CLIQUE TREE INEQUALITIES AND THE SYMMETRIC TRAVELLING SALESMAN PROBLEM

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The linear programming cutting plane approach for solving the travelling salesman problem has recently proven to be highly successful, cf. Crowder and Padberg (1980), Grotschel (1980a), Padberg and Hong (1980). One of the reasons for this success is certainly the fact that instead of ordinary cutting planes (Gomory cuts etc.) problem-specific cutting planes could be used which define facets of the underlying integer programming polytopes.

In this paper we shall define a new class of inequalities (clique tree inequalities) valid for the travelling salesman polytope which properly contains many of the known classes of inequalities (like subtour elimination constraints, 2-matching constraints, comb inequalities), and we show that all these new inequalities induce facets of the travelling salesman polytope. Since the general structure of these new inequalities is quite simple we hope that it will be possible to use the inequalities efficiently in cutting plane procedures for the travelling salesman problem.

1. Introduction and notation. The linear programming cutting plane approach for solving the travelling salesman problem has recently proven to be highly successful, cf. Crowder and Padberg (1980), Grotschel (1980a), Padberg and Hong (1980). One of the reasons for this success is certainly the fact that instead of ordinary cutting planes (Gomory cuts etc.) problem-specific cutting planes could be used which define facets of the underlying integer programming polytopes.

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Our purpose, therefore, is threefold. First we introduce this set of so-called clique tree inequalities which are valid for the travelling salesman polytope and its monotone extension. Second we show that these inequalities are facet-inducing for these polytopes and, almost always, induce distinct facets. Third, we show that in a sense made precise in §7, there is no facet-inducing generalization.

For our purposes, a graph \( G = (V,E) \) consists of a finite set \( V \) of nodes and a finite set \( E \) of two-element subsets of \( V \). The elements of \( E \) are called edges. If \( e \in E \) is an edge then the two nodes, say \( i \) and \( j \), contained in \( e \) are called the endpoints of \( e \). An edge is said to join or link \( i \) and \( j \), and \( i \) and \( j \) are called adjacent. For ease of notation we denote an edge \( e \) linking two nodes \( i \) and \( j \) by \( ij \).

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Note that our definition implies that (our) graphs have no loops or multiple edges. The cardinality of the node set of a graph is called the order of this graph. A graph in which every two nodes are adjacent is called complete. The complete graph of order $n$ is denoted by $K_n$.

If $G = \{V, E\}$ is a graph and $W \subseteq V$ then the set of edges in $G$ which have both endnodes in $W$ is denoted by $E(W)$. If $F \subseteq E$ then the set of nodes of $G$ which are contained in at least one edge $e \in F$ is denoted by $V(F)$. A graph $H = \{W, F\}$ is called a subgraph of $G = \{V, E\}$ if $W \subseteq V$ and $F \subseteq E(W)$. If $W \subseteq V$, $\emptyset \neq W \cap V$, then $E(W)$ is the set of edges with one endnode in $W$ and the other endnode in $W \cap V$. The edge set $E(W)$ is called a cut. We write $H \subseteq G$ instead of $V(F) \subseteq V(G)$ for $e \in E(F)$.

A clique in a graph $G = \{V, E\}$ is a set of nodes such that any two nodes in $W$ are adjacent. This definition implies that the graph $\{V, W, E(W)\}$ is a complete subgraph of $G$. In the sequel we shall use the clique word only for those cliques in a graph which are maximal with respect to set inclusion.

For any graph $G = \{V, E\}$, for all $S \subseteq V$, we let $G - S$ denote the subgraph of $G$ obtained by deleting the nodes of $S$, plus all incident edges. If $G_1 = \{V_1, E_1\}$ and $G_2 = \{V_2, E_2\}$ are subgraphs of a graph $G$ then the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, is the subgraph $\{V_1 \cup V_2, E_1 \cup E_2\}$ of $G$.

An articulation set of a graph $G = \{V, E\}$ is a minimal set $S$ of nodes such that $|V\setminus S| > |E\setminus V|$. An edge in $G$ is called a cut edge of $G$. Every subgraph $\{V, W, E(W)\}$ of $G$ which contains no cut edge and which is maximal with respect to property 2 is called a block of $G$.

A path $P$ in a graph $G = \{V, E\}$ is a sequence of edges $e_1, e_2, \ldots, e_k$, such that $e_1 = \{v_0, v_1\}, e_2 = \{v_1, v_2\}, \ldots, e_k = \{v_{k-1}, v_k\}$, and such that $e_i \neq \emptyset$ holds for $0 < i < k$. The nodes $v_0$ and $v_k$ are the endnodes of $P$. We say that $P$ goes from $v_0$ to $v_k$ or that $P$ links $v_0$ and $v_k$. The number $k$ of edges of $P$ is called the length of $P$. Since a path is uniquely determined by the sequence of its nodes we also write $[v_0, v_1, \ldots, v_k, v_{k+1}]$ to denote a path from $v_0$ to $v_{k+1}$. A graph $G$ is called connected if any two nodes of $G$ can be linked by a path in $G$. A path of length $|V| - 1$ is called a hamiltonian path of $G$.

It follows from these definitions that is quite unlikely that we will ever obtain a tractable linear description of $Q_F$ or $Q_T$, cf. Grötschel (1980b), Grötschel, Lovász and Schrijver (1981a), and Karp and Papadimitriou (1980). Nevertheless, even partial knowledge of such inequallities is desirable for use in the practical and algorithmic points of view.

The polyhedra $Q_F$ and $Q_T$ have been intensively studied in the past, and we shall survey some of the results known to date.
In order to obtain results about the facial structure of a polyhedron it is important to know its dimension. For the travelling salesman polytope we have
\[
\dim \mathcal{Q}_n = m = |E|, \tag{2.4}
\]
\[
\dim \mathcal{Q}_n = m - n = |E| - |V|, \tag{2.5}
\]
where (2.4) is obvious, and a proof of (2.5) can be found in Grötschel and Padberg (1979a). Thus, \(\mathcal{Q}_n\) is a face of \(\mathcal{Q}_m\) whose dimension is less than the dimension of \(\mathcal{Q}_n\).

By definition, \(\mathcal{Q}_n\) is contained in the unit hypercube which implies that the inequalities
\[
0 < x_i < 1 \quad \text{for all} \quad j \in E \tag{2.6}
\]
are valid for \(\mathcal{Q}_n\). In fact, these inequalities induce facets of \(\mathcal{Q}_n\) for all \(n \geq 3\). For the travelling salesman polytope \(\mathcal{Q}_n\) we have that the inequalities \(x_i < 1\) induce facets of \(\mathcal{Q}_n\) for all \(n \geq 5\), while the inequalities \(x_i < 1\) induce facets of \(\mathcal{Q}_n\) for all \(n \geq 4\) (in case \(n = 4\)) by induction using the inequalities of Grötschel (1977).

Since every node of \(K_n\) lies on exactly two edges of every tour, \(\mathcal{Q}_n\) is contained in the \(n\) hyperplanes defined by
\[
x(v) = \sum_{e \in E} x_e = 2 \quad \text{for all} \quad v \in V. \tag{2.7}
\]

The intersection of the hyperplanes defined by (2.7) is exactly the affine space spanned by \(\mathcal{Q}_n\). Let us denote the node-edge incidence matrix of \(K_n\) by \(A\) and let \(2\) be a \(n\)-vector all of whose components are one; then
\[
A x = 2 \tag{2.8}
\]
is a minimal system of equations whose solution set consists of \(\mathcal{Q}_n\). This implies (using Farkas' Lemma) that any facet inducing inequality of \(\mathcal{Q}_n\) is unique up to scaling and adding multiples of (2.8), i.e.

**Lemma 2.9.** Let \(a^T x < 0\) induce a nonempty proper face of \(\mathcal{Q}_n\) and suppose that \(b^T x < 0\) induces a facet of \(\mathcal{Q}_n\). Then \(a^T x < a_0^T x = 0\) and \(b^T x < b_0^T x = 0\), respectively, i.e., define the same facet of \(\mathcal{Q}_n\), if and only if there are \(\lambda \in \mathbb{R}\) and \(p > 0\) such that \(a^T = pb^T + \lambda A\).

Note that since both \(a^T x < a_0^T x = 0\) and \(b^T x < b_0^T x = 0\) define nonempty faces of \(\mathcal{Q}_n\), the condition \(a^T = pb^T + \lambda A\) implies \(a_0^T x = b_0^T x + \lambda\). Moreover, we can also express \(b\) in terms of \(a\) in a similar fashion since \(a^T = a^T/p = \lambda A/p\).

It is obvious that the inequalities
\[
x(v) < 2 \quad \text{for all} \quad v \in V \tag{2.10}
\]
are valid for \(\mathcal{Q}_n\), and it was shown in Grötschel (1977) that the inequalities (2.10) are facet-inducing for \(\mathcal{Q}_n\) for all \(n \geq 4\).

Dantzig, Fulkerson & Johnson (1954) introduced the so-called subtour elimination constraints
\[
x(E(W)) < |W| - 1 \quad \text{for all} \quad W \subseteq V, \emptyset \neq W \neq V \tag{2.11}
\]
which algebraically state that no tour can contain a subtour (a cycle of length |W|, \(W \neq V\)). It is clear that the inequalities (2.11) are valid for \(\mathcal{Q}_n\) and \(\mathcal{Q}_n\). The following result was shown by Grötschel (1977), resp. Grötschel and Padberg (1979b).

**Theorem 2.12.** (a) The subtour elimination constraints \(x(E(W)) < |W| - 1\) are facets inducing for \(\mathcal{Q}_n\), for all \(W \subseteq V\) with \(2 < |W| < n - 1\) and all \(n \geq 4\).
Figure 2.1 shows an example of a clique tree, where cliques are indicated by ellipse-shaped figures. Each ellipse containing a "*" is a tooth. The "*" indicates that there must be a node in the tooth which is not in any handle.

The graph of Figure 2.2 shows the smallest clique tree which is not a comb or a single clique. It is a graph on eleven nodes which has two handles (encircled) and five teeth, four of which have two nodes and one (the center) has three nodes.

We shall call those clique trees simple for which any handle and any tooth have at most one node in common. For example, the clique tree in Figure 2.2 is simple.

Suppose we have a clique tree \( C \) with handles \( H_1, H_2, \ldots, H_s \) and teeth \( T_1, T_2, \ldots, T_s \). We shall show in the sequel that the following clique tree inequality defines a facet of \( Q^*_C \) and \( Q^*_C \):

\[
\sum_{i=1}^{s} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) < \sum_{i=1}^{s} |H_i| + \sum_{j=1}^{s} |T_j| - s - 1 - \frac{L}{2} \tag{2.17}
\]

where for every tooth \( T_j \), the integer \( i \) denotes the number of handles which intersect \( T_j \). Note that in case there is a tooth \( T \) and a handle \( H \) with \( |H \cap T| > 2 \) then the coefficients on the left-hand side of (2.17) are 0, 1 and 2. The inequality (2.17) is a 0/1-inequality only if the clique tree is simple. If \( W \) is the set of all nodes of a clique tree, then for simple clique trees, inequality (2.17) can be written as:

\[
\sum_{i=1}^{s} x(E(H_i)) + \sum_{j=1}^{s} x(E(T_j)) < |W| - \frac{L + 1}{2} \tag{2.18}
\]

Note that the inequalities (2.11) and (2.14) are special cases of the clique tree inequalities (2.17). Any subtour elimination constraint can be considered as a clique

3. Construction of clique trees, validity. In this section we shall give constructive characterizations of clique trees ("gluing" and "splitting") and we shall prove that the clique tree inequalities (2.17) are valid with respect to \( Q^*_C \) and \( Q^*_C \).

The name clique tree reflects the fact that the structure of such a graph is "tree-like," namely, replacing every clique by a node and linking a pair of nodes whenever the corresponding cliques intersect we obtain a tree, the tree underlying the clique tree. It follows from Definition 2.16 that the partition of a clique tree into handles and teeth can be obtained with an analogue of "tree pruning" as follows.

