inequalities are equivalent if and only if there exist $\rho > 0$ and $\lambda \in \mathbb{R}^n$ such that $b^{\mathrm{T}} = \rho a^{\mathrm{T}} + \lambda^{\mathrm{T}} A$, in other words, for each edge $ij \in E$ we have $b_{ij} = \rho a_{ij} + \lambda_i + \lambda_j$.

THEOREM 5.1. *Let C_1 and C_2 be distinct clique trees. Then C_1 and C_2 are equivalent if and only if*

(a) *C_1 is a single clique W and C_2 is a single clique $V \setminus W$ or*

(b) *C_1 and C_2 are combs with identical sets of teeth and if W is the handle of C_1 then $V \setminus W$ is the handle of C_2.*

PROOF. Taking $\rho = 1$, $\lambda_i = -\frac{1}{2}$ for all $i \in W$, and $\lambda_i = \frac{1}{2}$ for all $i \in V \setminus W$ shows the equivalence of C_1 and C_2 if (a) or (b) holds. So we now prove the necessity of our conditions.

Let C_1 and C_2 be two different equivalent clique trees, and let $a^{\mathrm{T}}x \leq a_0$ and $b^{\mathrm{T}}x \leq b_0$ be the associated inequalities. If both C_1 and C_2 are single cliques, then unless $C_1 = V \setminus C_2$, there exists $\bar{x} \in Q_T^n$ satisfying $a^{\mathrm{T}}\bar{x} = a_0$, $b^{\mathrm{T}}\bar{x} < b_0$ so these inequalities induce distinct facets. So we assume that C_1 is not a single clique, i.e., C_1 contains at least four cliques satisfying Definition 2.16.

Let $\lambda \in \mathbb{R}^n$ and $\rho > 0$ be such that $b_{ij} = \rho a_{ij} + \lambda_i + \lambda_j$ for all $ij \in E$. Let $N := \{i \in V \mid \lambda_i < 0\}$. Since C_1 and C_2 are different we have $\lambda \neq 0$. If $\lambda \geq 0$ then for some i, $\lambda_i > 0$ and hence C_2 contains the star $\delta(i)$ which is impossible. So N is nonempty, and in fact $|N| \geq 2$. For, if there were a unique i with $\lambda_i < 0$ then there would be $u \in V$ with $\lambda_u > 0$ and hence $\delta(u) \setminus \{ui\}$ would be contained in C_2 which is impossible by the definition.

Since $b_{ij} = \rho a_{ij} + \lambda_i + \lambda_j \geq 0$ it is immediate that $a_{ij} \geq 1$ for all $i, j \in N$. Hence N is a clique (but not necessarily maximal) in C_1. This implies that there are at most two maximal cliques of C_1 which contain N. Let U be the set of nodes belonging to neither of these two cliques, i.e. $U = \{u \in V \mid \text{there exists } i \in N \text{ with } ui \text{ not in } C_1\}$. Clearly, $\lambda_u > 0$ for all $u \in U$, and hence U is a (not necessarily maximal) clique in C_2. Obviously, $|U| \geq 2$.

We now want to prove that either $\lambda_i = -\frac{1}{2}$ for all $i \in N$ and $\lambda_u = \frac{1}{2}$ for all $u \in U$ or $\lambda_i = -1$, $i \in N$, and $\lambda_u = 1$ for all $u \in U$.

Let $u, v, w \in V$. We say that u, v, w form a *zero-triangle in* C_1 if $a_{uv} = a_{uw} = a_{vw} = 0$. In this case, we have $b_{uv} = \lambda_u + \lambda_v$, $b_{uw} = \lambda_u + \lambda_w$, $b_{vw} = \lambda_v + \lambda_w$. Suppose that $\lambda_w < 0$, then in order for b_{uv}, b_{uw}, b_{vw} to be a possible set of values in a clique tree inequality we must have either $\lambda_w = -\frac{1}{2}$ and $\lambda_v = \lambda_u = \frac{1}{2}$ or $\lambda_w = -1$ and $\lambda_u = \lambda_v = 1$, which can be routinely verified.

Now choose any $w \in N$ and any $u \in U$ which is not adjacent to w in C_1. Since C_1 is a clique tree there exists a node $v \in U$ such that u, v, w form a zero-triangle in C_1. Therefore we have $-\lambda_w = \lambda_u = \lambda_v$. If N is contained in two cliques of C_1 we have immediately that $-\lambda_w = \lambda_u$ for all $w \in N$, $u \in U$. If N is contained in a single clique of C_1, then we may have to repeat our argument with a different node $w' \in N$ to arrive at the same conclusion. Thus we get two cases.

Case 1. $\lambda_w = -1$ for all $w \in N$, and $\lambda_u = 1$ for all $u \in U$.

Case 2. $\lambda_w = -\frac{1}{2}$ for all $w \in N$, and $\lambda_u = \frac{1}{2}$ for all $u \in U$.

Case 1. Suppose there is an edge $uv \in E(U)$ with $a_{uv} \geq 1$. Then $b_{uv} > 2$, a contradiction. Note also that since C_1 is a clique tree which is not a clique there must be an edge $up \in \delta(U)$ with $a_{up} \geq 1$.

Suppose there exists such a p with $p \in V \setminus (U \cup N)$, i.e. $\lambda_p \geq 0$, then we have $b_{up} = 2$. We furthermore know that $b_{uv} = 2$ for all $v \in U \setminus \{u\}$. Since no edge $uv \in E(U)$ belongs to C_1, there must be a node $v \in U$ with $a_{pv} = 0$. From $b_{uv} = b_{up} = 2$ we conclude $b_{vp} = 2$, and so $\lambda_p = b_{vp} - \rho a_{pv} - \lambda_v = 1$. Thus $\lambda_p + \lambda_u = 2 = b_{up}$ which implies $\rho = 0$, a contradiction.

Therefore, for every edge $up \in \delta(U)$ with $a_{up} \geq 1$ we must have $p \in N$. The properties of a clique tree now imply that there is a single clique K of C_1 that contains N, and that $U = V \setminus K$. Since no edge of $E(U)$ is an edge of C_1, C_1 is in fact a comb with handle K. As $\lambda_u = -\lambda_w$ for all $u \in U$, $w \in N$ we can conclude that for all edges uw with $u \in U$, $w \in N$ we have $b_{uw} = a_{uw}$. Therefore the teeth of C_1 give rise to at least

$a^{\mathrm{T}}x \leq a_0$ could not be support reduced, a contradiction. Therefore exactly one $v \in V$ satisfies $\lambda_v > 0$. But again, we cannot have $\delta(v) \subseteq E \setminus E^0$, so $k \in \delta(v)$, $a_k = 0$ and $a^{\mathrm{T}}x \leq a_0$ is of the form (a).

