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SHORT COMMUNICATION

A PROPERTY OF CONTINUOUS UNBOUNDED ALGORITHMS

Martin GRÖTSCHEL and Rabe VON RANDOW

University of Bonn, Bonn, F.R.G.

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Let $A : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous mapping, let $x_0 \in \mathbf{R}^n$, and define the sequence $\{x_i\}_{i\in\mathbf{N}}$ (where $\mathbf{N}_+ = \{1, 2, 3, \ldots\}$ and $\mathbf{N} = \mathbf{N}_+ \cup \{0\}$) as follows:

$$x_{i+1} = A(x_i), \quad i \in \mathbf{N}. \tag{1}$$

The question that we wish to consider is the following: if the sequence $\{x_i\}_{i\in\mathbf{N}}$ is unbounded, can it contain convergent subsequences?

This problem arises in the context of attempting to eliminate the compactness condition in the global convergence theorem of the theory of algorithms as it is formulated by Luenberger [1, Section 6.5 and Ex. 10, p. 132] or by Zangwill [2, Section 11.3].

The purpose of this note is to answer the above question in the affirmative by constructing an example of a continuous mapping A such that for a particular x_0 the sequence $\{x_i\}_{i\in\mathbf{N}}$ defined by (1) is unbounded and contains convergent subsequences. The example constructed below seems at first sight to be unduly complicated, but this is due to the fact that every such sequence $\{x_i\}_{i\in\mathbf{N}}$ possesses certain properties which we shall establish in the following three propositions.

From now on $\{x_i\}_{i\in\mathbf{N}}$ will denote a sequence defined by (1), where $A : \mathbf{R}^n \to \mathbf{R}^n$ is a continuous mapping and $x_0 \in \mathbf{R}^n$.

Lemma 1. *If the sequence $\{x_i\}_{i\in\mathbf{N}}$ is unbounded, then*

(a) *the terms of the sequence $\{x_i\}_{i\in\mathbf{N}}$ are pairwise distinct, and*

(b) *there exists a subsequence $\{x_{i_j}\}_{j\in N}$ of $\{x_i\}_{i\in N}$ which has no accumulation points.*

Proof. (a) If $x_i = x_j$ for $i \neq j$, then the sequence has only finitely many distinct terms, hence is bounded, which contradicts the hypothesis.

(b) This follows directly from the hypothesis.

Definition 2. Let $\{x_{k_i}\}_{i\in\mathbf{N}}$ be a subsequence of $\{x_i\}_{i\in\mathbf{N}}$. Define the sequence $\{r_i\}_{i\in\mathbf{N}}$ in $\mathbf{N}_+$ as follows:

$$r_i = k_{i+1} - k_i, \quad i \in \mathbf{N},$$

i.e.

$$x_{k_{i+1}} = A^{r_i}(x_{k_i}), \quad i \in \mathbf{N}.$$

Proposition 3. *Assume that*

(a) *the sequence $\{x_i\}_{i\in\mathbf{N}}$ is unbounded, and*

(b) *$\{x_{k_i}\}_{i\in\mathbf{N}}$ is a convergent subsequence of $\{x_i\}_{i\in\mathbf{N}}$. Then the sequence $\{r_i\}_{i\in\mathbf{N}}$ associated with $\{x_{k_i}\}_{i\in\mathbf{N}}$ by Definition 2 is unbounded.*

Proof. Suppose that the sequence $\{r_i\}_{i\in\mathbf{N}}$ is bounded, and let $r = \max\{r_i \mid i \in \mathbf{N}\}$. By Lemma 1(b) there exists a subsequence $\{x_{l_i}\}_{i\in\mathbf{N}}$ of $\{x_i\}_{i\in\mathbf{N}}$ which has no accumulation points. We can without loss of generality assume that $l_0 \geq k_0$. Define mappings $j : \mathbf{N} \to \mathbf{N}$ and $q : \mathbf{N} \to \{0, 1, 2, \ldots, r-1\}$ as follows:

$$j(i) = \max\{\alpha \in \mathbf{N} \mid l_i \geq k_\alpha\}, \quad \text{i.e.} \quad k_{j(i)} \leq l_i < k_{j(i)+1},$$

and

$$q(i) = l_i - k_{j(i)}, \quad \text{i.e.} \quad x_{l_i} = A^{q(i)}(x_{k_{j(i)}}),$$

$$0 \leq q(i) < r_{j(i)} \leq r, \quad i \in \mathbf{N}.$$

Thus every term of the subsequence $\{x_{l_i}\}_{i\in\mathbf{N}}$ is contained in one of the r subsequences $\{A^p(x_{k_i})\}_{i\in\mathbf{N}}$, $p \in \{0, 1, 2, \ldots, r-1\}$, and these converge by the continuity of A. Hence at least one of these r convergent subsequences contains infinitely many terms of $\{x_{l_i}\}_{i\in\mathbf{N}}$ and therefore the sequence $\{x_{l_i}\}_{i\in\mathbf{N}}$ has at least one accumulation point which contradicts the definition of $\{x_{l_i}\}_{i\in\mathbf{N}}$.