Let us call a clique "independent if it meets at most one other clique. By (5) of (2.16) no handle can be independent. Thus all the pendant cliques of a clique tree are teeth. If we remove these teeth from the clique tree the resulting graph will have new pendant cliques, called these handles. Removing these handles we obtain a further graph whose pendant cliques are called teeth, etc. Obviously, Definition 2.16 implies that this pruning procedure ends in a non-ambiguous way. We shall now describe how one can glue and split clique trees. These methods will be important tools in subsequent inductive proofs.

3.1. Gluing clique trees at a tooth. Let \( C' \) and \( C'' \) be two clique trees with node sets \( V' \) and \( V'' \), and suppose \( T' \subseteq V' \cap V'' \) is a tooth of \( C' \) and \( C'' \), that \( T \) contains a node not in any handle of \( C' \) and \( C'' \), and that the handles of \( C' \) and \( C'' \) do not intersect. Then \( C' \cup C'' \) is a clique tree called the clique tree obtained from \( C' \) and \( C'' \) by gluing at tooth \( T \).

3.2. Splitting a clique tree at a tooth and a handle. Let \( C \) be a clique tree and \( T \) a tooth of \( C \). Let \( H \) be a handle of \( C \) intersecting \( T \). Delete the nodes \( H \setminus T \) from \( C \) and let \( C' \) be the component of \( C \) (\( H \setminus T \)) containing \( T \). Delete the nodes from \( C \) which are in handles meeting \( T \) but not in \( T \) or \( H \). Let \( C' \) be the component of this graph containing \( T \). Then \( C' \) and \( C'' \) are clique trees called the clique trees obtained from \( C \) by splitting at \( T \) and \( H \).

It is clear from Definition 2.16 and the operations defined in (3.1) and (3.2) that the graphs resulting from gluing and splitting are indeed clique trees. Figure 3.1 shows a gluing operation, and Figure 3.2 a splitting operation. In case \( T \) is a tooth intersecting one handle \( H \) only, we obtain the original clique tree and the complete graph.
\[ [T, E(T)] \text{ as the result of splitting at } T \text{ and } H. \text{ If } C \text{ is a clique tree which we split at a tooth } T \text{ and a handle } H \text{ to obtain two clique trees } C' \text{ and } C^* \text{ then, by gluing } C' \text{ and } C^* \text{ at } T, \text{ the original clique tree } C \text{ is reproduced. The gluing operation can be used to define clique trees inductively as follows:}

**Definition 3.3.** (a) Complete graphs with at least two and at most } n - 2 \text{ nodes are clique trees, and combs are clique trees.}

(b) If } C' \text{ and } C^* \text{ are clique trees satisfying the assumptions of (3.1) then the graph resulting from gluing at tooth } T \text{ is a clique tree.}

(c) All graphs that can be generated by (a) and (b) are called clique trees.

**Lemma 3.4.** The clique tree Definitions 3.3 and 2.16 are equivalent.

**Proof.** For clique trees which are complete graphs there is nothing to prove. All clique trees in the sense of Definition 2.16 with one handle are combs, so they are clique trees in the sense of Definition 3.3. Suppose we have shown that the definitions are equivalent for all clique trees with } r > 1 \text{ handles.}

Assume that } C \text{ is a clique tree in the sense of Definition 2.16 with } r + 1 \text{ handles. Then there is a tooth } T \text{ meeting at least two handles. Let } H \text{ be one of the handles and split at } T \text{ and } H. \text{ The resulting clique trees } C' \text{ and } C^* \text{ in the sense of Definition 2.16 are clique trees in the sense of Definition 3.3 by the induction hypothesis. By gluing } C' \text{ and } C^* \text{ at } T \text{ we get back our original clique tree } C. \text{ Thus by axiom (b) of Definition 3.3 } C \text{ is a clique tree.}

The gluing operation 3.1 shows that clique trees in the sense of Definition 3.3 are clique trees in the sense of Definition 2.16. It is easy to see that inductive definitions even more restrictive than Definition 3.3 would generate all clique trees. For example, requiring that one of the clique trees } C' \text{ or } C^* \text{ in axiom (b) be a comb would result in the same class of graphs. So all clique trees can be obtained by starting with a comb or a clique with } k \text{ nodes, } 2 \leq k \leq n - 2, \text{ and iteratively gluing combs onto the previously constructed clique tree.}

Another splitting operation on clique trees is the following:

3.5. **Splitting a clique tree at a handle.** Let } C \text{ be a clique tree and } H \text{ a handle of } C. \text{ Let } T_1, \ldots, T_r \text{ be the teeth of } C \text{ which intersect } H. \text{ For every tooth } T_i, i \in \{ 1, \ldots, r \}, \text{ let } C_i \text{ be the clique tree not containing } H \text{ obtained from } C \text{ by splitting at } T_i \text{ and } H. \text{ Then the clique trees } C_1, \ldots, C_r \text{ are called the clique trees obtained from } C \text{ by splitting at } H. \text{ Figure 3.3 shows the result of a splitting operation 3.5. It is easy to see how the sizes of the clique trees obtained by gluing or splitting relate to the sizes of the initial clique trees.}

**Remark 3.6.** (a) Let } C \text{ be the clique tree obtained by gluing (3.1) clique trees } C' \text{ and } C^* \text{ at tooth } T. \text{ Then}

\[ s(C') + s(C^*) = s(C) + |T| - 1. \]

(b) Let } C' \text{ and } C^* \text{ be the clique trees obtained from } C \text{ by splitting (3.2) at tooth } T \text{ and handle } H. \text{ Then}

\[ s(C') + s(C^*) = s(C) + |T| - 1. \]

(c) Let } C \text{ be a clique tree and } H \text{ a handle of } C \text{ intersecting } k \text{ teeth. Let } C_1, \ldots, C_k \text{ be the clique trees obtained from } C \text{ by splitting (3.5) at handle } H. \text{ Then}

\[ \sum_{i=1}^{k} s(C_i) = s(C) - |H| + \frac{k + 1}{2}. \]

**Theorem 3.7.** Let } C \text{ be a clique tree in } K_n \text{ with handles } H_1, \ldots, H_r \text{ and teeth } T_1, \ldots, T_r. \text{ Then the clique tree inequality}

\[ \sum_{i=1}^{r} x(E(H_i)) + \sum_{i=1}^{r} x(E(T_i)) < \sum_{i=1}^{r} |H_i| + \sum_{i=1}^{r} (|T_i| - 1) - \frac{k + 1}{2} = s(C) \]

is valid with respect to } \Omega \phi \text{ (and hence with respect to } \Omega \phi \).

**Proof.** We prove by induction on the number of handles. If } C \text{ has no handle then the clique tree inequality is a subtour elimination constraint and there is nothing to prove.

Suppose the claim is true for all clique trees with } r \text{ handles, and assume } C \text{ is a clique tree with } r + 1 \text{ handles. Pick any handle } H \text{ of } C \text{ and denote the other handles of } C \text{ by } H_1, \ldots, H_r. \text{ Let } T_1, \ldots, T_r \text{ be the teeth of } C \text{ intersecting } H, \text{ and let } C_1, \ldots, C_r \text{ be the clique trees obtained from } C \text{ by splitting at } H. \text{ Every such clique tree has at most } r \text{ handles. We assume that } C \text{ contains } T_i, i = 1, \ldots, k. \text{ Let } \delta^2 x < s(C_i) \text{ be the corresponding clique tree inequalities.}

For every clique tree } C_i, i \in \{ 1, \ldots, k \}, \text{ let } C_i \text{ be the clique tree obtained from } C_i \text{ by replacing } T_i \text{ with } T_i \setminus H, \text{ and let } \delta^2 x < s(C_i) \text{ be the corresponding clique tree inequality.}

By Remark 3.6 we have

\[ \sum_{i=1}^{k} s(C_i) = s(C) - |H| + \frac{k + 1}{2}. \]
which implies
\[
\sum_{i=1}^{k} x(C_i) = s(C) - |H| - \sum_{i=1}^{k} |H \cap T_i| + \frac{k + 1}{2}.
\]
From this we obtain (setting \(H_{\text{sat}} := H\))
\[
2\left(\sum_{i=1}^{k} x(E(H_i)) + \sum_{i=1}^{k} x(E(T_i))\right) < \sum_{i=1}^{k} \left(\alpha x_i + \alpha i x_i + x(E(H_i \cap T_i))\right) + \sum_{i \in H} x(\beta(x_i))
\]
\[
\leq \sum_{i=1}^{k} \left(\alpha x_i + \alpha i x_i + |H \cap T_i| - 1\right) + 2|H|
\]
\[
= 2s(C) + 1.
\]
For every incidence vector of a path system the left-hand side above is an even integer. So, dividing by two and rounding down the right-hand side we get the desired result.

4. CLIQUE TREES INDUCE FACETS. In this section we prove that clique trees define facets of \(Q^2\). The proof of the main theorem, Theorem 4.7, proceeds by induction on the number of handles and is rather long. Much of the proof relies on the existence of path systems satisfying several properties. We have separated the proofs of the existence of these path systems into several technical lemmas which precede Theorem 4.7.

In order to simplify the following we first introduce some new notions.

DEFINITION 4.1. Let \(C\) be a clique tree in \(K\).
(a) Let \(T\) be a tooth of \(C\) and \(H_1, \ldots, H_k\) be the handles of \(C\) that intersect \(T\). We say that \(T\) \(saturates\) \(T_i\) if the following holds
\[
|S \cap E(T_i)| = |T_i| - 1,
\]
\[
|S \cap E(H_i \cap T)| = |H_i \cap T| - 1
\]
for \(i = 1, \ldots, k\).

(b) A path system \(P\) is said to \(saturate\) a clique tree \(C\) if its incidence vector satisfies the clique tree inequality (2.17) for \(C\) with equality. If in addition all edges in the path system are edges of \(C\), then we call \(P\) a \(path\ cover\ of \(C\).

We shall need the following construction several times in the sequel.

CONSTRUCTION 4.2. Suppose \(C\) is a comb with handle \(H\) and teeth \(T_1, \ldots, T_k\). Pick any tooth, say \(T_1\), and any (possibly empty) subset \(S\) of \(H \setminus (T_2 \cup \cdots \cup T_k)\).

Remove \(S \setminus (T_1 \cup T_2 \cup \cdots \cup T_k)\) from \(C\).

Now pick any pair of remaining teeth, say \(T_2\) and \(T_3\), and construct the following path. Start in some node of \(T_2\), go through all nodes of \(T_2 \cap H\), then go to a node in \(T_2 \cap H\), go through all nodes of \(T_2 \cap H\), and if there is any node in \(H\), then go through all nodes of \(H\), go to a node in \(T_2 \cap H\), go through all nodes of \(T_2 \cap H\), and end in some node of \(T_3 \cap H\).

Continue this pairing of teeth with \(T_1\) and \(T_i\), \(i \in \{4, 6, \ldots, k - 1\}\) as just described for \(T_3\) and \(T_4\) the only difference being that we go directly from \(T_i \cap H\) to \(T_{i+1} \cap H\) without running through \(H\). Call this construction \(pairing\ of\ an\ even\ number\ of\ teeth\ of\ a\ comb\).
nodes in $T'' \setminus H'$ ending in a node of $T'' \setminus H'$. We remove all nodes from $C'$ which are in $P_i$ or in the extension of $P_j$, just described before. Then we pair the remaining even number (possibly 0) of teeth of $C'$ using Construction 4.2.

In all three cases we have constructed a path cover of $C'$ containing a path $P_i$ ending at $i$. It is easy to see that the union of the path cover of $D$ and any of the three path covers of $C$ (including the edge from $j$ to a node in $T'' \setminus H'$) gives a path cover of $C$ with the required properties.