Suppose that exactly one $v \in V$ satisfies $\lambda_v < 0$. Then every $w \in V \setminus \{v\}$ must satisfy $\lambda_w \geq -\lambda_v$. Therefore k is the only possible edge of $E(V\setminus\{v\})$ that can have $a_k = 0$, so for $a^{\mathrm{T}}x \leq a_0$ to be support reduced we must have $\lambda_w = -\lambda_v$ for all $w \in V\setminus\{v\}$, $k \in E(V\setminus\{v\})$ and $a_k = 0$ which implies that $\lambda_w = \frac{1}{2}$ for all $w \in V\setminus\{v\}$ and $\lambda_v = -\frac{1}{2}$ and we have the form (b). ∎

Now we prove the main lemma.

LEMMA 6.3. *Let $a^{\mathrm{T}}x \leq a_0$ be a support reduced facet inducing inequality for Q_T^n not of the form* (a) *or* (b) *of* (6.2). *Then $a^{\mathrm{T}}x \leq a_0$ is facet inducing for $\tilde{Q}_T^n$.*

PROOF. Since $a \geq 0$, the inequality is valid for $\tilde{Q}_T^n$. Let X be the set of all incidence vectors x of tours of K_n that satisfy $a^{\mathrm{T}}x = a_0$. Since $a^{\mathrm{T}}x \leq a_0$ is facet inducing for Q_T^n there exists an affinely independent set $\hat{X} \subseteq X$ satisfying $|\hat{X}| = \dim Q_T^n = \binom{n}{2} - n$. Let B be a basis of $\mathscr{M}(K_n)$ contained in $E^0(a)$.

If there exists $k \in E$ such that $x_k = 0$ for all $x \in X$ then $a^{\mathrm{T}}x \leq a_0$ induces the trivial facet $x_k \geq 0$, a contradiction. Hence for each $j \in B$ there exists $x^j \in X$ such that $x_j^j = 1$. Let $\tilde{x}^j$ be obtained from x^j for all $j \in B$ by setting $\tilde{x}_j^j = 0$. Then each $\tilde{x}^j$ is the incidence vector of a hamiltonian path of K_n, and $a^{\mathrm{T}}\tilde{x}^j = a_0$ for all $j \in B$. Let $\tilde{X} := \{\tilde{x}^j : j \in B\}$. We will show that $\hat{X} \cup \tilde{X}$ is affinely independent, and then since $\hat{X} \cup \tilde{X} \subseteq \tilde{Q}_T^n$ and $|\hat{X} \cup \tilde{X}| = \binom{n}{2} = \dim \tilde{Q}_T^n$ it will follow that $a^{\mathrm{T}}x \leq a_0$ is facet inducing.

First we observe that for all $x \in \hat{X}$, $x(\delta(v)) = 2$ for all $v \in V$ and that for all $\tilde{x}^j \in \tilde{X}$, $\tilde{x}^j(\delta(u)) = 2$ for $u \in V\setminus\{v,w\}$ and $\tilde{x}^j(\delta(v)) = \tilde{x}^j(\delta(w)) = 1$ where $j = vw$.

Now suppose that $\hat{X} \cup \tilde{X}$ is affinely dependent. Then there exists $\mu \neq 0$ satisfying

$$\Sigma(\mu_x : x \in \hat{X}) = \Sigma(\mu_j : j \in B) = 0 \quad \text{and}$$
$$\Sigma(\mu_x x : x \in \hat{X}) + \Sigma(\mu_j \tilde{x}^j : j \in B) = 0. \tag{6.4}$$

This implies, in particular,

$$\text{for any } v \in V, \quad \Sigma(\mu_x x(\delta(v)) : x \in \hat{X}) + \Sigma(\mu_j \tilde{x}^j(\delta(v)) : j \in B) = 0. \tag{6.5}$$

Since $\hat{X}$ is affinely independent there exists $k \in B$ such that $\mu_k \neq 0$. Choose such a k which is at maximum distance in $[V,B]$ from the odd cycle of the component of $[V,B]$ that contains k. If k is not an edge of the cycle, then let v be the node of k furthest from the cycle. Then $x^k(\delta(v)) = 1$ but $x(\delta(v)) = 2$ for all $x \in \hat{X} \cup \tilde{X}\setminus\{x^k\}$ having nonzero μ. This contradicts (6.4) and (6.5). So k must be an edge of an odd cycle C of B, and $\mu_j = 0$ for every edge j of B not in C, but incident with a node of C. Hence if $C = j_1, j_2, \ldots, j_{2r+1}$ we have by (6.4) and (6.5) that $\mu_{j_1} = -\mu_{j_2} = \mu_{j_3} = -\mu_{j_4} = \cdots = -\mu_{j_{2r+1}} = -\mu_{j_1}$. But this means $\mu_j = 0$ for every edge j of C, a contradiction to $\mu_k \neq 0$. Therefore no such μ exists, $\hat{X} \cup \tilde{X}$ is affinely independent and the result follows. ∎

Using Theorem 4.7 and Lemmas 6.2 and 6.3 it is now easy to show that clique tree inequalities are facet inducing for $\tilde{Q}_T^n$.

THEOREM 6.6. *Let C be a clique tree in $K_n = [V,E]$. Then the corresponding clique tree inequality* (2.17) *induces a nontrivial facet of $\tilde{Q}_T^n$.*

PROOF. By Theorem 4.7 every clique tree inequality $a^{\mathrm{T}}x \leq a_0$ is facet inducing for Q_T^n. If C consists of a single tooth T, i.e. $a^{\mathrm{T}}x \leq a_0$ is a subtour elimination constraint, then by Definition 2.16(4) there exist distinct $u, v \in V\setminus T$ and $\delta(u) \cup \delta(v) \subseteq E^0(a)$ so $E^0(a)$ contains a basis of $\mathscr{M}(K_n)$. If C contains three teeth T_1, T_2, T_3 then there exist nodes $t_i \in T_i$ belonging to no other clique of C, so the edges $\{t_1u \mid u \in V\setminus T_1\} \cup \{t_2u \mid u \in T_1\} \cup \{t_2t_3\}$ form a basis of $\mathscr{M}(K_n)$ contained in $E^0(a)$.

Therefore $a^{\mathrm{T}}x \leq a_0$ is support reduced. By Lemma 6.2 it cannot induce a trivial facet of Q_T^n so by Lemma 6.3 it induces a nontrivial facet. ∎

By Theorem 2.12(a), for every $W \subseteq V$ with $|W| = n-1$, the subtour elimination constraints $x(E(W)) \leq n-2$ induce facets of $\tilde{Q}_T^n$. Since $Q_T^n \subseteq \{x \mid x(E(W)) = n-2$ for all $W \subseteq V$, $|W| = n-1\} = \{x \mid Ax = 2\}$ these subtour elimination constraints do not induce facets of Q_T^n. Hence this is an example where Lemma 6.3 resp. Theorem 6.6 cannot be used to derive a facet-result for $\tilde{Q}_T^n$ from one for Q_T^n. (This case made the restriction to teeth of at most $n-2$ nodes in Definition 2.16(4) necessary.) In general, however, Lemma 6.3 seems to be a very powerful tool for proving results about $\tilde{Q}_T^n$.