Proposition 4. *Assume that*

(a) *the sequence $\{x_i\}_{i\in\mathbf{N}}$ is unbounded, and*

(b) *$\{x_{k_i}\}_{i\in\mathbf{N}}$ is a convergent subsequence of $\{x_i\}_{i\in\mathbf{N}}$ such that the associated sequence $\{r_i\}_{i\in\mathbf{N}}$ (cf. Definition 2) has the property $r_i > i$ for all $i \in \mathbf{N}$.*

Then $\{A^j(x_{k_i})\}_{i\geq j}$, $j \in \mathbf{N}$, is a family of countably many pairwise disjoint convergent subsequences of $\{x_i\}_{i\in\mathbf{N}}$. Moreover, if $\{x_{l_i}\}_{i\in\mathbf{N}}$ is a subsequence of $\{x_i\}_{i\in\mathbf{N}}$ without accumulation points (such a subsequence exists by Lemma 1(b)), then no member of the above family contains infinitely many terms of $\{x_{l_i}\}_{i\in\mathbf{N}}$.

Remark 5. If the sequence $\{x_i\}_{i\in\mathbf{N}}$ contains a convergent subsequence, then clearly it contains one with the property stated in hypothesis (b) of Proposition 4.

Proof. Clearly the subsequences $\{A^j(x_{k_i})\}_{i\geq j}$, $j \in \mathbf{N}_+$, converge by the continuity of A. We now prove that they are pairwise disjoint. Suppose there exist $m, n, p, q \in \mathbf{N}$ with $m < n$, $p \geq m$, $q \geq n$, such that $A^m(x_{k_p}) = A^n(x_{k_q})$. Then by Lemma 1(a) there exists a uniquely determined $t \in \mathbf{N}$ such that $A^m(x_{k_p}) = A^n(x_{k_q}) = x_t$, i.e., $t = k_p + m = k_q + n$. Hence $p > q$, and $n - m = k_p - k_q = \sum_{i=q}^{p-1} r_i \geq r_q > q \geq n$, i.e., $n > m + n$ which is a contradiction. The second statement of Proposition 4 follows as in the proof of Proposition 3.

Proposition 6. *Assume that*

(a) *the sequence $\{x_i\}_{i\in\mathbf{N}}$ is unbounded, and*

(b) *$\{x_{l_i}\}_{i\in\mathbf{N}}$ is a subsequence of $\{x_i\}_{i\in\mathbf{N}}$ without accumulation points and such that its associated sequence $\{s_i\}_{i\in\mathbf{N}}$ (i.e., $s_i = l_{i+1} - l_i$, $i \in \mathbf{N}$, cf. Definition 2) has the property $s_i > i + 1$ for all $i \in \mathbf{N}$.*

Then $\{x_{l_i-j}\}_{i\geq j}$, $j \in \mathbf{N}$, is a family of countably many pairwise disjoint subsequences of $\{x_i\}_{i\in\mathbf{N}}$ without accumulation points.

Remark 7. By hypothesis (a) of Proposition 6 and Lemma 1(b) there exists a subsequence of $\{x_i\}_{i\in\mathbf{N}}$ without accumulation points. Clearly we can assume without loss of generality that it has the property stated in hypothesis (b) of Proposition 6.

Proof. Each of the subsequences $\{x_{l_i-j}\}_{i\geq j}$, $j \in \mathbf{N}_+$, has no accumulation points by the continuity of A, because $A^j(x_{l_i-j}) = x_{l_i}$, $i \in \mathbf{N}$. We now prove that they are pairwise disjoint. Suppose there exist $m, n, p, q \in \mathbf{N}$ with $m < n$, $p \geq m$, $q \geq n$, such that $x_{l_p-m} = x_{l_q-n}$. Then by Lemma 1(a) there exists a uniquely determined $t \in \mathbf{N}$ such that $x_{l_p-m} = x_{l_q-n} = x_t$, i.e., $t = l_p - m = l_q - n$. Hence $p < q$, and $n - m = l_q - l_p = \sum_{i=p}^{q-1} s_i \geq s_{q-1} > q \geq n$, i.e., $n > m + n$, which is a contradiction.