**Lemma 4.4.** Let $C$ be a clique tree, $T$ be a tooth of $C$ and $\pi$ be a path saturating $T$. (a) For every node $i$ of $C$ which is not in $T$ and not in any handle intersecting $T$ there is a path cover $P$ of $C$ containing $\pi$ as a path and containing a path $P_i$ ending at $i$.

(b) Let $u$ and $v$ be the endpoints of $\pi$ and suppose $i$ is in a handle $H$. Then for every $i \in H \setminus T$ there exists a path cover $P$ of $C$ which contains a path $P_i$ ending at $i$ and a path $P_u$, ending at $u$ and containing $i$ such that $P_P$ and $P_u$ are disjoint.

**Proof.** If $T$ is a pendant tooth then the claim follows from Lemma 4.3. So suppose $T$ is not pendant and the node $i$ satisfies the requirements of (a) or (b).

Let $H_1, \ldots, H_k$ be the handles intersecting $T$. Denote by $C_j'$, $C_j''$ the clique trees obtained from splitting (3.2) at $T$ and $H_j$, $j = 1, \ldots, k$. By construction, the tooth $T$ is pendant in every clique tree $C_j'$. If $i$ is in $H_j \setminus T$ then the node $i$ must be in $H_j$. Applying Lemma 4.3(a) or (b) we get $C_j'$ contains a path containing $\pi$ as a path and containing a path $P_i$ ending at $i$ and a path $P_u$, ending at $u$ and containing $i$ such that $P_P$ and $P_u$ are disjoint.

For every other clique tree $C_j'$, $j = 2, \ldots, k$, we choose a node $i_j$ which is in a tooth $C_j' \setminus T$ and not in a handle. Then we apply Lemma 4.3(a) or (b) to $C_j'$, $T$, $H_j$, and $i_j$ to obtain a path cover $P_j$ of $C_j'$ which contains $\pi$ as a path.

The union of the path covers $P_1, \ldots, P_k$ of $C_1', \ldots, C_k'$ is a path cover of $C$ (this is easy to see by comparing the sizes of $C_1', \ldots, C_k'$) which has the desired properties.

**Lemma 4.5.** Let $C$ be a clique tree, $H$ a handle of $C$ and $T$ a tooth intersecting $H$. Let $p, q$ be two nodes in $H$ which are in different teeth intersecting $H$ but not in $T$. Let $\pi$ be a path whose endpoints are $p$ and $q$ which contains all nodes of $H$ which are not in teeth, all nodes in $H \setminus T$ but no other nodes and which satisfies $[\pi \cap E(H \cap T)] = |H \cap T| - 1$. Then there exists a path cover $P$ of $C$ containing $\pi$ as a subpath and which contains two different paths each having at most one endnode in a (pendent) tooth of $C$. (In case $s = 3$ one of these two paths may be degenerate.)

**Proof.** Let $T_1, \ldots, T_s$ be the teeth intersecting $H$. Let us assume without loss of generality that $T = T_i$, $\pi \subseteq T_i$ and $\pi \subseteq T_i$. For every tooth $T_j$, $i = 2, \ldots, k$, construct a path $\pi_j$ that saturates $T_j$ and has one endnode in $H \cap T_j$. In case $i = 2$ or 3 this endnode should be $p$ resp. $q$.

Let $C_1', C_2', \ldots, C_s'$ be the clique trees obtained by splitting (3.5) at $H$ where $T_i \setminus T_j$ is in $C_j$, $j = 1, \ldots, s$. For $i = 2, \ldots, s$, use Lemma 4.4 and a node $j_i$ in a (pendent) tooth of $C_i'$ different from $T_i$ to obtain a path cover of $C_i'$ containing $\pi_i$ as a path. Note that if $C_i'$ is $T_i$, we set $j_i = i$.

Let $C_i'$ be the clique tree obtained from $C_i'$ by replacing $T_i$ by $T_i \setminus H$ and let $P_i$ be a path cover of $C_i'$ in which one path ends at a node in a (pendent) tooth. Note that in case $i = 1$, $T_i$ is pendant in $C_i'$ and $|T_i| = 2$, then $P_i$ is degenerate. Let $P$ be the union of the path systems $P_1, \ldots, P_s$, and $\pi$ and the edges $e_{i_j}, i = 4, k, k - 1$, whose endnodes are the endnodes of $\pi_j$ and $\pi_{j+1}$ in $H \cap T_j$ resp. $H \cap T_{j+1}$.

By construction $P$ is a path system, contains $\pi$ as a subpath and contains at least two different paths ending at a node in a (pendent) tooth of $C$. Let $a' \neq x = x(C)$ be the clique tree inequalities for $C$ and $C_i$, $i = 1, 2, \ldots, k$. If $P$ is the set of edges in $E(H)$ which do not have both endnodes in the same tooth and $H$ is the set of nodes in $H$ which are in no tooth, obviously $|F \cap P| = |F| + (k + 1)/2$. Note also that $a' = x = x(C_i - 1)$, then using Remark 3.6(c)

$$a'x' = \sum_{i \in C} a'x'_i + \sum_{i \in C} |F \cap E(H \cap T_i)| + |F \cap P|$$

$$= \sum_{i \in C} x(C_i) - 1 + \sum_{i \in C} |H \cap T_i| - 1 + |H| + k + 1/2$$

$$= x(C) - |H| + \frac{k + 1}{2} + 1 - |H| - k + \frac{k + 1}{2}$$

$$= x(C).$$

Hence $P$ saturates $C$.

**Lemma 4.6.** Let $C$ be a clique tree in $K_n$, $T$ a tooth of $C$, and $V'$ the node set of $C$. Let $H_1, \ldots, H_k$ be the handles intersecting $T$, $T' = T \cap \bigcup_{i=1}^k H_i$, and let $a' \neq x = x(C)$ be the clique tree inequalities for $C$. Then there are two nodes $u, v \in V' \setminus T$ which are in different teeth of $C$ (these teeth may be chosen to be pendant) such that there is a path $P$ with endnodes $u$ and $v$, containing all nodes of $(V' \setminus T)$ and satisfying

(a) $[P \cap E(H \cap T)] = |H \cap T| - 1$, $j = 1, \ldots, k$.

(b) $a'x' = x(C) - |T'| + 1$.

**Proof.** Let $C$ be the clique tree obtained from $C$ by splitting (3.2) at $T$ and $H_j$ which contains $T$ and $H_j$, for $j = 1, \ldots, k$.

For every $j \in \{1, \ldots, k\}$ pick two nodes $p_j, q_j$ in $H_j$ and a path $\pi_j$ such that all requirements of Lemma 4.5 are satisfied with respect to the tooth $T_j$ of $C$. Then for every $j = 1, \ldots, k$ there exists a path cover $P_j$ of $C_j'$ with the properties specified in Lemma 4.5.

Clearly, $P_j$ contains a Hamiltonian path in $T \cap H_j$. Remove this path from $P_j$ and call the remaining path system $P_j'$. Let $P'$ be the union of the path systems $P_j'$, $j = 1, \ldots, k$, then it is obvious from the construction that $|P' \cap E(H \cap T)| = |H \cap T| - 1, j = 1, \ldots, k$, and $a'x' = x(C) - |T'| + 1$. By adding appropriate edges (from $K_n$, not in $E(C)$) we can easily turn $P'$ into a path $P$ with the required properties.

Using Lemmas 4.3, 4.4, 4.6 we shall now prove our main result.

**Theorem 4.7.** Let $C$ be a clique tree in $K_n = (V,E)$. Then the corresponding clique tree inequality (2.17) defines a facet of $Q'_n$.

(Note that for $n < 3$, $K_n$ contains no clique trees. For $n = 4, 5$, the only clique trees consist of single cliques and their corresponding inequalities are equal to or equivalent to (see Theorem 2.10) the trivial inequalities $x_1 \leq 1$ which we have already remarked are facet inducing. So although Theorem 4.7 is true for all values of $n$, it is of most interest for $n \geq 6$ and, in particular, for $n > 11$ when clique trees which are neither single cliques nor combs exist.)

**Proof.** 1. We prove the theorem by induction on the number of handles. If $C$ has no handle then $C$ is a complete graph and (2.17) is a subtour elimination constraint. If
C has one handle then C is a comb. It was shown in Grötschel and Padberg (1979b) that subtour elimination constraints and comb inequalities define facets of \(Q_T\).

2. Now we assume that the theorem is true for all clique trees with \(r \geq 1\) handles and assume that C is a clique tree with \(r + 1\) handles. The corresponding clique tree inequality, denoted by \(a^T x \leq \alpha\) in the sequel, is valid with respect to \(Q_T\) by Theorem 3.5. Moreover, it is easy to find a tour in \(K_{C}\) whose incidence vector does not satisfy \(a^T x \leq \alpha\) with equality. Thus, \(F_{C} := Q_T \cap \{x \in \mathbb{R}^n | a^T x = \alpha\}\) is a proper facet of \(Q_T\).

Suppose now that \(d^T x < \delta\) is a facet defining inequality for \(Q_T\) which satisfies \(F_{C} \subseteq F_{T} := Q_T \cap \{x | d^T x = \delta\}\). We shall show that there exist \(\bar{v} > 0\) and a vector \(X \in \mathbb{R}^n\) with \(X = d^T + \lambda \delta A\), where \(A\) is the node-edge incidence matrix of \(K_{C}\), cf. (2.8). By Lemma 2.9 this will prove our claim.

3. As can be expected our proof will be rather technical. We have to perform various splits, tours and coefficient calculations to obtain the desired result. We now introduce the various clique trees, teeth, handles and node sets we are going to need in the sequel. In part 5 of this proof we introduce an inequality \(b^T x \leq \delta\) which is equivalent to \(d^T x \leq \delta\) by construction. The purpose of the remaining parts \(6, \ldots, 14\) is to show that \(b^T x \leq \delta\) holds for some \(\bar{v} > 0\). The best way to follow the stages of the proof is to keep track of the coefficients of the clique tree \(C\) presently under consideration and recording the changes of \(b\) already calculated.

By assumption, \(C\) has at least two handles. Thus C is a tooth, \(T\), which intersects at least two handles. Choose one of these handles and call it \(H_{1}\). Denote by \(T\) the set of nodes of \(T\) which are in no handle of \(C\). Note that \(T\) is nonempty by Definition 2.16 (4), and let \(w\) be a fixed node in \(T\).

Split (3.2) clique tree \(C\) at tooth \(T\) and handle \(H_{1}\) to obtain clique trees \(C'\) and \(C''\).

Definition. \(K_{C'}\) contains \(T\) and \(H_{1}\). \(K_{C''}\) contains \(T\) and \(H_{1}\).

We now modify \(C'\) and \(C''\) slightly. First denote by \(H\) the set of nodes in \(C\) which are in handles intersecting \(T\) but not in \(H_{1}\). Now replace in \(C'\) the tooth \(T\) by the tooth \(T_{1} = T \setminus H\) and call the new clique tree \(C_{1}\). In \(C''\) the tooth \(T\) is replaced by \(T_{2} = T \setminus H\), and the new clique tree is called \(C_{2}\).

The set of nodes of \(K_{C'}\) is as usual denoted by \(V\). The set of nodes which are not contained in \(C\) is denoted by \(V_{C}\), and the node set of \(C_{1}\) resp. \(C_{2}\) is denoted by \(V_{C_{1}}\) resp. \(V_{C_{2}}\).

Let \(K^2\) be the complete graph on the node set \(V_{C}\) and \(K^2\) be the complete graph on the node set \(V_{C_{1}} \cup V_{C_{2}}\).

4. By Lemma 4.6 there are two nodes, say \(u\) and \(v\), which are in two different teeth (\(= T_{1}\)) of \(C_{1}\) but in no handle of \(C_{2}\) such that there is a hamiltonian path \(P_{1}\) in \(K^2\) with ends \(u, v\) and \(u^*\) and \(v^*\) satisfying \(|P_{1} \cap E(H_{1} \cup T_{1})| = |H_{1} \cup T_{1}| - 1\) and \(a^T x_{P_{1}} = s(C_{1}) - |T_{1}| + 1\).