Usually it is technically much simpler to obtain results about facets for $\tilde{Q}_T^n$ than for Q_T^n, cf. Grötschel (1977). Therefore it would be nice to have a converse of Lemma 6.3 stating under which assumptions an inequality inducing a facet of $\tilde{Q}_T^n$ also induces a facet of Q_T^n. However, we do not know a reasonable result of this type.

7. Completeness of simple clique trees.

We have seen that clique tree inequalities are facet inducing for Q_T^n and $\tilde{Q}_T^n$, and that with two exceptions, each such inequality induces a different facet. In this section we show that clique tree inequalities are the most general facet inducing inequalities of Q_T^n, of a certain type.

Let $G = [V', E']$ be a subgraph of $K_n = [V,E]$. We let $r(G) := \max\{x(E') : x \in Q_T^n\}$. The value $r(G)$ is commonly called the *rank* of G and equals the maximum number of edges of G that can be contained in a path system of G, or equivalently, in a hamilton cycle of K_n. We may write $r(E')$ for $r(G)$.

If an inequality $x(E') \leq \alpha$ is to be valid for Q_T^n then we must have $\alpha \geq r(E')$, and if in addition there is to be $\hat{x} \in Q_T^n$ satisfying $\hat{x}(E') = \alpha$, we must have $\alpha = r(E)$. Thus we say that the *inequality induced by the subgraph* $G = [V', E']$ of K_n is the inequality $x(E') \leq r(E')$. (Of course it is in general a very hard problem to compute $r(E')$. However we do not worry about that at present. It is sufficient here that the value exists.) We say that G is *facet inducing* for Q_T^n if the inequality induced by G is facet inducing.

Let $G = [V', E']$ be a connected graph such that every block is a clique. That is, G has a treelike structure but need not satisfy the properties of Definition 2.16 that we require of clique trees. (See Figure 7.1.) We have seen two cases in which such a graph is facet inducing for Q_T^n. The first of course is when G is a simple clique tree (see §2 and Theorem 4.7) and the second is when $E' = \delta(v)\setminus\{j\}$ for some node v of K_n and some $j \in \delta(v)$ (see Lemma 6.2). In the former case, G induces a nontrivial facet, and in the latter case, G induces a trivial facet. We call a subgraph $G = [V', E']$ of K_n a *clique structure* if G is connected, every block is a clique, and $E' \subsetneq \delta(v)$ for any node v of K_n. The main theorem of this section is that if G is a facet inducing clique structure for K_n, then G is a simple clique tree.

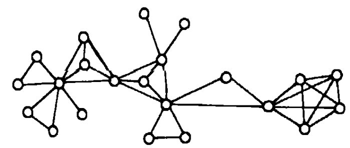


FIGURE 7.1. A clique structure.

LEMMA 7.1. *Let $G = [V', E']$ be a clique structure consisting of $s \geq 2$ cliques joined by a common cutnode v. Then $r(G) = |V'| - s + 1$.*

PROOF. It is easily seen that if $G_1, G_2, \dots, G_s$ are the pieces joined by v then $r(G) = \sum_{i=1}^{s} r(G_i) - (s-2)$ since a saturating path cover of G will only be able to saturate two of the G_i. Since $r(G_i) = |V(G_i)| - 1$ the result follows. ∎

LEMMA 7.2. *Let $G = [V', E']$ be a facet inducing clique structure for Q_T^n. Then for any $v \in V'$ there exists an integral vector $\bar{x} \in Q_T^n$ satisfying $\bar{x}(E') = r(G)$ and $\bar{x}(\delta(v) \cap E') \leq 1$.*

PROOF. Suppose that for some $v \in V'$ every saturating path cover of G contained two edges incident with v. If $\delta(v) \cap E' \subsetneq \delta(v)$ then $x(\delta(v) \cap E') \leq 2$ induces a proper face of Q_T^n. Since $E' \not\subseteq \delta(v)$, it is easy to see that there exists $x \in Q_T^n$ satisfying $x(\delta(v) \cap E') = 2$ but $x(E') < r(G)$. This contradicts G being facet inducing. Suppose $\delta(v) \cap E' = \delta(v)$. If G consisted of a single clique then $G = K_n$ and every $x \in Q_T^n$ satisfies $x(E') = r(G)$ so G does not induce a proper face, a contradiction. Otherwise v is a cutnode joining $s \geq 2$ cliques $G_1, G_2, \dots, G_s$. If $|V(G_i)| \geq 3$ for some $i \in \{1, 2, \dots, s\}$ then every x satisfying $x(E') = r(G)$ also satisfies $x(E(G_i \setminus \{v\})) = r(G_i \setminus \{v\})$ but the converse is not true. Since this latter inequality induces a proper face of Q_T^n, this contradicts G being facet inducing. If $|V(G_i)| = 2$ for all $i \in \{1, 2, \dots, s\}$ then $E' = \delta(v)$ so we contradict G being a clique structure. Therefore there is a saturating path cover containing at most one edge incident with v. The incidence vector of any tour containing this path cover provides the required $\bar{x}$. ∎

Let $G = [V', E']$ be a graph containing a cutnode v. Let $C_1, C_2, \dots, C_r$ be the nodesets of the components of $G - \{v\}$. Then we call the graphs $G[C_i \cup \{v\}]$ for $i = 1, 2, \dots, r$ the *pieces of G joined by v*.

LEMMA 7.3. *Let $G = [V', E']$ be a facet inducing clique structure for Q_T^n and let v be any cutnode of G. If $G_1, G_2, \dots, G_s$ are the pieces of G joined by v, then $r(G) = r(G_1) + r(\bigcup_{i=2}^{s} G_i) - 1$.*

PROOF. Let $\bar{G} := \bigcup_{i=2}^{s} G_i$. Then clearly $r(G) \leq r(G_1) + r(\bar{G})$. If $r(G) = r(G_1) + r(\bar{G})$ then the inequality $x(E') \leq r(G)$ is the sum of the inequalities $x(E(G_1)) \leq r(G_1)$ and $x(E(\bar{G})) \leq r(\bar{G})$. It is straightforward to verify that either of these inequalities induces a proper face of Q_T^n which properly contains the face induced by $x(E') \leq r(G)$, contradicting G being facet inducing so

$$r(G) \leq r(G_1) + r(\bar{G}) - 1. \tag{7.4}$$

Let P_1 be a saturating path cover of G_1 and let $\bar{P}$ be a saturating path cover of $\bar{G}$. Then at most two edges of P_1 are incident with v. Deleting these edges and then taking the union with $\bar{P}$ we obtain a feasible path cover of G. Hence $r(G) \geq r(G_1) + r(\bar{G}) - 2$. Suppose