Example 8. The unbounded sequence $\{x_i\}_{i\in\mathbf{N}}$ is given in Table 1 in the form of a triangular array in which the jth column, $j \in \mathbf{N}$, represents the convergent subsequence $\{A^j(x_{k_i})\}_{i\geq j}$, and the upper diagonal represents the subsequence $\{x_{l_i}\}_{i\in\mathbf{N}}$ without accumulation points. We note that the columns are pairwise disjoint, and that the parallel diagonals represent the pairwise disjoint subsequences $\{x_{l_i-j}\}_{i\geq j}$, $j \in \mathbf{N}$, without accumulation points. An explicit formula for x_i, $i \in \mathbf{N}$, is: $x_i = (i - \frac{1}{2}n(n+1)) + 1/(n+2)$, where $n \in \mathbf{N}$ is such that $n(n+1) \leq 2i < (n+1)(n+2)$.

The function $A : [0, \infty) \to [0, \infty)$, which gives rise to this sequence when one

Table 1
The sequence $\{x_i\}_{i\in\mathbf{N}}$

						$\frac{1}{2}$					
x_0						$\frac{1}{3}$	$1\frac{1}{3}$				
x_1	x_2					$\frac{1}{4}$	$1\frac{1}{4}$	$2\frac{1}{4}$			
x_3	x_4	x_5				$\frac{1}{5}$	$1\frac{1}{5}$	$2\frac{1}{5}$	$3\frac{1}{5}$		
x_6	x_7	x_8	x_9			$\frac{1}{6}$	$1\frac{1}{6}$	$2\frac{1}{6}$	$3\frac{1}{6}$	$4\frac{1}{6}$	$\searrow$
x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	$\searrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		0	1	2	3	4	∞
x	$A(x)$	$A^2(x)$	$A^3(x)$	$A^4(x)$	∞						

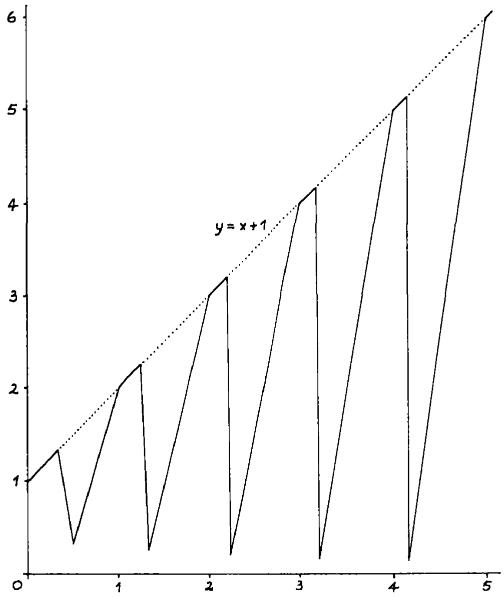

Fig. 1. Graph of the function A.

sets $x_0 = \frac{1}{2}$, has the graph depicted in Fig. 1. An explicit formula for A is:

$$A(x) = \begin{cases} x+1, & x \in \left[n, n+\dfrac{1}{n+3}\right], \\ -(n+1)(n+2)(n+3)x + (n+1)^3(n+3) + \dfrac{1}{n+3}, & x \in \left[n+\dfrac{1}{n+3}, n+\dfrac{1}{n+2}\right], \\ \dfrac{n+2}{n+3}\left\{\dfrac{1}{n+1}(n^2+5n+5)x - (n^2+4n+2)\right\}, & x \in \left[n+\dfrac{1}{n+2}, n+1\right], \end{cases}$$

where $n \in \mathbf{N}$. The straight line $y = x+1$ joins the points $(n, n+1)$ and $(n + 1/(n+3), n+1+1/(n+3))$, $n \in \mathbf{N}$. The local minima are the points $(n + 1/(n+2), 1/(n+3))$, $n \in \mathbf{N}$.

Clearly the function A is continuous. To verify that A gives rise to the above sequence $\{x_i\}_{i\in\mathbf{N}}$ when one sets $x_0 = \frac{1}{2}$, we need to show:

(a) $A(n + 1/(n+2)) = 1/(n+3)$ for all $n \in \mathbf{N}$,

(b) $A(n + 1/r) = (n+1) + 1/r$ for all $n, r \in \mathbf{N}$ with $r \geq 3$ and $n+3 \leq r$.

Proof. (a) $A(n + 1/(n+2)) = A((n+1)^2/(n+2)) = 1/(n+3)$ by the definition of A.

(b) We have $n < n + 1/r \leq n + 1/(n+3)$, hence $A(n + 1/r) = (n+1) + 1/r$ by the definition of A.

Remark 9. The function A can be made C^∞ by "smoothing the corners". Hence the above results hold for C^∞-algorithms.

References

[1] D.G. Luenberger, *Introduction to linear and nonlinear programming* (Addison-Wesley, Reading, MA, 1973).
[2] W.I. Zangwill, *Nonlinear programming: a unified approach* (Prentice-Hall, Englewood Cliffs, NJ, 1969).