Similarly, by Lemma 4.6 there are two nodes and \(u^*\) and \(v^*\) which are in two different teeth (\(= T_{2}\)) of \(C_{2}\) but in no handle of \(C_{1}\) such that there is a hamiltonian path \(P_{2}\) in \(V_{C_{2}} \setminus T_{2}\) with ends \(u, v\) and \(u^*\) satisfying \(|P_{2} \cap E(H_{1} \cup T_{2})| = |H_{1} \cup T_{2}| - 1\) for all handles \(H_{1}\) of \(C_{2}\) intersecting \(T_{2}\) and such that \(a^T x_{P_{2}} = s(C_{2}) - |T_{2}| + 1\).

Using (3.2) and (4) we obtain

\[\begin{align*}
0 &= b^T x - b^T x_{P_{1}} + b_{uv} - b_{uw} - b_{uw} = b_{uv} - b_{uw} - b_{uw} = b_{uv}.
\end{align*}\]
If \( i \) is a node in \( T \) we use the node \( v \) instead of \( v^* \) in the construction above to obtain \( b_{ij} = 0 \). If \( i \) is a node in \( \bar{V} \) then a similar (but simpler) construction gives the same result. Therefore we get

\[
b_{ij} = a_{ij} = 0 \quad \text{for all } i \in V_2 \cup \bar{V}.
\]

Exchanging the roles of the clique trees \( C_1 \) and \( C_2 \), the nodes \( u, u' \) and \( u, v' \), and the paths \( P_1 \) and \( P_2 \) we obtain by symmetry

\[
b_{ij} = a_{ij} = 0 \quad \text{for all } i \in \bar{V} \setminus T.
\]

7. We shall now apply the induction hypothesis to \( C_2 \) to calculate the coefficients \( b_i \) for \( i, j \in V_2 \cup \bar{V} \).

Consider the clique tree \( C_2 \) as a clique tree in the complete graph, say \( K^2 \), on the node set \( V_2 \cup \bar{V} \cup \{u\} \), and let \( Q_2 \) be the corresponding travelling salesman polytope, \( n_2 = |V_2| + |\bar{V}| + 1 \). Let \( \Delta x < s(C_2) \) be the clique tree inequality for \( C_2 \) with respect to \( Q_2^p \), i.e. \( \Delta \) is a vector in \( R^M \) with \( m_i = n_2(n_2 - 1)/2 \). Moreover, let \( b \) be the vector obtained from \( b \) by removing all components which do not correspond to an edge in \( E^1 \).

Suppose \( S_2 \) is any hamiltonian cycle in \( K^2 \) saturating \( C_2 \). Since \( u \) is in \( K^2 \), \( S_2 \) contains an edge \( u, i \in V_2 \cup \bar{V} \). Removing this edge from \( S_2 \), adding the edge \( u' \) and the path \( P_1 \), cf. 4, we obtain a tour in \( K_u \) by construction saturates \( C \). From (1) and (6) we have \( b_u = b_{u'} = 0 \) for all \( i \in V_2 \cup \bar{V} \). Thus we obtain that every tour \( S_2 \) in \( K^2 \) saturating \( C_2 \) satisfies

\[
\Delta x \cdot b = b^T x^* = : \beta.
\]

If \( \Delta x < \beta \) were not valid with respect to \( Q_2^p \), then we could pick a tour in \( K^2 \) violating this inequality, extend it to a tour in \( K_u \) as above and obtain a tour in \( K_u \) whose incidence vector violated \( \Delta x < \beta \). This implies that \( F_2 = \{ x \in Q_2^p | \Delta x < \beta \} \) is a face of \( Q_2^p \). (Using the same extension argument it is easy to see that \( F_2 \neq Q_2^p \).)

Now by our induction hypothesis, the clique tree inequality for \( C_1 \) defines a facet \( F_1 = \{ x \in Q_1 | \Delta x < s(C_1) \} \) of \( Q_1^p \) (8) implies that \( F_1 \subset F_2 \). Since \( F_2 \neq Q_2^p \) we necessarily have \( F_2 = F_1 \). By \( \Delta x < \beta \) are equivalent with respect to \( Q_2^p \). Thus Lemma 2.9 implies that there are \( \lambda_0 \in V_2 \cup \bar{V} \) and \( \rho > 0 \) such that

\[
b_i = \rho \lambda_0 + \lambda_i \quad \text{for all } i \in V_2 \cup \bar{V} \cup \{u\}.
\]

From (1) we know that \( b_u = a_{ij} = 0 \) for all \( i \in V_2 \cup \bar{V} \), therefore (9) implies \( \lambda_0 = \lambda_i \), for all \( i \in V_2 \cup \bar{V} \). By (3) we have \( b_u = a_{ij} = 0 \). Since \( e, v \in V_2 \) we get from (9) \( \lambda_0 = \lambda_e = \lambda_v = 0 \). Hence \( \lambda_i = 0 \) and therefore

\[
b_i = \rho \lambda_0 = 0 \quad \text{for all } i \in V_2 \cup \bar{V} \cup \{u\}.
\]

8. The induction hypothesis is now applied to the "other side" of \( C \). Let \( K^1 \) be the complete graph on \( V_1 \cup \{v\} \), where \( v \) is defined in 4, and let \( Q_1^p \) be the corresponding travelling salesman polytope. We consider \( C_1 \) as a clique tree in \( K^1 \) and denote the clique tree inequality for \( C \) with respect to \( Q_1^p \) by \( \Delta x < s(C_1) \). Similarly let \( b \) be the vector obtained from \( b \) by removing all components not corresponding to edges in \( E^1 \).

Let \( S_1 \) be any hamiltonian cycle in \( K^1 \) saturating \( C_1 \). Node \( v \) has two neighbours in \( S_1 \). At most one of these two nodes is \( T \), otherwise it would be easy to construct a tour \( S' \) in \( K^1 \) with \( \Delta x < s(C_1) \), a contradiction. Suppose \( i \) is a neighbour of \( v \) in \( S_1 \), which is not in \( T \). Removing edge of \( S_1 \) adding \( v' \) and the path

\[
P_1 \setminus \{u, u'\}, \quad \text{for all } i \in V_2 \cup \bar{V} \cup \{u\}.
\]

By our induction hypothesis, \( \Delta x < s(C_1) \) defines a facet; thus, using the same argument as in (7), we obtain

\[
b_i = \rho \lambda_0 v_0 + \lambda_i \quad \text{for all } i \in V_1 \cup \{v\}.
\]

By (2) \( b_u = a_{ij} = 0 \) and hence (12) gives \( \mu_i = -\mu_j \) for all \( i \in V \setminus T \). By (1) \( b_u = a_{ij} = 0 \) for all \( i \in T \); thus we get from (12) \( \mu_i = \mu_j \) for all \( i \in T \). Moreover (3) implies \( \mu_i = 0 \). Since \( \Delta x < \beta \) we get \( -\mu_i = -\mu_j \). Thus (12) implies \( \mu_i = -\mu_j \). Hence \( \mu_i = -\mu_j \) which implies \( \mu_i = 0 \) and therefore \( \beta = 0 \) for all \( i \in V_2 \cup \{u\} \). Thus we get from (12)

\[
b_i = \rho \lambda_0 v_0 \quad \text{for all } i \in V_1 \cup \{v\}.
\]

9. Since \( \Delta x < \beta \) we have \( \mu_i = 0 \) in case \( \beta \geq 2 \).

Suppose \( \beta = 2 \). Let \( e \in E \cup \{u, u'\} \cup \{v, v'\} \) and let \( w \) be a hamiltonian path in \( T \) starting at \( w_0 \) going to \( f \) and ending at \( k \) which saturates \( C \) with respect to \( C \). This implies that \( e \) saturates \( T \) also with respect to the clique trees \( C' \) and \( C'' \), see 3.

Apply Lemma 4.4(a) to \( C' \) so as to obtain a path \( C' \) containing \( e \) as a path and having a path ending at \( u \). The paths different from \( e \) can be linked up to give a single path \( C \) in \( \beta \) ending at \( u \) and some other node, say \( v \). Similarly, we apply Lemma 4.4(a) to \( C'' \) and thereby obtain a path \( C'' \) having one end equal to \( v \) and the other to \( e \), say, that \( Q_2 \) contains all nodes not in \( V \cup \bar{V} \) and that \( Q_2 \) saturates \( C'' \). Note that by the construction used in the proof of Lemma 4.4(a) we may assume that \( e \) is in \( H \) and \( e \) does not belong to the handle in \( C'' \) containing \( e \).

We now combine \( Q_2, Q_0 \) and \( e \) to form a tour \( S_1 \) by adding the edges \( sv, sv', \bar{v}e \). By construction, \( S_1 \) saturates \( C \). Note that \( b_u = b_v = b_{u'} = b_{v'} = 0 \). The tour \( S_2 = S_1 \setminus \{u, u', v, v'\} \cup \{w, w, w, \bar{e}, \bar{e}, \bar{v}, \bar{v}, 0\} \) also saturates \( C \) and we have \( b_u = b_v = b_{u'} = b_{v'} = 0 \). Thus we get \( \beta = \beta^* - \beta^* e = \beta^* \). i.e.

\[
p := p_1 = p_2.
\]

Note that if \( S \) is a tour in \( K \) constructed in 7 or 8 by removing an edge from a tour on \( K^1 \) or \( K^2 \) and adding the path \( P_1 \) or \( P_2 \) and an edge \( u' \) or \( v' \), then the incidence vector of \( S \) satisfies \( \Delta x = a \) and \( \beta x = \beta \), and, moreover, (1), (2), . . . , (14) imply that \( b_i = \rho a_i \) for all \( e \in S \). Thus \( b x = \rho a x = \rho a x = \rho a x = \rho \beta \) which gives

\[
\beta = \rho a.
\]
three cliques of $C_i$ each meeting exactly one node of $U$. This contradicts $b_m = 2$ for all $u, v \in U$. Thus, the first case cannot occur.

Case 2. We claim that $V(N \cup U) = \emptyset$. If not, choose any node $p$ in this set.

Since $C_i$ is a clique tree there must be a node $w \in U$ with $a_{wp} = 1$. Since $\lambda_i = 1$ and $\lambda_j > 0$ we must have $b_p = 1$ or $b_w = 2$, and so $\lambda_p = 1$ or $\lambda_w = 2$.

If $\lambda_p = 1$, then for any $v \in U$ we have $b_v = 2$, hence $b_w = 2$ for any $v, w \in U$. But this means that $a_{wp} = 1$ for all $v, w \in U$. Thus $U$ is a clique of $C_i$ which is impossible.

Therefore we have $\lambda_i = 1$. By definition, in $C_i$ node $p$ is adjacent to every node $w \in N$ which implies $a_{wp} > 1$ and thus $b_p = 1$. Moreover, since $\lambda_j = 1$ we have $b_j > 1$ for all $j \in U$. But then $C_j$ contains the star $\delta(p)$ which is impossible. Therefore we conclude that $V(N \cup U) = \emptyset$ and that $N = V = U$ is a maximal clique of $C_i$.

Suppose that $N$ is a tooth of $C_i$. Then $U$ contains two teeth $T_1$ and $T_2$ of $C_i$ which meet the same handle $H$. Let $w \in T_1 \cap H$ and $v \in T_2 \cap H$ and let $w \in T_1 \cap H$, then $b_w = b_v = 2$ and $b_u = 1$ which is impossible.

Suppose that $N$ is a handle. If $C_i$ is a comb then we have alternative (b) of our theorem. Otherwise, in $C_i$ a tooth $T$ joins $N$ to a handle which is completely contained in $U$. By Definition 2.16 there are nodes $u \in T \cap (N \cup U), v \in T \cap H$ and $w \in H \cup T$ and we can conclude that $b_u = b_v = 2$ and $b_w = 1$ which is impossible in a clique tree. This finishes the second case and therefore the proof of Theorem 5.1.1.