$$r(G) = r(G_1) + r(\bar{G}) - 2. \tag{7.5}$$

We will show that this implies that every $x \in Q_T^n$ satisfying $x(E') = r(G)$ also satisfies $x(\delta(v) \cap E') = 2$. For suppose there exists $\bar{x}$ satisfying $\bar{x}(E') = r(G)$ and $\bar{x}(\delta(v) \cap E') \leq 1$. Then $\bar{x}(\delta(v) \cap E^*) = 0$, where E^* is the edge set of either $\bar{G}$ or of G_1. If $\bar{x}(E^*) \leq r(E^*) - 2$ then let x^* be the incidence vector of a saturating path cover for E^*. At most two edges j incident with v have $x_j^* = 1$, so by setting one of these x_j^* to 0, we obtain $\tilde{x}$ satisfying $\tilde{x}(E^*) = r(E^*) - 1$. But if we then replace $\bar{x}$ with $\tilde{x}$ for the edges of E^*, we get a vector $\hat{x} \in Q_T^n$ satisfying $\hat{x}(E') \geq \bar{x}(E') + 1 = r(G) + 1$, a contradiction. Therefore $\bar{x}(E^*) \geq r(E^*) - 1$. But then let $E^{**} \equiv E' \setminus E^*$ and let x^{**} satisfy $x^{**}(E^{**}) = r(E^{**})$. If we change $\bar{x}$ to agree with x^{**} on E^{**} we get a vector

$\hat{x} \in Q_T^n$ satisfying $\hat{x}(E') \geq r(G) + 1$, again a contradiction. Therefore no $x \in Q_T^n$ satisfies $x(E') = r(G)$ and $x(\delta(v) \cap E') \leq 1$, and we contradict Lemma 7.2. Therefore (7.5) cannot hold, so $r(G) \geq r(G_1) + r(\bar{G}) - 1$ which combined with (7.4) gives the result. ∎

LEMMA 7.6. *Let $G = [V', E']$ be a facet inducing clique structure for Q_T^n. Then for any $j \in E$ there exists an integral vector $\bar{x} \in Q_T^n$ satisfying $\bar{x}_j = 1$ and $\bar{x}(E') = r(G)$.*

(Note that this edge j need not belong to E', but can in fact be any edge of G.)

PROOF. Suppose that every integer $\bar{x} \in Q_T^n$ which satisfies $\bar{x}(E') = r(G)$ also satisfies $\bar{x}_j = 0$. Then the inequality $x(E') \leq r(E')$ must induce the same facet as the trivial inequality $x_j \geq 0$. We will show that the inequality $x(E') \leq r(E')$ is support reduced (see §6) and the result will then follow from Lemma 6.2. Showing that this inequality is support reduced consists simply of showing that $E \setminus E'$ contains a basis of $\mathscr{M}(K_n)$.

If G contains a single clique, then either $|V \setminus V'| \geq 2$, in which case $E \setminus E'$ contains a basis of $\mathscr{M}(K_n)$, or $|V \setminus V'| \leq 1$ and every $x \in Q_T^n$ satisfies $x(E') = r(E')$, contradictory to this inequality inducing a proper face. If G contains at least two cutnodes and hence two node disjoint cliques, then it is easy to find a basis of $\mathscr{M}(K_n)$ in $E \setminus E'$ unless G consists of exactly three cliques B_1, B_2, B_3 with a cutnode $u \in V(B_1) \cap V(B_2)$ and another cutnode $v \in V(B_2) \cap V(B_3)$. In this case $r(G) = |V(G)| - 1$, but $r(B_1) = |V(B_1)| - 1$ and $r(B_2 \cup B_3) = |V(B_2)| + |V(B_3)| - 2$. Thus we contradict Lemma 7.3 for u. Finally, suppose that G contains a single cutnode v joining $s \geq 2$ cliques. If there exists $w \in V \setminus V'$, then $\delta(w)$ plus any edge of $E(V \setminus \{v\}) \setminus E'$ gives a basis of $\mathscr{M}(K_n)$ contained in $E \setminus E'$. If no such w exists, then $\delta(v) \subseteq E'$ so every $x \in Q_T^n$ satisfies $x(\delta(v) \cap E') = 2$ and we contradict Lemma 7.2.

Therefore $x(E') \leq r(E')$ is support reduced and since it induces a trivial facet of Q_T^n, it follows from Lemma 6.2 that it must either be of the form (6.2a) or (6.2b). However, a clique structure cannot be of the form (6.2a), so G must be of the form (6.2b), i.e., $E' = E \setminus (\delta(v) \cup \{k\})$ for some $v \in V$ and $k \in E(V \setminus \{v\})$. Since $E' \subsetneq E(V \setminus \{v\})$, G must contain at least two cliques. If there were more than two cliques or if any clique contained more than two nodes, then we would contradict $E(V \setminus \{v\}) \setminus E' = \{k\}$. But this means that G consists of two cliques, each consisting of two nodes, and $K_n = K_4$. But then $E' \subseteq \delta(v)$ where v is the cutnode of G, so we contradict G being a clique structure. ∎

Now we prove a first necessary condition for a clique structure to be facet inducing.

PROPOSITION 7.7. *Let $G = [V', E']$ be a facet inducing clique structure for Q_T^n. Then every cutnode of G joins exactly two pieces.*

PROOF. Suppose to the contrary that a cutnode v joins pieces $G_1, G_2, \dots, G_s$ for $s \geq 3$. Suppose that some integral $x \in Q_T^n$ satisfies $x(E(G_t)) = r(G_t)$ and $x(\delta(v) \cap E(G_t)) = 0$ for some t, $1 \leq t \leq s$. Let $\bar{G} := \bigcup(G_i : i = 1, 2, \dots, s;\ i \neq t)$. Choose integer $x' \in Q_T^n$ satisfying $x'(E(\bar{G})) = r(\bar{G})$. Then we can find a new $x^* \in Q_T^n$ equal to x on $E(G_t)$ and equal to x' on $E(\bar{G})$, so $x^*(E') = r(\bar{G}) + r(G_t)$ so $r(G) \geq r(\bar{G}) + r(G_t)$ contradicting Lemma 7.3. Therefore

$$\text{for any } t = 1, 2, \dots, s, \text{ for any integer } x \in Q_T^n, \text{ if } x \text{ satisfies } x(E(G_t)) = r(G_t) \text{ then } x(\delta(v) \cap E(G_t)) \geq 1. \tag{7.8}$$

By Lemma 7.2 there exists integer $\hat{x} \in Q_T^n$ satisfying $\hat{x}(E') = r(G)$ and $\hat{x}(\delta(v) \cap E') \leq 1$. If $\hat{x}(\delta(v) \cap E') = 0$ then by (7.8) we would have $\hat{x}(E(G_i)) < r(G_i)$ for all $i \in \{1, 2, \dots, s\}$, so we could change $\hat{x}$ on $E(G_i)$ for some i and obtain $\hat{\hat{x}} \in Q_T^n$ satisfying $\hat{\hat{x}}(E') > \hat{x}(E')$, contradictory to $\hat{x}(E') = r(E')$. Therefore $\hat{x}(\delta(v) \cap E') = 1$ and there is one piece, say G_s, containing an edge $j \in \delta(v) \cap E'$ with $\hat{x}_j = 1$. If

