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Decomposition and Optimization over Cycles in Binary Matroids*

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For $k = 2$ and 3, we define several k-sums of binary matroids and of polytopes arising from cycles of binary matroids. We then establish relationships between these k-sums, and use these results to give a direct proof that a certain LP-relaxation of the cycle polytope is the polytope itself if and only if M does not have certain minors. The latter theorem was proved earlier by Barahona and Grötschel via Seymour's deep theorem characterizing the matroids with the sum of circuits property. We also exploit the relationships between matroid and polytope k-sums to construct polynomial time algorithms for the solution of the maximum weight cycle problem for some classes of binary matroids and for the solution of the separation problem of the LP-relaxation mentioned above. © 1989 Academic Press, Inc.

1. INTRODUCTION

Let M be a binary matroid on an m-element ground set E. A cycle of M is a disjoint union of circuits of M. Let $P(M)$ denote the convex hull of the incidence vectors of the cycles of M, i.e.,

$$P(M) = \text{conv}\{\chi^C \in \mathbb{R}^E \mid C \text{ is a cycle of } M\}. \tag{1.1}$$

This polytope has been studied in Barahona and Grötschel [2]. Its dimension, several classes of facets, and the vertex adjacency have been determined. We continue this investigation and also focus on the combinatorial optimization problem

$$\max\{c(C) \mid C \text{ is a cycle in } M\}, \tag{1.2}$$

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where $c \in \mathbb{R}^E$ is a given objective function and $c(C)$ stands for the sum $\sum_{e \in C} c_e$. We call this problem the **maximum weight cycle problem**, or just the **cycle problem of binary matroids**. Clearly, (1.2) is equivalent to the linear program

$$\max\{c^T x \mid x \in P(M)\}, \tag{1.3}$$

since every optimal solution of (1.2) yields an optimal vertex solution of (1.3) and vice versa. Problem (1.2) includes, among other interesting combinatorial optimization problems, the max-cut problem in graphs (if M is the cographic matroid of a graph G, then the cycles of M are the cuts of G) and the Eulerian subgraph problem (if M is the graphic matroid of a graph G, then the cycles of M are the (not necessarily connected) Eulerian subgraphs of G). Since the max-cut problem is $\mathcal{NP}$-hard, the maximum cycle problem (1.2) is $\mathcal{NP}$-hard as well.

We use matroidal and polyhedral k-sums, $k = 2, 3$, to obtain a complete description of $P(M)$, in case M can be k-separated into M_1 and M_2 and complete descriptions of $P(M_1)$ and $P(M_2)$ are known. We also prove that particular matroidal k-sums correspond to polyhedral k-sums of a certain LP-relaxation of $P(M)$. These composition results are then combined with the characterization of the Euler subgraph polytope by Edmonds and Johnson [7] and with two decomposition theorems by Seymour [15] and Wagner [21] to a direct proof that the aforementioned LP-relaxation is $P(M)$ itself if and only if M does not have certain minors. The latter theorem was proved earlier by Barahona and Grötschel [2] via the difficult characterization of the matroids with the sum of circuits property of Seymour [16]. Finally we use decomposition and composition techniques to design polynomial time combinatorial algorithms for the solution of (1.2) for certain classes of binary matroids. Among these are the just-mentioned matroids with the sum of circuits property. Finally, we describe polynomial time separation algorithms for certain LP-relaxations of (1.3). This way we obtain—via the ellipsoid method—polynomial time algorithms for (1.3) for further classes of binary matroids.

To begin with, let us quote some of the results of Barahona and Grötschel [2] which we will use. (These results were first proved for the cographic case by Barahona and Mahjoub [4].) Since $P(M)$ is contained in the unit hypercube, the **trivial inequalities**

$$0 \leqslant x_e \leqslant 1, \qquad \text{for all} \quad e \in E \tag{1.4}$$

are valid for $P(M)$. If e is neither a coloop nor contained in a triad (i.e., a cocircuit with three elements), then the inequalities (1.4) define facets of $P(M)$.

Note that a coloop is never contained in a cycle, and that for two coparallel elements, any cycle contains both of them or none. These observations yield that every point in $P(M)$ satisfies the system of equations

$$x_e = 0 \qquad \text{for all coloops } e \in E$$

$$x_e - x_f = 0 \qquad \text{for all coparallel elements } e, f \in E.$$

In fact, these equations define the affine hull of $P(M)$, so the dimension of $P(M)$ is equal to the number of coparallel classes of M.