6. The relationship of $Q^2$ and $Q^2_i$. We noted in the introduction that the travelling salesman polytope $Q^2$ is a face of the monotone travelling salesman polytope $Q^2_i$, namely the face obtained by requiring that $A$ is the node-edge incidence matrix of $G$. Therefore, for each facet $F$ of $Q^2$ there exist one or more facets $F'$ of $Q^2_i$ such that $F' \subseteq F$. In this section we consider the question of when a facet inducing inequality $a' < a_i$ of $Q^2_i$ is also a facet inducing for $Q^2$. We prove a sufficient condition for this to be the case and show that this condition is indeed the case for clique tree inequalities. We also describe how any facet inducing inequality can be transformed into an equivalent facet inducing inequality which will also be facet inducing for $Q^2_i$.

Let $a < a_i$ be a facet inducing for $Q^2_i$, $a > 0$ then the inequality is valid for $Q^2_i$. If not, by Lemma 2.9 we can let $b' := a + \lambda A$ and $a' = a + \lambda A^2$ for sufficiently large $\lambda$. This inequality is obtained by $b < b'$ and $a < a_i + \lambda A^2$.

Let $A$ be the family of all subsets of the edges for which the corresponding subsets of the columns of $A$ are linearly independent. Then $(A, E)$ is a matroid defined on $E$, which we denote by $\mathcal{M}(G)$. This matroid is sometimes called the real matroid of $G$. It is well known, and easily verified, that $J \subseteq E$ is in $\mathcal{M}(G)$ (i.e. the corresponding columns of $A$ are linearly independent) if and only if each component of the graph $(V, E)$ contains no even cycle and at most one odd cycle. The circuits or minimal dependent sets of $\mathcal{M}(G)$ are the sets $C \subseteq E$ such that $C$ is an even cycle of $G$ or else $C$ is the edge set of two edge disjoint odd cycles of $G$.

Figure 6.1.

It is easy to see that $J \subseteq E$ is a basis of $\mathcal{M}(G)$ for $n \geq 3$ if and only if every component of $(V, E)$ has exactly one odd cycle and no even cycle. For example, if $T$ is the edge set of a spanning tree of $G$ and $j \in E \setminus V(T)$ creates an odd cycle when added to $T$, then $T \cup (j)$ is a basis.

For $J \subseteq E$, let $A_j$ denote the corresponding column of $A$. For $S \subseteq E$, let $A_S$ denote the $(n \times |S|)$-submatrix of $A$ consisting of those columns corresponding to members of $S$. The set $S$ contains a basis of $\mathcal{M}(G)$ if and only if the rows of $A_S$ are linearly independent, i.e., for any $\lambda \in \mathbb{R}^n$ satisfying $\lambda A_S = 0$ we must have $\lambda = 0$. Since the rows of $A$ are linearly independent, $A \in \mathbb{R}^n$ satisfies $\lambda A = 0$ if and only if $A = 0$.

We have the following:

**Lemma 6.1.** The set $S \subseteq E$ contains no basis of $\mathcal{M}(G)$ if and only if there exists $\lambda \in \mathbb{R}^n$ such that $\lambda A = 0$ but $\lambda A_{i,j} = 0$ for all $i \in S$.

(In fact this is a specialization of the theory of chain groups developed by Tutte (1965) as an alternative representation for matroids defined by matrices. The original form, considering linear independence of columns as we did here, appeared in Whitney (1935), the original paper on matroids.)

Let $a > 0$ such that $a' < a_i$ induces a facet of $Q^2_i$. Then $E \setminus E(a)$ is used in the support of $A_k$. If $E \setminus E(a)$ contains no basis of $\mathcal{M}(G)$ then by Lemma 6.1 there exists $\lambda$ such that $\lambda A = 0$ for all $i \in E \setminus E(a)$ but $\lambda A' = 0$ is not identically zero. By subtracting an appropriate multiple of $\lambda A'$ from $a$ and $\lambda A^2$ from $a_i$ we obtain (using Lemma 2.9) an inequality $a' < a_i$ that induces the same facet of $Q^2_i$ that satisfies $\lambda A' = 0$ and $\lambda A^2 = 0$. We call this procedure reducing the inequality $a' < a_i$ to $a' < a_i$. Clearly we need only reduce an inequality at most $E$ times before obtaining an equivalent inequality $a' < a_i$ with $E > 0$ and such that $E(a)$ contains a basis of $\mathcal{M}(G)$. Finally we scale the inequality so that the smallest nonzero coefficient of $a$ has value one. We call such an inequality $a' < a_i$ induced by an inequality $a' > 0$ for some $k \in E$ and nontrivial otherwise. First we show that it is easy to recognize when a support reduced inequality induces a trivial facet.

**Lemma 6.2.** Let $a' < a_i$ be a support reduced facet inducing inequality for $Q^2_i$. Then $a' < a_i$ induces a trivial facet of $Q^2_i$ if and only if

(a) $E \setminus E(a) = E_0(k)$ for some $k \in V$ and $E \setminus E_0(k)$, and in this case $a_k = 1$ for all $j \in E \setminus E_0(k)$ and $a_j = 2$ for all $j \in E \setminus E_0(k)$,

(b) $E \setminus E(a) \subseteq E $ for some $v \in V$ and $k \in E \setminus E_0(k)$, and in this case $a_k = 1$ for all $j \in E \setminus E_0(k)$ and $a_j = 2$ for all $j \in E \setminus E_0(k)$.

**Proof.** By Lemma 2.9 $a' < a_i$ induces a trivial facet if and only if for some $k \in E$ there exist $\lambda, \lambda_k > 0$ such that $a_k = \lambda k$, for $j \in E \setminus E_0(k)$ and $a_j = \lambda_j = \lambda - \lambda_k$. The sufficiency of (a) follows by defining $\lambda_1 := 1$, $\lambda_j := 0$ for $j \in E \setminus E_0(k)$, and $\lambda_k$.

The sufficiency of (b) follows by defining $\lambda_1 := 1$, $\lambda_j := 0$ for $j \in E \setminus E_0(k)$, and $\lambda_k$. We now prove the necessity. Suppose $a' < a_i$ is equivalent to $-x_0 < 0$. Set $E' := E \setminus E_0(k)$. If there exist distinct $v, w \in V$ such that $a_v, a_w < 0$ then $a_v, a_w < 0$, a contradiction. So there is at most one $v \in V$ with $a_v < 0$.

Suppose that $0$ satisfies $a_0 < 0$. Then if there exist distinct $v, w \in V$ with $a_v = 0$ and $a_w > 0$, we would have $E_0(k) \subseteq E_0(k')$ contained in $E \setminus E_0(k)$. Therefore at least one of $(v, w)$ is contained in a component of $(V, E)$ containing no odd cycle, so

![Figure 6.1. Two circuits of $\mathcal{M}(G)$.](image-url)
First suppose \( j \in V_j \cap T \) and let \( H^* \) be the handle intersecting \( T \) for which removal of \( H^* \cap T \) disconnects \( j \) from \( T \). Let \( \nu \) be a path saturating \( T \) ending in \( H^* \) and \( H^* \). Using Lemma 4.4 extend \( \nu \) to a path cover \( \nu^* \) of \( C \) containing a path \( \nu_i \) ending at \( i \) and extend \( \nu \) to a path cover \( \nu^* \) of \( C \) containing a path \( \nu_j \) ending at \( j \). It is obvious that \( \nu^* \cup \nu^* \) can be extended to a hamiltonian path in \( K \), with endpoints \( i \) and \( j \) such that the edges added to \( \nu^* \cup \nu^* \) do not belong to those given in (16). Then adding \( \nu \cup \epsilon \) we obtain a tour \( S \) saturating \( C \) such that all values \( b_x, x \in S \), are known to equal \( \rho \), except for \( \epsilon \in \nu \). The case \( j \in T \) is treated similarly. By the argument of 10, we obtain

\[
b_y = \rho a_y \quad \text{for all } i \in V_i \setminus T, \quad j \in (\bar{V} \cup V_i \setminus T). \tag{17}
\]

The other cases one has to consider are

12. \( j \in V_i \setminus T, j \in V_j \setminus T \)

13. \( i \in V_i \cap T, j \in (\bar{V} \cup V_i \setminus T) \)

14. \( j \in V \setminus T, j \in V_i \setminus T \).

Cases 12 and 13 are symmetric. The construction of the desired tour is essentially the same as in 11. In case 14 one starts with a path \( i \) from \( i \) to \( j \) saturating \( T \) and extends it to a tour \( S \) in \( K \), saturating \( C \). Then the edges in \( S \) from \( i \) and \( j \) to nodes not in \( T \), say \( y \) and \( z \), are removed as well as an edge from \( x \) to some other node in \( T \), say \( y \). Then the edges \( y \) and \( z \) are added to obtain a tour \( S \) saturating \( C \). In order to verify that \( S \) saturates \( C \), we want to use \( a_y = a_z = a_y \) and \( \lambda = 0 \). It follows from the fact that \( \lambda = 0 \) and (13) that \( a_
u = 0 \). The construction of \( S \) indicated above ensures this for the edges \( y \) and \( z \) as well. Then \( \lambda \) can be shown to equal \( \rho \), using the argument of 10.

Altogether we have shown that \( a = \rho b + \rho \Lambda \Lambda \) and \( \lambda = \rho \Lambda \lambda \). This completes the proof of our theorem.

\section{5. Equivalence of clique trees}

We say that clique trees \( C \) and \( C \) are equivalent if their respective inequalities \( a < \alpha \), \( b < \beta \) and \( \lambda < \lambda \) induce the same facet of \( \mathcal{Q}_2 \). In this section we show that there are only two cases of equivalent clique trees. Otherwise, every distinct clique tree inequality induces a distinct facet of \( \mathcal{Q}_2 \). In fact, the facets induced by all clique tree inequalities which are neither comb inequalities nor submodular elimination inequalities are all distinct.

The importance of this result is that it shows that from the point of view of cutting planes, the set of clique tree inequalities is essentially minimal. That is, there is no small subset of the clique trees that suffices to provide all the cuts provided by the whole set.

Theorem 2.12(b) asserted that distinct subgraph elimination constraints are equivalent if and only if they are of the form \( x(E(W)) < W - 1 \) and \( x(E(V \setminus W)) \in \{P \mid W \setminus 1 \} \). Theorem 2.15(c) asserted that two distinct facet-inducing comb inequalities are equivalent if and only if the two comb constraints have the same set of teeth and complementary handles. Moreover, no comb inequality is equivalent to a submodular elimination inequality. The results of this section not only provide proofs of these results, but also show that these are the only cases of equivalence of clique trees.

Let \( a < \alpha \), \( b < \beta \) and \( \lambda < \lambda \) be two facet-inducing inequalities for \( \mathcal{Q}_2 \). By Lemma 2.9 these inequalities are equivalent if and only if there exist \( \rho > 0 \) and \( \lambda \in \mathbb{R}^+ \) such that \( b = \rho a + \lambda \Lambda \lambda \), in other words, for each edge \( i \in E \) we have \( b_i = \rho a_i + \lambda \Lambda \lambda_i \).

\textbf{Theorem 5.1.} Let \( C \) and \( C \) be distinct clique trees. Then \( C \) and \( C \) are equivalent if and only if

\[ (a) \; C \text{ is a single clique } W \text{ and } C \text{ is a single clique } V \setminus W \]

(b) \( C \) and \( C \) are with identical sets of teeth and if \( W \) is the handle of \( C \), then \( V \setminus W \) is the handle of \( C \).

\textbf{Proof.} Taking \( \rho = 1, \lambda = -\frac{1}{\lambda} \) for all \( i \in W \), and \( \lambda = \frac{1}{\lambda} \) for all \( i \in V \setminus W \) shows the equivalence of \( C \) and \( C \) if (a) or (b) holds. We now prove the necessity of our conditions.