$\hat{x}(E(G_s)) < r(G_s)$, then $\hat{x}(E(G_s))$ can be increased, without decreasing $\hat{x}(E(G_i))$ for $i = 1,2,\dots,s-1$, contradicting $\hat{x}(E') = r(E')$. Thus we have

$$\hat{x}(E(G_s)) = r(G_s) \quad \text{and} \quad \hat{x}(\delta(v) \cap E(G_s)) = 1. \tag{7.9}$$

By (7.8), for each $i \in \{1,2,\dots,s-1\}$, we have $\hat{x}(E(G_i)) \leq r(G_i) - 1$. In fact, we must have equality, for if $\hat{x}(E(G_i)) = r(G_i) - 2$, then we could increase $\hat{x}(E(G_i))$ and again violate $\hat{x}(E') = r(E')$. So we have

$$\hat{x}(E(G_i)) = r(G_i) - 1 \quad \text{for} \quad i \in \{1,2,\dots,s-1\}. \tag{7.10}$$

Combining (7.9) and (7.10) we have

$$r(G) = \sum_{i=1}^{s} (r(G_i) - 1) + 1. \tag{7.11}$$

If it were possible to find some integral $\hat{\hat{x}} \in Q_T^n$ and $i \in \{1,2,\dots,s-1\}$ such that $\hat{\hat{x}}(E(G_i)) = r(G_i)$ and $\hat{\hat{x}}(\delta(v) \cap E(G_i)) \leq 1$, then we could replace $\hat{x}$ with $\hat{\hat{x}}$ for the edges of $E(G_i)$, and make some changes to $\hat{x}_k$ for $k \in E \setminus E'$, and contradict $\hat{x}(E') = r(E')$. Therefore

$$\text{for } i \in \{1,2,\dots,s-1\}, \text{ and any integer } x \in Q_T^n \text{ satisfying } x(E(G_i)) = r(G_i), \text{ we must have } x(\delta(v) \cap E(G_i)) = 2. \tag{7.12}$$

Combining (7.8), (7.11) and (7.12) we obtain

$$\text{for any integer } x \in Q_T^n \text{ satisfying } x(E') = r(E') \text{ there will be a unique } t \in \{1,2,\dots,s\} \text{ such that } x(E(G_t)) = r(G_t) \text{ and for every other } i \in \{1,2,\dots,s\} \setminus \{t\}, \text{ we will have } x(E(G_i)) = r(G_i) - 1. \tag{7.13}$$

Let B_1 and B_2 be the blocks of G_1 and G_2 respectively that contain v. There exists $k \in E \setminus E'$ joining $u_1 \in V(B_1)\setminus\{v\}$ and $u_2 \in V(B_2)\setminus\{v\}$. By Lemma 7.6 there exists integer $x^* \in Q_T^n$ satisfying $x_k^* = 1$ and $x^*(E') = r(G)$. By (7.13) there is a unique $t \in \{1,2,\dots,s\}$ such that $x^*(E(G_t)) = r(G_t)$. We consider two cases.

Case 1. $t = s$. By (7.8) $x^*(\delta(v) \cap E(G_s)) \geq 1$, so $x^*(\delta(v) \cap E(G_i)) = 1$ for at most one $i \in \{1,2,\dots,s-1\}$. Replace x_j^* with $\hat{x}_j$ for $j \in E(G_s) \cup E(G_i)$. For the edges of at least one of G_1, G_2, say G_1, the value x^* has been unchanged. We now set $x_k^* := 0$ and set $x_j^* := 1$ where j is the edge of B_1 joining u_1 to v. Then by appropriately modifying x_l^* for $l \in E \setminus E'$, we obtain $x^{**} \in Q_T^n$ satisfying $x^{**}(E') = x^*(E') + 1$, contradictory to $x^*(E') = r(G)$.

Case 2. $t \in \{1,2,\dots,s-1\}$. By (7.12) and (7.13), we have $x^*(\delta(v) \cap E(G_t)) = 2$, so for at least one of G_1, G_2, say G_1, we have $x^*(\delta(v) \cap E(G_1)) = 0$. If $t = 2$ and $x_h^* = 1$ for the edge h of B_2 joining v and u_2, let $l := h$. Otherwise, let l be any edge of $\delta(v)$ for which $x_l^* = 1$. Let j be the edge of B_1 joining u_1 and v. If we set $x_k^* := x_l^* := 0$ and set $x_j^* := 1$ then the resulting x^{**}, restricted to E', is the incidence vector of a path cover of G satisfying $x^{**}(E') = x^*(E')$.

Thus we can redefine x_h appropriately for $h \in E \setminus E'$ and x^{**} will satisfy $x^{**} \in Q_T^n$ and $x^{**}(E') = r(G)$. But by (7.8) and (7.12), there is no $t \in \{1,2,\dots,s\}$ for which $x^*(E(G_t)) = r(G_t)$ so we contradict (7.13).

This final contradiction shows that we must have $s = 2$, and so the proof is complete. ∎

In view of this result, we will always have $s = 2$ in Lemma 7.3. Therefore it specializes to the following

COROLLARY 7.14. *Let $G = [V', E']$ be a facet inducing clique structure for Q_T^n, let v be a cutnode of G and let $x \in Q_T^n$ be integral and satisfy $x(E') = r(G)$. Then v joins exactly two pieces G_1 and G_2, and either $x(E(G_1)) = r(G_1)$ and $x(E(G_2)) = r(G_2) - 1$ or $x(E(G_1)) = r(G_1) - 1$ and $x(E(G_2)) = r(G_2)$.*

We now prove two last technical lemmas before proving the main result of this section.

LEMMA 7.15. *Let v be a cutnode contained in a handle H of a simple clique tree $C = [V', E']$. Let integral $x \in Q_T^n$ satisfy $x(E') = r(C)(= s(C))$ and $x(\delta(v) \cap E(H)) = 1$. Then there exists integral $x' \in Q_T^n$ satisfying $x'(E') = r(C)$, $x'(\delta(v) \cap E(H)) = 0$ and $x'_j = x_j$ for every edge j belonging to the piece of C joined by v that does not contain H.*

PROOF. Let x be as in the statement of the lemma, let C_1 and C_2 be the pieces joined by v, where H is contained in the node set of C_1. Then C_2 is a clique tree and by Lemma 4.4 it is possible to obtain a saturating path cover of C_2 which has one edge incident with v. We can define $\bar{x}$ to be equal to the incidence vector of this path cover for $E(C_2)$, equal to x for $E(C_1)$ and suitably defined for $E \setminus E'$ and we have $\bar{x} \in Q_T^n$ satisfying $\bar{x}(E') = r(C)$ and $\bar{x}(E(C_2)) = r(C_2)$. Therefore, by Corollary 7.14, we must have $x(E(C_1)) = r(C_1) - 1$.