Let \( C \) and \( C \) be two different equivalent clique trees, i.e., \( C \) and \( C \) have the same set of teeth, \( a \) and \( b \). \( a \) and \( b \) are the associated inequalities. If both \( C \) and \( C \) are single cliques, then unless \( C \) and \( C \) have the same \( \lambda \), there exists \( \lambda \in \mathbb{R}^+ \) satisfying \( \lambda \neq \alpha \beta \). \( b \) and \( \beta \) have these inequalities induce distinct facets. Therefore we assume that if \( C \), then \( C \) is not a single clique, i.e., \( C \) contains at least four cliques satisfying Definition 2.16.

Let \( \lambda \in \mathbb{R}^+ \), \( \beta \) be such that \( b \), \( a \), \( a \), \( b \), \( a \), \( b \) are the associated inequalities. If \( \lambda \) is the smallest \( \lambda \) such that \( \lambda < \alpha \beta \) is possible. \( \lambda \) is nonempty, and in fact \( |\lambda| > 2 \). For, if there were a unique \( \lambda \), \( \lambda \leq 0 \), then there would be a \( \lambda \in \lambda \) if \( \lambda = 0 \) and hence \( \beta \) is not possible. \( \lambda \) is nonempty, and in fact \( |\lambda| > 2 \). For, if there were a unique \( \lambda \), \( \lambda < 0 \) then there would be a \( \lambda \in \lambda \) if \( \lambda = 0 \) and hence \( \beta \) is not possible. \( \lambda \) is nonempty, and in fact \( |\lambda| > 2 \).
a^* < a_0$ could not be support reduced, a contradiction. Therefore exactly one $o \in Y$ satisfies $\lambda = 0$. But again, we cannot have $\delta(o) \subseteq \mathbb{R}^2$, so $s \in \delta(o)$, $a_0 = 0$ and $a^* < a_0$ in the form (a).

Suppose that exactly one $o \in Y$ satisfies $\lambda = 0$. Then every $w \in V \setminus Y$ must satisfy $\lambda = 1$. Therefore $k$ is the only possible edge of $\delta(Y \setminus \{o\})$ that can have $a_0 > 0$, so for $a^* < a_0$ to be support reduced we must have $\lambda = 0$ for all $w \in V \setminus Y$, $k \in \delta(Y \setminus \{o\})$ and $a_0 = 0$ which implies that $\lambda = 1$ for all $w \in V \setminus Y$ and $a = 0$ and we have the form (b).

Now we prove the main lemma.

**Lemma 6.3.** Let $a^* < a_0$ be a support reduced facet inducing inequality for $QF$ not of the form (a) or (b) of (6.2). Then $a^* < a_0$ is facet inducing for $QF$.

**Proof.** Since $a > 0$, the inequality is valid for $QF$. Let $X$ be the set of all incidence vectors $x$ of tours of $K_n$ that satisfy $a^* < a_0$. Since $a^* < a_0$ is facet inducing for $QF$, there exists an affinely independent set $\mathbb{X} \subseteq X$ satisfying $|\mathbb{X}| - \dim \delta(Y) = n$. Let $\mathbb{X}$ be a basis of $\mathbb{A}(K_n)$ contained in $E^2(Y)$.

If there exists $k \in E$ such that $\lambda = 0$ for all $s \in X$ then $a^* < a_0$ induces the trivial facet $\mathbb{X}_0 > 0$, a contradiction. Hence for each $j \in B$ there exists $x' \in X$ such that $a_j = 1$. Let $x''$ be obtained from $x'$ for all $s \in B$ by setting $a_j = 0$. Then each $X''$ is the incidence vector of a hamiltonian path of $K_n$, and $a^* < a_0$ for all $s \in B$. Let $X' = \mathbb{X} \setminus \{s\}$ and $X'' = \mathbb{X} \setminus \{s\}$ for all $j \in B$. Let $X' \subseteq X''$ be affinely independent and then since $\mathbb{X} \subseteq X''$ it will follow that $a^* < a_0$ is facet inducing.

First we observe that for all $x \in X$, $x(s) = 2$ for all $o \in V$ and that for all $x \in X$, $x(x(s)) = 1$ for all $s \in V$.

Now suppose that $\mathbb{X} \subseteq X''$ and $\mathbb{X} \subseteq X'$ is affinely independent. Then there exists $\mu \neq 0$ satisfying $\sum_{x \in \mathbb{X}} x(s) - \sum_{j \in B} x(j) = 0$ and $\sum_{x \in \mathbb{X}} x(s) - \sum_{j \in B} x(j) = 0$.

This implies, in particular,

$$\sum_{o \in V} x(s) - \sum_{j \in B} x(j) = 0 \quad (6.4)$$

Since $\mathbb{X} \subseteq X''$ is affinely independent there exists $k \in B$ such that $\mu_k = 0$. Choose such a $k$ which is at maximum distance in $[V, B]$ from the odd cycle of the component of $[V, B]$ that contains $k$. If $k$ is not an edge of the cycle, then let $x$ be the node of $k$ furthest from the cycle. Then $x(s) = 1$ but $x(s_k) = 0$ for all $x \in X \setminus \mathbb{X}$, having nonzero $\mu$. This contradicts (6.4) and (6.5). So $k$ must be an edge of an odd cycle $C$ of $B$, and $\mu_k = 0$ for every edge $j$ of $B$ not in $C$, but incident with a node of $C$. Hence if $C = v_1, \ldots, v_{2k - 1}$, we have by (6.2) and (6.5) that $\mu_{v_1} = \mu_{v_2} = \cdots = \mu_{v_{2k - 1}} = -\mu_{v_{2k}} = -\mu_{v_{2k - 2}} = \cdots = -\mu_{v_2} = -\mu_{v_1}$. But this means $\mu_k = 0$ for every edge $j$ of $C$, a contradiction to $\mu_k \neq 0$. Therefore no such $x$ exists, $\mathbb{X} \subseteq X''$ is affinely independent and the result follows.

Using Theorem 4.7 and Lemma 6.2 and 6.3 it is now easy to show that clique tree inequalities are facet inducing for $QF$.

**Theorem 6.6.** Let $C$ be a clique tree in $K_n = [V, E]$. Then the corresponding clique tree inequality (2.17) induces a nontrivial facet of $QF$.

**Proof.** By Theorem 4.7 every clique tree inequality $a^* < a_0$ is facet inducing for $QF$. If $C$ consists of a single tooth $T$, i.e., $\mathbb{X} \subseteq a_0$ is a subtour elimination constraint, then by Definition 2.16(4) there exist distinct $u, v \in V \setminus T$ and $\delta(u) \cap \delta(v) \subseteq \mathbb{X}(u)$.
LEMMA 7.1. Let $G = (V', E')$ be a clique structure consisting of $s + 2$ cliques joined by a common cutset $u$. Then $r(G) = |V'| - s + 1$.

PROOF. It is easily seen that if $G_1, G_2, \ldots, G_s$ are the pieces joined by $u$ then $r(G) = \sum r(G_i) - (s - 2)$ since a saturating path cover of $G$ will only be able to saturate two of the $G_i$. Since $r(G) = |V'| - 1$ the result follows.

LEMMA 7.2. Let $G = (V', E')$ be a facet inducing clique structure for $Q$. Then for any $v \neq E'$ there exists an integral vector $\mathbf{x} \in Q$ satisfying $x(E') = r(G)$ and $x((v) \cap E') < 1$.

PROOF. Suppose that for some $v \neq E'$ every saturating path cover of $G$ contains two edges incident with $v$. If $8(0) \cap E' = \emptyset$ then $x(8(0) \cap E') < 2$ does not induce a proper face of $Q$. Since $E' \cap 8(0) = \emptyset$, it is easy to see that there exists $x \in Q$ satisfying $x(8(0) \cap E') = 2$ but $x(E') < r(G)$. This contradicts G being facet inducing. Suppose $8(0) \cap E' = \emptyset$. If $G$ consisted of a single clique then $G = K_3$, and every $x \in Q$ satisfies $x(E') = r(G)$ so $G$ does not induce a proper face, a contradiction. Otherwise $v$ is a cutnode joining $2$ or $3$ cliques $G_1, G_2, \ldots, G_s$. If $|V(G_i)| \geq 3$ for some $i$ then every $x$ satisfying $x(E') = r(G)$ also satisfies $x((v) \cap E') = r(G) \cap (v)$ but the converse is not true. Since this latter inequality induces a proper face of $Q$, this contradicts G being facet inducing. If $|V(G_i)| = 2$ for all $i \neq 1,2,3$, then $E' = \emptyset$ so we contradict G being a clique structure. Therefore there is a saturating path cover containing at most one edge incident with $v$. The incidence vector of any tour containing this path cover provides the required $x$.

Let $G = (V', E')$ be a graph containing a cutnode $v$. Let $C_1, C_2, \ldots, C_s$ be the nodest of the components of $G \setminus v$. Then we call the graphs $G(C_i \cup \{v\})$ for $i = 1,2,\ldots, s$ the pieces of $G$ joined by $v$.

LEMMA 7.3. Let $G = (V', E')$ be a facet inducing clique structure for $Q$ and let $v$ be any cutnode of $G$. If $G_1, G_2, \ldots, G_s$ are the pieces of $G$ joined by $v$, then $r(G) = r(G_1) + r(G_2) + \cdots + r(G_s) - 1$.

PROOF. Let $G = \bigcup G_i$. Then clearly $r(G) \leq r(G_1) + r(G_2) + \cdots + r(G_s)$. If $r(G) = r(G_1) + r(G_2) + \cdots + r(G_s)$ then the inequality $x(E') \leq r(G)$ is the sum of the inequalities $x(E(8(G_i)) < r(G_i)$ and $x((v) \cap E') \leq r(G)$, contradicting G being facet inducing so $r(G) < r(G_1) + r(G_2) + \cdots + r(G_s) - 1$.

LEMMA 7.4. Let $P_1$ be a saturating path cover of $G$, and let $F$ be a saturating path cover of $G$. Then at most two edges of $F$, are incident with $v$. Deleting these edges and then taking the union with $F'$ we obtain a feasible path cover of $G$. Hence $r(G) \geq r(F) + r(F) - 2$.

PROOF. Suppose $r(G) = r(F) + r(F) - 2$. Then we will show that this implies that every $x \in Q$ satisfying $x(E') = r(G)$ also satisfies $x(\delta(v) \cap E') = 2$. For suppose there exists $x$ satisfying $x(E') = r(G)$ and $x(\delta(v) \cap E') < 1$. Then $x(\delta(v) \cap E') < |E'|$, which is the edge set of either $G$ or $G_i$. Therefore $x(E') < |E'|$ and let $x^*$ be the incidence vector of a saturating path cover for $E^*$. At most two edges of $F$ are incident with $x^*$, so by setting one of these $x^*$ to 0, we obtain $x$ satisfying $x(E^*) = r(E^*)$. But if we then replace $x^*$ with the edges of $F$, we get a vector $x \in Q$ satisfying $x(E') > x(E^*) + 1 + r(G) < 1$, a contradiction. Therefore $x\cap E' = E'$ and $x^* \cap E'$ = $E^*$. If we change $x$ to agree with $x^*$ on $E'$ we get a vector $x \in Q$ satisfying $x(E') > r(G) + 1$, again a contradiction. Therefore $x \in Q$ satisfies $x(E') = r(G)$ and $x((v) \cap E') < 1$, and we contradict Lemma 7.2. Therefore (7.5) cannot hold, so $r(G) > r(G) + r(G) - 1$ which combined with (7.4) gives the result.

LEMMA 7.5. Let $G = (V', E')$ be a facet inducing clique structure for $Q$. Then for any $v \neq E'$ there exists an integral vector $x \in Q$ satisfying $x(E') = r(G)$ and $x((v) \cap E') < 1$.