Now C_1 is not a clique tree, but if we add a new tooth T^* containing two nodes not in C_1 and the cutnode v, then we have a clique tree C^*. Let π be a saturating path of T^* in which v has degree two. By Lemma 4.3 we can extend π to a path cover P of C^*. Let x^* be the incidence vector of a tour of K_n that contains P. Then $x^*(E(C^*)) = r(C^*)$ and $x^*(E(T^*)) = r(T^*)$, so by Corollary 7.14, $x^*(E(C_1)) = r(C_1) - 1$ and clearly $x^*(\delta(v) \cap E(C_1)) = 0$.

Thus we define $x'_j := x_j$ for $j \in E(C_2)$, $x'_j := x_j^*$ for $j \in E(C_1)$ and set x'_j to some suitable value for $j \in E \setminus E'$, we obtain the required x'. ∎

LEMMA 7.16. *Let T be a tooth of a clique tree $C = [V', E']$ and let $\mathring{T}$ be the set of those nodes of T which belong to no other clique. If $|\mathring{T}| > 1$ then every $x \in Q_T^n$ satisfying $x(E') = r(C)$ must also satisfy $x(\delta(v) \cap E') > 1$ for every $v \in \mathring{T}$. If $|\mathring{T}| = 1$, then there does exist $x' \in Q_T^n$ satisfying $x'(E') = r(C)$ and $x'(\delta(v) \cap E') = 0$, where $\mathring{T} = \{v\}$.*

PROOF. Suppose that there exist distinct v', $v \in \mathring{T}$ and $x \in Q_T^n$ satisfies $x(E') = r(C)$ and $x(\delta(v) \cap E') = 0$. If $x(\delta(v') \cap E') < 2$, then we could set $x_{vv'} := 1$ and contradict $x(E') = r(C)$. Therefore $x(\delta(v') \cap E') = 2$ and so if P is the path cover of C induced by x then P contains a path π which contains v' as an internal node.

If an end, say u, of π were in T, then by setting $x_{uv} := 1$ we would again contradict $x(E') = r(C)$. Therefore neither end is in T, so there exists a handle H such that a cutnode w of C joins H and T and $x(\delta(w) \cap E(H)) = 1$. By Lemma 7.15 we can modify x_j for edges j in $E(C')$, where C' is the piece of C joined by v that contains H, plus $j \in E \setminus E'$ so that the resulting x' satisfies $x'(\delta(w) \cap E(H)) = 0$, $x'(E') = x(E')$, and $x'_j = x_j$ for all $j \in E \setminus (E' \cup E(C'))$. Then by setting $x'_{wv} := 1$ we contradict $x'(E') = r(C)$.

Now suppose $|\mathring{T}| = 1$, and let $\{v\} = \mathring{T}$. Let $H_1, H_2, \dots, H_k$ for $k > 1$ be the handles intersecting T. For each $i = 1,2,\dots,k$ let v_i be the cutnode of C joining H_i and T, let C_i be the piece of C joined by v_i containing H_i and let C'_i be the clique tree obtained by adding v and an edge joining v_i and v to C_i. By Remark 3.6(b), $r(C) = s(C) = \sum_{i=1}^{k} s(C'_i)$. Since C is facet inducing, by Lemma 7.3, $r(C_i) = r(C'_i) = s(C'_i)$ for $i = 1,2,\dots,k$. Therefore if we let x' be the incidence vector of a tour containing saturating path covers $P_1, P_2, \dots, P_k$ of $C_1, C_2, \dots, C_k$, then $x'(E') = r(C)$ and $x'(\delta(v) \cap E') = 0$. ∎

Now we are ready to show that any facet-inducing clique structure G is a clique tree. Since each cutnode of G must join exactly two blocks, what we must show is the following:

(i) The blocks of G can be partitioned into sets $\mathscr{T}$ of teeth and $\mathscr{H}$ of handles such that the members of each $\mathscr{T}$ and $\mathscr{H}$ are pairwise disjoint.

(ii) Every handle meets an odd number of at least three teeth.

(iii) Every tooth contains a node which belongs to no handle.

We describe a pruning process that will either establish (i), (ii) and (iii) or else will show that G is not facet inducing. A clique of G is called *pendent* if it has nonempty intersection with at most one other clique.

7.17. *Pruning process.*

Step 0. Initially $\mathscr{T} := \mathscr{H} := \emptyset$. Let $\bar{G} := G$. Then $\bar{G}$ is the subgraph of G consisting of all those cliques not yet allocated to $\mathscr{T}$ or $\mathscr{H}$.

Step 1 [Tooth allocation]. If

(7.18) a pendent clique of $\bar{G}$ intersects a clique in $\mathscr{T}$, or

(7.19) $\bar{G}$ consists of exactly two cliques, or

(7.20) for some pendent clique K of $\bar{G}$, every node of K belongs to some other clique

then G is not facet inducing and we stop. Otherwise, remove every pendent clique from $\bar{G}$ and add it to $\mathscr{T}$. If $\bar{G}$ is now empty, then G is a clique tree and we stop. If not, go to Step 2.

Step 2 [Handle allocation]. If

(7.21) a pendent clique of $\bar{G}$ intersects a clique in $\mathscr{H}$, or

(7.22) $\bar{G}$ consists of exactly two cliques, or

(7.23) $\bar{G}$ consists of a single clique which meets an even number of members of $\mathscr{T}$,

or

(7.24) $\bar{G}$ consists of more than one clique, but a pendent clique K meets an odd number of members of $\mathscr{T}$,

then G is not facet inducing and we stop. Otherwise remove every pendent clique from $\bar{G}$ and add it to $\mathscr{H}$. If $\bar{G}$ is now empty then G is a clique tree and we stop. Otherwise go to Step 1. ∎

It can be routinely verified that if (7.17) terminates with the conclusion that G is a clique tree, then this is indeed the case, and moreover, that $\mathscr{T}$ is the set of teeth and $\mathscr{H}$ is the set of handles. In fact we will always have the following properties satisfied following each completion of Step 1 or 2.

(7.25) For any clique T added to $\mathscr{T}$ in the previous execution of Step 1, the clique T together with the portion of G disconnected from $\bar{G}$ by deleting the edges of $E(T)$ is a valid clique tree, having T as a tooth.

(7.26) For any clique H added to $\mathscr{H}$ in the previous execution of Step 2, if the resulting $\bar{G}$ is not empty, then the clique H, plus the intersecting clique of $\bar{G}$ plus the portion of G disconnected from $\bar{G}$ by deleting the edges of $E(H)$, is a valid clique tree having H as a handle.

Now we show that if one of (7.18)–(7.24) holds, then G cannot be facet inducing.

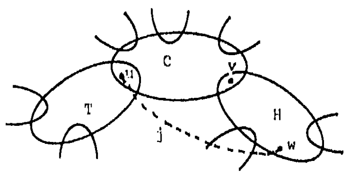


FIGURE 7.2

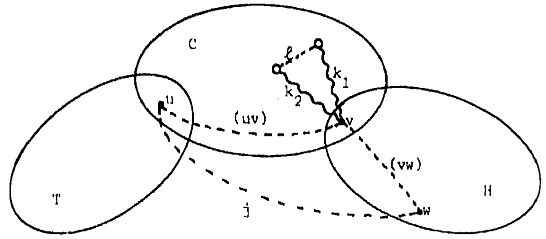


FIGURE 7.3

PROPOSITION 7.27. *If* (7.18) *or* (7.21) *holds then G cannot be facet inducing.*

PROOF. In either case, a pendent clique C of $\bar{G}$ intersects a clique $T \in \mathscr{T}$ and a clique $H \in \mathscr{H}$. (It intersects one of these because of (7.18) or (7.21) and the other because this must be the first time that C was pendent, so at the previous step it intersected one of the other type.) Let u be the cutnode joining T and C, let v be the cutnode joining C and H and let w be a node of H different from v. (See Figure 7.2.)

Let j be the edge of K_n joining u and w. Then $j \notin E'$, but by Lemma 7.6 if G is facet inducing then there is $x \in Q_T^n$ satisfying $x_j = 1$ and $x(E') = r(G)$. The edges k of G having $x_k = 1$ are the edges of a path cover of G, and both u and w are path ends *of different paths*, since otherwise x would not be the incidence vector of a tour. We now obtain a contradiction by constructing another $x' \in Q_T^n$ for which $x'(E') > x(E') + 1$. First, if $x(\delta(v) \cap E') \leq 1$ then we can obtain such an x' by redefining one of x_{uv} or x_{vw} to be equal to one, and suitably modifying x_k for $k \in E \setminus E'$. Therefore we suppose that $x(\delta(v) \cap E') = 2$. If there are edges $k_1, k_2 \in \delta(v) \cap E(C)$ for which $x_{k_1} = x_{k_2} = 1$, then we can obtain the desired x' from x by setting $x'_{k_1} := x'_{k_2} := 0$, $x'_{uv} := x'_{vw} := 1$ and $x'_l := 1$, where l is the edge of $E(C)$ joining the ends of k_1, k_2 distinct from v (see Figure 7.3), and then suitably defining x'_k for $k \in E \setminus E'$. If $x(\delta(v) \cap E(H)) = 2$ then we can construct the desired x' in a completely analogous fashion, so we suppose there are edges $k_1 \in \delta(v) \cap E(C)$ and $k_2 \in \delta(v) \cap E(H)$ with $x_{k_1} = x_{k_2} = 1$. Let P be the path cover of C induced by x and let π be the path containing the edges k_1 and k_2. If u is not an end of π, and hence does not appear in π (since u meets at most one edge of a path in P) then we construct x' by using Lemma 7.15 to modify x so that $x'(\delta(v) \cap E(H)) = 0$, then setting $x'_{uv} := 1$, then setting x'_k for $k \in E \setminus E'$ appropriately. Therefore we can assume that u is one end of π. But this means that $x(\delta(u) \cap E(T)) = 0$. Since $T \in \mathscr{T}$, there is some $u' \in V(T)$ which belongs to no other clique of G, so if C_1 is the piece of G joined by u, then C_1 is a clique tree containing the tooth T and both u and u' belong to T but no other clique of C_1. Therefore by Lemma 7.16, since $x(\delta(u) \cap E(T)) = 0$, we have $x(E(C_1)) = r(C_1) - 1$. But by using Lemma 4.4, we can construct a saturating path cover P of C_1 having exactly one edge incident with u. Then, by modifying x so that it is the incidence vector of P for edges of $E(C_1)$ and by suitably changing x_j for $j \in E \setminus E'$, we obtain the desired x'. ∎

PROPOSITION 7.28. *If neither* (7.18) *nor* (7.21) *holds, but* (7.19) *or* (7.22) *holds, then* $G = [V', E']$ *cannot be facet inducing.*

PROOF. In this case $\bar{G}$ consists of two cliques K_1 and K_2 joined by a cutnode v. If $G = \bar{G}$, then by Lemma 7.1 $r(G) = r(K_1) + r(K_2)$, so by Lemma 7.3, G cannot be facet inducing. So assume $G \neq \bar{G}$, and consequently there are other cliques intersecting K_1 and K_2. Since (7.18) and (7.21) do not hold, either each of these cliques is in $\mathscr{H}$ or each is in $\mathscr{T}$. Let u be a cutnode of K_1 joining K_1 to a clique K_4 other than K_2, and let

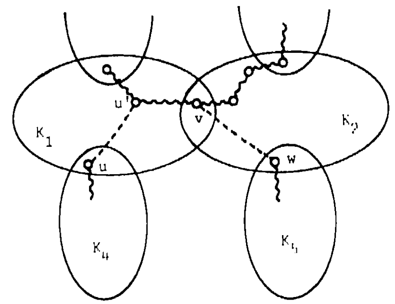


FIGURE 7.4

w be a cutnode of K_2 joining K_2 to a clique K_5 other than K_1. Let j be the edge of $E \setminus E'$ joining u and w and suppose that G is facet inducing. By Lemma 7.6 there exists integer $x \in Q_T^n$ satisfying $x_j = 1$ and $x(E') = r(G)$ so we must have $x(\delta(u) \cap E') \leq 1$ and $x(\delta(w) \cap E') \leq 1$.

Exactly as in the preceding proof we can show that we must have $x(\delta(v) \cap E') = 2$ and moreover, that there exist $k_1 \in E(K_1) \cap \delta(v)$ and $k_2 \in E(K_2) \cap \delta(v)$ such that $x_{k_1} = x_{k_2} = 1$. Let P be the path cover of G induced by x and let π be the path containing k_1 and k_2. If π had an end s either different from u and in K_1 or different from w and in K_2, then by defining either $x_{su} := 1$ or $x_{sw} := 1$ we would contradict $x(E') = r(G)$. Since $x_j = 1$, at least one end of π does not belong to $\{u, w\}$, so some clique K_3 is joined by a cutnode v' to either K_1 or K_2 and $E(K_3)$ contains at least one edge of π. Thus $x(\delta(v') \cap E(K_3)) = 1$. If $K_3 \in \mathscr{H}$, then by Lemma 7.15 we could modify x for the piece of G joined by v' containing K_3 and then set $x_{v'u}$ or $x_{v'w}$ to one (depending on whether $v' \in V(K_1)$ or $v' \in V(K_2)$) and again contradict $x(E') = r(G)$. Therefore K_3, and hence all cliques intersecting K_1 and K_2, belong to $\mathscr{T}$.