PROOF. Suppose that every integer $x \in Q$ which satisfies $x(E') = r(G)$ also satisfies $x((v) \cap E') = 0$. Then the inequality $x(E') < r(G)$ must induce the same facet as the trivial inequality $x < 1$. We will show that the inequality $x(E') < r(G)$ is support reduced (see (6.2)) and the result then follows from Lemma 6.2. Showing that this inequality is support reduced consists simply of showing that $E' \cap E'$ contains a basis of $\mathcal{L}(K_s)$. If $G$ contains a single clique, then either $|V' \setminus E'| > 2$, in which case $E' \cap E'$ contains a basis of $\mathcal{L}(K_s)$, or $|V' \setminus E'| < 2$ and every $x \in Q$ satisfies $x(E') = r(G)$, contradicting to the trivial inequality inducing a proper face. If $G$ contains at least two cutnodes and hence two node disjoint cliques, then it is easy to find a basis of $\mathcal{L}(K_s)$ in $E'$ 

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[Text continues]
\(\delta(E(G)) < r(G)\), then \(\delta(E(G))\) can be increased, without decreasing \(\delta(E(G))\) for \(i = 1, 2, \ldots, s - 1\), contradicting \(\delta(E') = r(E')\). Thus we have
\[
\delta(E(G)) = r(G) \quad \text{and} \quad \delta(E(G) \cap E(G)) = 1. \tag{7.9}
\]

By (7.8), for each \(i \in \{1, 2, \ldots, s - 1\}\), we have \(\delta(E(G)) < r(G) - 1\). In fact, we must have equality, for if \(\delta(E(G)) = r(G) - 1\), then we could increase \(\delta(E(G))\) and again violate \(\delta(E') = r(E')\). So we have
\[
\delta(E(G)) = r(G) - 1 \quad \text{for} \quad i \in \{1, 2, \ldots, s - 1\}. \tag{7.10}
\]

Combining (7.9) and (7.10) we have
\[
\delta(E(G)) = \delta(E(G) \cap E(G)) = 1. \tag{7.11}
\]

If \(\delta(E(G)) = r(G)\) and \(\delta(E(\mathcal{C}) \cap E(G)) = 1\), then we can replace \(\delta(E(G))\) for the edges of \(E(G)\), and make some changes to \(X_G\) for \(t \in E(E')\); and contradict \(\delta(E') = r(E')\). Therefore
\[
\delta(E(G)) = \delta(E(G) \cap E(G)) = 1. \tag{7.11}
\]

If \(\delta(E(G)) = r(G)\) and \(\delta(E(\mathcal{C}) \cap E(G)) = 1\), then we can replace \(\delta(E(G))\) for the edges of \(E(G)\), and make some changes to \(X_G\) for \(t \in E(E')\); and contradict \(\delta(E') = r(E')\). Therefore
\[
\delta(E(G)) = \delta(E(G) \cap E(G)) = 1. \tag{7.11}
\]

For any integer \(i \in \{1, 2, \ldots, s - 1\}\), and any integer \(x \in Q^s\) of \(x(E(G)) = r(G)\), we must have \(\delta(\beta(i) - i) \geq E(G)) = r(G)\), and find \(\delta(E(G)) = r(G)\).

Case 1. \(i = 0\). For \(i \in \{1, 2, \ldots, s - 1\}\), we have \(\delta(E(G)) = r(G)\).

Case 2. \(i \in \{1, 2, \ldots, s - 1\}\). By (7.12) and (7.13), we have \(\delta(E(G)) = r(G)\).

This final contradiction shows that we must have \(x = 2\), and so the proof is complete.

In view of this result, we will always have \(x = 2\) in Lemma 7.3. Therefore it specializes to the following

**Corollary 7.14.** Let \(G = [V, E']\) be a facet inducing clique structure for \(Q^s\), let \(v\) be a cutnode of \(G\) and let \(x \in Q^s\) be integral and satisfy \(x(E') = r(E')\). Then \(v\) joins exactly two pieces \(G_1\) and \(G_2\), and either \(x(E(G_1)) = r(G_1)\) and \(x(E(G_2)) = r(G_2) - 1\) or \(x(E(G_1)) = r(G_1) - 1\) and \(x(E(G_2)) = r(G_2)\).

We now prove two last technical lemmas before proving the main result of this section.

**Lemma 7.15.** Let \(v\) be a cutnode contained in a handle \(H\) of a simple clique tree \(C = [V, E']\). Let integral \(x \in Q^s\) satisfy \(x(E') = r(C) = s(C)\) and \(x(E(H)) = 1\). Then there exists integral \(x' \in Q^s\) satisfying \(x(E') = r(C)\), \(x(E(H)) = 0\) and \(x' = x_f\) for every edge \(f\) belonging to the piece of \(C\) joined by \(v\) that does not contain \(H\).

**Proof.** Let \(x_1, x_2, \ldots, x_n\) be the statement of the lemma, let \(C_1\) and \(C_2\) be the pieces joined by \(v\), where \(H\) is contained in the node set of \(C_1\). Then \(C_2\) is a clique tree and by Lemma 4.4 it is possible to obtain a saturating path cover of \(C_2\) which has one edge incident with \(v\). We can define \(x_1\) to be equal to the incidence vector of this path cover for \(C_2\), equal to \(x\) for \(E(C_1)\) and suitably defined for \(E'(E')\) and we have \(x \in Q^s\) satisfying \(x(E') = r(C)\) and \(x(E(C_1)) = r(C_1)\). Therefore, by Corollary 7.14, we must have \(x(E(C_1)) = r(C_1) - 1\).

Now \(C_1\) is not a clique tree, but if we add a new tooth \(T\) containing two nodes not in \(C_1\) and the cutnode \(v\), then we have a clique tree \(C^*\). Let \(v\) be a saturating path of \(T^*\) in which \(v\) has degree two. By Lemma 4.3 we can extend \(v\) to a path cover of \(C^*\). Let \(x'\) be the incidence vector of a path of \(K_v\) that contains \(v\). Then \((x'(E^*) - r(C^*))\)

**Lemma 7.16.** Let \(T\) be a tooth of a clique tree \(C = [V, E']\) and let \(v\) be the set of the nodes of \(T\) which belong to no other clique. If \(|T| > 1\) then every \(x \in Q^s\) satisfying \(x(E') = r(C)\) must also satisfy \(x(E(C)) \cap E(C) \geq 1\) for every \(v \in \mathcal{C} \setminus T\), if \(|T| = 1\), then there exist \(x' \in Q^s\) satisfying \(x(E') = r(C)\) and \(x(E(C)) \cap E(C) \geq 0\), where \(|T| = 0\).

**Proof.** Suppose that \(x = x_1 = x_2\) satisfies \(x(E') = r(C)\) and \(x(E(C)) \cap E(C) \geq 0\). If \(x(E(C)) \cap E(C) < 2\), then we could set \(x_1 = 1\) and contradict \(x(E(C)) = r(C)\). Therefore \(x(E(C)) \cap E(C) \geq 2\) and so if \(P\) is the path cover of \(C\) induced by \(x\) then \(P\) contains a path \(w\) which contains \(v\) as an internal node.

If an end, say, of \(v\) were not in \(T\), then \(v\) would again contradict \(x(E') = r(C)\). Therefore either end is in \(T\), so there exists a handle \(H\) such that a cutnode \(w\) of \(C\) joins \(H\) and \(T\) and \(x(E(C)) \cap E(C) \geq 1\). By Lemma 7.15 we can modify \(x\) for edges \(j \in E(C')\), where \(C'\) is the piece of \(C\) containing \(v\) that contains \(H\), and \(v \in E(E')\) so that the resulting \(x'\) satisfies \(x(E(C)) \cap E(C) \geq 0\), and \(x' = x_f\) for all \(j \in E(E(C))\). Then by setting \(x_E = 1\) we contradict \(x(E') = r(C)\).

Now suppose \(|T| = 1\), and let \(v = T\). Let \(H_1, H_2, \ldots, H_k\) be the handles intersecting \(T\). For each \(l = 1, 2, \ldots, k\), let \(v_l\) be the cutnode of \(C\) joining \(H_l\) and let \(H_1\) be the piece of \(C\) containing \(v\) and \(v_1\). By Lemma 3.6(b), \(r(C) = \Sigma_{i=1}^{k} r(C_i)\). Since \(C\) is face inducing, by Lemma 7.3, \(r(C_1) = r(C)\). Therefore \(x(E(C)) \cap E(C) \geq 1\).
Now we are ready to show that any facet-inducing clique structure $G$ is a clique tree.
Since each cutnode of $G$ must join exactly two blocks, what we must show is the following:
(i) The blocks of $G$ can be partitioned into sets $\mathcal{T}$ of teeth and $\mathcal{R}$ of handles such that the members of each $\mathcal{T}$ and $\mathcal{R}$ are pairwise disjoint.
(ii) Every handle meets an odd number of at least three teeth.
(iii) Every tooth contains a node which belongs to no handle.

We describe a pruning process that will either establish (i), (ii) and (iii) or else will show that $G$ is not facet inducing. A clique of $G$ is called pendant if it has no nonempty intersection with at most one other clique.

7.17. Pruning process.

Step 0. Initially $\mathcal{T} := \mathcal{R} := \emptyset$. Let $G := G$. Then $G$ is the subgraph of $G$ consisting of all those cliques not yet allocated to $\mathcal{T}$ or $\mathcal{R}$.

Step 1 [Tooth allocation]. If 

    (7.18) $G$ consists of exactly two cliques, or

    (7.19) $G$ consists of exactly two cliques, or

    (7.20) for some pendant clique $K$ of $G$, every node of $K$ belongs to some other clique then $G$ is not facet inducing and we stop. Otherwise, remove every pendant clique from $G$ and add it to $\mathcal{T}$. If $G$ is now empty, then $G$ is a clique tree and we stop. If not, go to Step 2.

Step 2 [Handle allocation]. If 

    (7.21) a pendant clique of $G$ intersects a clique in $\mathcal{T}$, or

    (7.22) $G$ consists of exactly two cliques, or

    (7.23) $G$ consists of a single clique which meets an even number of members of $\mathcal{R}$, or

    (7.24) $G$ consists of more than one clique, but a pendant clique $K$ meets an odd number of members of $\mathcal{R}$, then $G$ is not facet inducing and we stop. Otherwise remove every pendant clique from $G$ and add it to $\mathcal{R}$. If $G$ is now empty then $G$ is a clique tree and we stop. Otherwise go to Step 1.

It can be readily verified that if (7.17) terminates with the conclusion that $G$ is a clique tree, then this is indeed the case, and moreover, that $\mathcal{R}$ is the set of teeth and $\mathcal{R}$ is the set of handles. In fact we will always have the following properties satisfied following each completion of Step 1 or 2.

(7.25) For any clique $T$ added to $\mathcal{T}$ in the previous execution of Step 1, the clique $T$ together with the portion of $G$ disconnected from $G$ by deleting the edges of $E(T)$ is a valid clique tree, having $T$ as a tooth.

(7.26) For any clique $H$ added to $\mathcal{R}$ in the previous execution of Step 2, if the resulting $G$ is not empty, then the clique $H$, plus the intersecting clique of $G$ plus the portion of $G$ disconnected from $G$ by deleting the edges of $E(H)$, is a valid clique tree having $H$ as a handle.

Now we show that if one of (7.18)-(7.24) holds, then $G$ cannot be facet inducing.