If u were an end of π, then $x(E(K_4) \cap \delta(u)) = 0$, so if we let C_1 be the piece of G joined by u that contains K_4, then C_1 is a clique tree in which K_4 is a tooth, so by Lemmas 7.16 and 4.4, we could redefine x for $E(C_1)$ and $E \setminus E'$ and contradict $x(E') = r(G)$. Therefore, u is not an end of π, and similarly, w is not an end of π. Let u' be the end of k_1 different from v. (See Figure 7.4.) If we obtain x' from x by letting $x'_{k_1} := 0$, $x'_{uu'} := x'_{vw} := 1$, and x'_k be suitably redefined for $k \in E \setminus E'$, then $x' \in Q_T^n$ but $x'(E') = x(E') + 1$ so we contradict $x(E') = r(G)$, which completes the proof. ∎

PROPOSITION 7.29. *If neither* (7.18) *nor* (7.19) *hold but* (7.20) *holds then G is not facet inducing.*

PROOF. A pendent clique K is joined by some node v to another clique K' which is not in $\mathscr{T} \cup \mathscr{H}$, and every other node of K is a cutnode joining a different member of $\mathscr{H}$. Let C_1 be the piece of G joined by v that contains K. Then C_1 is a clique tree and K is a tooth containing exactly one node (v) belonging to no handle. By Lemma 7.16 there is a saturating path cover P_1 of C_1 containing no edge incident with v. Let P_2 be any saturating path cover of the other piece C_2 of G joined by v. Then P_1 and P_2 provide, together, a path cover of G that shows that $r(G) > r(C_2) + r(C_1)$ so by Lemma 7.3, G cannot be facet inducing. ∎

PROPOSITION 7.30. *If* (7.21) *does not hold, but* (7.23) *does hold, then G is not facet inducing.*

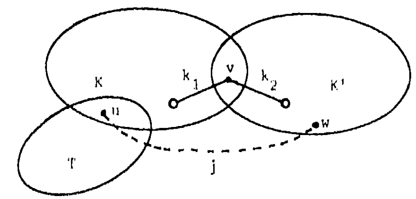


FIGURE 7.5

PROOF. G consists of a clique K joined to an even number (≥ 2) of members of $\mathscr{T}$. Let T be such a member of $\mathscr{T}$ joined to K by a cutnode v. Let C_1 and C_2 be the pieces of G joined by v. Since both are clique trees, and hence facet inducing, by Lemma 7.2 there exist saturating path covers P_1 and P_2 of C_1 and C_2, each of which has a single edge incident with v. Therefore, $r(G) > r(C_1) + r(C_2)$ so by Lemma 7.3, G cannot be facet inducing.

PROPOSITION 7.31. *If neither* (7.21) *nor* (7.22) *holds, but* (7.24) *holds, then G is not facet inducing.*

PROOF. A pendent clique K meets one clique K' not in $\mathscr{T} \cup \mathscr{H}$, plus an odd number of members of $\mathscr{T}$. Let v be the cutnode joining K and K', let u be a cutnode joining K and some tooth T and let w be any node of K' different from v. (See Figure 7.5.) Let j be the edge of $E \setminus E'$ joining u and w and suppose that G is facet inducing. By Lemma 7.6 there exists $x \in Q_T^n$ satisfying $x(E') = r(G)$ and $x_j = 1$. Therefore $x(\delta(u) \cap E') \leq 1$ and $x(\delta(w) \cap E') \leq 1$. As in the proofs of Lemmas 7.27 and 7.28 we can see that if P is the path cover of G induced by x, then there must be a path π containing $k_1 \in \delta(v) \cap E(K)$ and $k_2 \in \delta(v) \cap E(K')$. Now suppose that $x(\delta(u) \cap E(T)) = 0$. Let C_1 be the piece of G joined by u containing T. Then C_1 is a clique tree containing the tooth T which contains at least two nodes belonging to no other clique. By Lemma 7.16 $x(E(C_1)) < r(C_1)$, and by Lemma 4.4 we can find a saturating path cover of C_1 with exactly one edge incident with u. Then if x' is the incidence vector of a tour containing this path cover and P restricted to the other piece of G, we have $x'(E') > x(E') = r(G)$, a contradiction. Therefore the single edge k_3 of $E' \cap \delta(u)$ in P is an edge of $E(T)$, and in particular, u is not an end of π.

If some other tooth T' joined by a cutnode u' to K satisfied $x(\delta(u') \cap E(T')) = 2$, then using Lemma 4.4 we could modify x for the edges of the piece C' of G joined by u' containing T' so as to have $x(E(C')) = r(C')$ and $x(\delta(u') \cap E(T')) = 1$ and then set $x_{uu'} := 1$, contradicting $x(E') = r(G)$. Therefore, for each such T' and u', we have $x(\delta(u') \cap E(T')) \leq 1$. Similarly, if $x(\delta(u') \cap E(K)) = 2$, then, using Lemma 7.16, $x(C') < r(C')$ so we could first modify x for the edges of C' so that $x(E(C')) = r(C')$ and $x(\delta(u') \cap E(T')) = 1$. Then, for $l \in E(K)$ joining u' to some u'' and having $x_l = 1$, we could let $x_l := 0$ and $x_{uu''} := 1$ and contradict $x(E') \leq r(G)$.

But now we are done, for K contains an odd number of cutnodes other than u, and each is contained in a path of P that enters K. But this means that some path has an end s in K, so by setting $x_{su} := 1$, we contradict $x(E') \leq r(G)$ and complete the proof. ∎

So finally, by combining Propositions 7.27–7.31 we have completed the justification of Procedure 7.17 and indeed have proved the following.

THEOREM 7.32. *Let G be a facet inducing clique structure for Q_T^n. Then G is a simple clique tree.*

8. Concluding remarks. We have shown that clique tree inequalities provide a very large set of facet inducing inequalities of Q_T^n and $\tilde{Q}_T^n$, and that they include subtour elimination constraints and comb inequalities as special subclasses. We have shown that there is no more general structure composed of cliques joined by cutnodes for which the corresponding rank inequality is facet inducing for Q_T^n. It appears likely that Theorem 7.32 could be appropriately generalized to allow general clique structures where cliques are joined by articulation sets instead of cutnodes. However, for the present, we have not tried to prove such a theorem.

Clique tree inequalities have several interesting properties with respect to the rank function defined by Chvátal (1973b). For example, a clique tree with more than one handle is of rank at least two, and it appears that as the number of handles increases, so too does the rank although not linearly. This will be treated in a subsequent paper.

However, probably the main outstanding question is whether efficient routines can be developed for the clique tree separation problem. That is, given $\hat{x} \in \mathbb{R}^n$, either find a clique tree inequality violated by $\hat{x}$ or else show that $\hat{x}$ satisfies all clique tree inequalities. By virtue of Grötschel, Lovász and Schrijver (1981), a polynomial algorithm for this problem would result in a polynomial algorithm for solving the linear program:

$$\begin{aligned} &\text{minimize } cx\\ &0 \leq x \leq 1,\\ &Ax = 2,\\ ax \leq \alpha \quad &\text{for every clique tree inequality,}\end{aligned}$$

where A is the node edge incidence matrix of K_n. In almost all cases, it seems that the solution to this linear program would give a very good lower bound on the solution to the TSP with costs c. Presently the only solvable cases of this separation problem are for subtour elimination constraints and for (simple) combs in which each tooth has two nodes (Padberg and Rao 1982).

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