**Proposition 7.27.** If (7.18) or (7.21) holds then $G$ cannot be facet inducing.

**Proof.** In either case, a pendant clique $C$ of $G$ intersects a clique $T$ in $\mathcal{T}$ and a clique $H$ in $\mathcal{R}$. It intersects one of these because of (7.18) or (7.21) and the other because this must be the first time that $C$ was pendant, so at the previous step it intersected one of the other type. Let $u$ be the cutnode joining $T$ and $C$, let $v$ be the cutnode joining $C$ and $H$ and let $w$ be a node of $H$ different from $v$. (See Figure 7.2.) Let $\beta$ be the edge of $K$, joining $u$ and $v$. Then $\beta \notin \mathcal{E}$, but by Lemma 7.6 if $G$ is facet inducing then there is $x \in Q_2$ satisfying $x_1 = 1$ and $x(E') = r(G)$. The edges $G$ having $x_1 = 1$ are the edges of a path cover of $G$, and both $u$ and $w$ are path ends of different paths, since otherwise $x_1 = 1$ would be the incidence vector of a tour. We now obtain a contradiction by constructing another $x' \in Q_2$ for which $x(E') = x(E') + 1$. First, if $x_1 = 1$ then we can obtain such an $x'$ by redefining one of the $x_{uv}$ or $x_{uw}$ to be equal to one, and suitably modifying $x_k$ for $k \in E \setminus E'$. Therefore we suppose that $x_1 = 1$. If there are edges $k_1 \in E \setminus E'$ for which $x_{k_1} = 1$, then we can obtain the desired $x'$ from $x$ by setting $x_{uv} := x_{uv} - 1$, $x_{uw} := x_{uw} - 1$ and $x_{k_1} := 1$, where $\beta$ is the edge of $E(C)$ joining the ends of $C_1, C_2$ distinct from $u$ (see Figure 7.3), and then suitably defining $x_k$ for $k \in E \setminus E'$. If $x(E(C)) = 2$ then we can construct the desired $x'$ in a completely analogous fashion, so we suppose there are edges $k_1 \in E \setminus E'$ and $k_2 \in E \setminus E'$ with $x_{k_1} = x_{k_2} = 1$. Let $P$ be the path cover of $C$ induced by $x$ and let $\beta$ be the path containing the edges $k_1, k_2$ of $P$. If $u$ is not an end of $\beta$, and hence does not appear in $\beta$ (since $u$ meets at most one edge of a path in $P$) then we construct $x'$ by using Lemma 7.15 to modify $x$ so that $x(E') = E'(u) = 0$, setting $x_{uv} := 1$, then setting $x_{uv} := 1$. Therefore we can assume that $u$ is one end of $\beta$. But this means that $x(E(C)) = 2$. Since $T \in \mathcal{T}$, there is some $\alpha \in V(T)$ which belongs to no other clique of $G$, so $C_1$ is the piece of $G$ joined by $u$, then $C_1$ is a clique tree containing the tooth $T$ and both $u$ and $u'$ belong to $T$ but no other clique of $C$. Therefore by Lemma 7.16, since $x((E(C)) = E(T)) = 0$, we have $x(E(C)) = r(C) - 1$. But by using Lemma 4.4, we can construct a saturating path cover $\beta'$ of $C_1$ having exactly one edge incident with $u$. Then, by modifying $\beta$ so that it is the incidence vector of $\beta$ for edges of $E(C)$ and by suitably changing $x_k$ for $k \in E \setminus E'$, we obtain the desired $x'$.

**Proposition 7.28.** If neither (7.18) nor (7.21) holds, then $G$ is [7.19] or (7.22)] cannot be facet inducing.

**Proof.** In this case $G$ consists of two cliques $K_1$ and $K_2$ joined by a cutnode $u$. If $G = G$, then by Lemma 7.1 $r(G) = r(K_1) + r(K_2)$, so by Lemma 7.3, $G$ cannot be facet inducing. So assume $G \neq G$, and consequently there are other cliques intersecting $K_1$ and $K_2$. Since (7.18) and (7.21) do not hold, either each of these cliques is in $\mathcal{R}$ or each is in $\mathcal{T}$. Let $u$ be a cutnode of $K_1$, joining $K_1$ to a clique $K_2$ other than $K_2$, and let
Let $u$ be a cutnode of $K_1$ joining $K_2$ to a clique $K_3$ other than $K_1$. Let $j$ be the edge of $E \setminus E'$ joining $w$ and $u$ and suppose that $G$ is facet inducing. By Lemma 7.6 there exists integer $x \in Q^+_F$ satisfying $x_j = 1$ and $x(E') = r(G)$ so we must have $x((u) \cap E') = 1$ and $x((w) \cap E') = 1$.

Exactly as in the preceding proof we can show that we must have $x((u) \cap E') = 2$ and $x((w) \cap E') = 1$ and that there exist $k_1 \in E(K_1) \cap E(u)$ and $k_2 \in E(K_2) \cap E(u)$ such that $x_{k_1} = x_{k_2} = 1$. Let $P$ be the path cover of $G$ induced by $x$ and let $\pi$ be the path containing $K_1$ and $K_2$. If $\pi$ has an end vertex different from $u$ and in $K_2$, then by defining either $x_{k_1} := 1$ or $x_{k_2} := 1$ we would contradict $x(E') = r(G)$. Since $x_j = 1$, at least one end of $\pi$ does not belong to $(u, w)$, so some clique $K_3$ is joined by a cutnode $v'$ to either $K_1$ or $K_2$ and $E(K_3)$ contains at least one edge of $\pi$. Thus $x((v') \cap E(K_3)) = 1$. If $K_3 \subseteq E^\pi$, then by Lemma 7.15 we could modify $x$ for the piece of $G$ joined by $v'$ containing $K_3$ and then set $x_{k_1}$ or $x_{k_2}$ to one (depending on whether $v' \in V(K_1)$ or $v' \in V(K_2)$) and again contradict $x(E') = r(G)$.

Therefore $K_1$, and hence all cliques intersecting $K_2$ and $K_3$, belong to $\mathcal{I}$. If $u$ were an end of $v$, then $x(E(K_3) \cap \delta(u)) = 0$, so if we let $C$ be the piece of $G$ joined by $u$ that contains $K_1$, then $C$ is a clique tree in which $K_3$ is a tooth, so by Lemmas 1.16 and 4.4, we could redefine $x$ for $E(C)$ and $E \setminus E'$ and contradict $x(E') = r(G)$. Therefore, $u$ is not an end of $\pi$, and similarly, $w$ is not an end of $\pi$. Let $u'$ be the end of $k_1$, different from $u$. (See Figure 7.4.) If we obtain $x'$ from $x$ by letting $x_{u'} := 0$, $x_{u}^{old} := x_{u}^{new} = 1$, and $x_{k_1}^{old} := x_{k_1}^{new} = 1$, and $x'_{k_1}$ is suitably redefined for $k \in E \setminus E'$, then $x'(E') = x(E') + 1$ so we contradict $x(E') = r(G)$, which completes the proof.

Proposition 7.30. If neither (7.18) nor (7.19) hold but (7.20) holds then $G$ is not facet inducing.

Proof. A pendant clique $K$ is joined by some node $v$ to another clique $K'$ which is not in $\mathcal{I}$ and every other node of $K$ is a cutnode joining a different member of $\mathcal{I}$. Let $C$ be the piece of $G$ joined by $u$ that contains $K$. Then $C$ is a clique tree and $K$ is a tooth containing exactly one node $v$ belonging to no handle. By Lemma 7.16 there is a saturating path cover $P$ of $C$ containing no edge incident with $v$. Let $P_1$ be any saturating path cover of the other cutnode $K'$ of $G$ joined by $u$. Then $P_1$ and $P_2$ provide, together, a path cover of $G$ that shows that $r(G) > r(C) + r(C')$ so by Lemma 7.3, $G$ cannot be facet inducing.

Proposition 7.31. If (7.21) does not hold, but (7.22) does hold, then $G$ is not facet inducing.

Proof. A pendant clique $K$ meets no clique $K'$ not in $\mathcal{I}$, plus an odd number of members of $\mathcal{I}$. Let $v$ be the cutnode joining $K$ and $K'$, let $w$ be a cutnode joining $K$ and some other clique $K''$ and let $T$ and $T'$ be any node of $K''$ different from $v$. (See Figure 7.5.) Let $j$ be the edge of $E \setminus E'$ joining $u$ and $w$ and suppose that $G$ is facet inducing. By Lemma 7.6 there exists $x \in Q^+_F$ satisfying $x(E') = r(G)$ and $x_j = 1$. Therefore $x((u) \cap E') < 1$ and $x((w) \cap E') < 1$. As in the proof of Lemmas 7.27 and 7.28 we can see that if $P$ is the path cover of $G$ induced by $x$, then there must be a path $\pi$ containing $k_1 \in E(u) \cap E(K)$ and $k_1 \in E(w) \cap E(K'')$. Now suppose that $x((u) \cap E(T)) = 0$. Let $C_i$ be the piece of $G$ joined by $u$ containing $T$. Then $C_i$ is a clique tree containing the tooth $T$ which contains at least two nodes belonging to no other clique. By Lemma 7.16 $x(E(C_i)) < r(C_i)$, and by Lemma 4.4 we can find a saturating path cover $C$ of $G$ with exactly one edge incident with $C$. Then if $x'$ is the incidence vector of a tour containing this path cover and $P$ restricted to the other piece of $G$, we have $x(E') > x(E') = r(G)$, a contradiction. Therefore the single edge $k_1 \in E' \cap \delta(u)$ in $P$ is an edge of $E(T)$, and in particular, $v$ is not an end of $\pi$.

If some other tooth $T'$ joins a cutnode $u'$ to $K$ to $K'$ and another cutnode $u''$ to the clique $K''$ of $G$ joined by $u$, then using Lemma 4.4 we could modify $x$ for the edges of the piece $C'$ of $G$ joined by $u'$ containing $T'$ so as to have $x(E(C')) = r(C')$ and $x((u') \cap E(T')) = 1$ and then set $x_{u'} := 1$, contradicting $x(E') = r(G)$. Therefore, for each such $T'$ and $u'$, we have $x((u') \cap E(T')) < 1$. Similarly, if $x((u') \cap E(K)) = 2$, then, using Lemma 7.3, $x(C') < r(C')$ so we could first modify $x$ for the edges of $C'$ so that $x(E(C')) = r(C')$ and $x((u') \cap E(T')) = 1$. Then, for $i \in E(K)$, joining $u'$ to some other $u''$ having $x_i = 1$, we could let $x_i := 0$ and $x_{u'} := 1$ and contradict $x(E') < r(G)$.

But now we are done, for $K$ contains an odd number of cutnodes, other than $u$ and each is contained in a path of $G$ that ends in $K$, so by setting $x_{u'} := 1$, we contradict $x(E') < r(G)$ and complete the proof.

So finally, by combining Propositions 7.27–7.31 we have completed the justification of Procedure 7.17 and indeed have proved the following.

Theorem 7.32. Let $G$ be a facet inducing clique structure for $Q^+_F$. Then $G$ is a simple clique tree.
8. Concluding remarks. We have shown that clique tree inequalities provide a very large set of facet inducing inequalities of $Q_n^f$ and $Q_n^t$, and that they include subtour elimination constraints and comb inequalities as special subclasses. We have shown that there is no more general structure composed of cliques joined by cutnodes for which the corresponding rank inequality is facet inducing for $Q_n^f$. It appears likely that Theorem 7.32 could be appropriately generalized to allow general clique structures where cliques are joined by articulation sets instead of cutnodes. However, for the present, we have not tried to prove such a theorem.

Clique tree inequalities have several interesting properties with respect to the rank function defined by Chvátal (1973b). For example, a clique tree with more than one handle is of rank at least two, and it appears that as the number of handles increases, so too does the rank although not linearly. This will be treated in a subsequent paper.

However, probably the main outstanding question is whether efficient routines can be developed for the clique tree separation problem. That is, given $\delta \in R^r$, either find a clique tree inequality violated by $\delta$ or else show that $\delta$ satisfies all clique tree inequalities. By virtue of Grötschel, Lovász and Schrijver (1981), a polynomial algorithm for this problem would result in a polynomial algorithm for solving the linear program:

$$\begin{align*}
\text{minimize } & cx \\
0 & \leq x \leq 1, \\
Ax & = 2,
\end{align*}$$

where $A$ is the node edge incidence matrix of $K_n$. In almost all cases, it seems that the solution to this linear program would give a very good lower bound on the solution to the TSP with costs $c$. Presently the only solvable cases of this separation problem are for subtour elimination constraints and for (simple) combs in which each tooth has two nodes (Padberg and Rao 1982).

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