In a binary matroid the cardinality of the intersection of a cycle and a cocycle is even. Thus the **odd cocircuit inequalities**

$$x(F) - x(C \setminus F) \leq |F| - 1 \qquad \text{for all cocircuits } C \subseteq E \text{ and all } F \subseteq C, |F| \text{ odd} \tag{1.5}$$

are valid for $P(M)$. (Observe that the equation system for $P(M)$ given above is implicitly contained in the inequality system (1.4), (1.5).) An odd cocircuit inequality defines a facet of $P(M)$ if C has at least three elements and no chord and M has no F_7^* minor, where F_7^* denotes the dual Fano matroid. Let us define

$$Q(M) := \{x \in \mathbb{R}^E \mid x \text{ satisfies (1.4) and (1.5)}\}. \tag{1.6}$$

Clearly, $Q(M) \supseteq P(M)$. When does equality hold? The answer was given in Barahona and Grötschel [2] using a theorem of Seymour [16]. In the latter reference Seymour defines a **sum of circuits property** for matroids by demanding certain polyhedral integrality properties. Specifically, a matroid M on a set E has the sum of circuits property if the cone generated by the incidence vectors of the circuits (which is equal to cone($P(M)$)) is given by the following set of inequalities:

$$x_e \geq 0 \qquad \text{for all} \quad e \in E,$$

$$x_e - x(C \setminus \{e\}) \leq 0 \qquad \text{for all cocircuits } C \subseteq E \text{ and all } e \in C.$$

Seymour proved that a binary matroid M has the sum of circuits property if and only if M has no F_7^*, $M(K_5)^*$, or R_{10} minor, where $M(K_5)^*$ denotes the cographic matroid of the complete graph K_5 on five nodes and R_{10} is the binary matroid associated with the (5, 10)-matrix whose columns are the ten 0/1-vectors with three 1's and two 0's. Exploiting an extraordinary symmetry of the facial structure of $P(M)$, Barahona and Grötschel [2] deduced the following theorem from this characterization.

(1.7) THEOREM. *For a binary matroid M the following statements are equivalent*:

(i) $P(M)=Q(M)$.

(ii) *M has the sum of circuits property.*

(iii) *M has no F_7^*, $M(K_5)^*$, or R_{10} minor.*

Two examples of matroids M with $P(M)=Q(M)$ are as follows. The graphic matroid $M(G)$ of a graph $G=(V, E)$ has no F_7^*, $M(K_5)^*$, or R_{10} minor, so $P(M(G))$ is the convex hull of the incidence vectors of the Eulerian subgraphs of G and is given by the trivial inequalities and the inequalities

$$x(F)-x(\delta(W)\setminus F)\leqslant |F|-1 \qquad \text{for all } W\subseteq V \text{ and all } F\subseteq\delta(W),\ |F| \text{ odd}, \qquad (1.8)$$

where $\delta(W)=\{uv\in E \mid u\in W,\ v\in V\setminus W\}$ is the cut (or coboundary) induced by W. Though (1.8) is a consequence of Theorem (1.7), we should mention that it was first proved to be a description of $P(M(G))$ by Edmonds and Johnson [7]. For the second example let M be the cographic matroid of a graph G. Then M has no F_7^* or R_{10} minor, and Theorem (1.7) implies that $P(M)=Q(M)$ holds if and only if G is not contractible to K_5. This result is due to Barahona and Mahjoub [4].

The presentation proceeds as follows. In Section 2 we investigate binary matroid k-sums for $k=2$ and 3. The closely related polyhedral F-sums are introduced in Section 3. Properties of polyhedral F-sums with components of type $Q(\cdot)$ or $P(\cdot)$ are developed in Section 4 and lead to a direct proof of the equivalence of (i) and (iii) of Theorem (1.7). The final two sections are devoted to optimization aspects. In Section 5 we describe a polynomial time separation algorithm for certain $Q(\cdot)$ polytopes. Finally in Section 6 we develop a polynomial time optimization algorithm for certain $P(\cdot)$ polytopes.

Throughout, we use standard matroid terminology as defined in Welsh [22]. In particular the prefix "co" dualizes a term. For the algorithmic part that follows, we will assume that all given vectors $x=(x_1, ..., x_n)^T$ are rational and that each component $x_i=p/q$ is given by an encoding of the two integers p and q. A binary matroid M on E is given by a 0/1-matrix with columns indexed by E with the property that a subset $S\subseteq E$ is independent in M if and only if the columns indexed by S are linearly independent over $GF(2)$. In the theory of matroid algorithms it is also customary to define a matroid via an independence oracle. It is well known that in the case of a binary matroid, specification by an independence oracle is polynomially equivalent to specification by a binary matrix.

2. Binary Matroid k-Sums

In this section we define several sums of binary matroids and describe some of their elementary properties. The notation and approach closely follows Truemper [18].

Throughout, M is a connected binary matroid on a set E. If X is a basis of M, then M has a 0/1 **standard representation matrix** $\hat{B}$ over $GF(2)$ (short: **representation matrix**),

$$\hat{B} = \begin{array}{|c|c|} \multicolumn{1}{c}{X} & \multicolumn{1}{c}{Y} \\ \hline I & B \\ \hline \end{array}$$

where $E = X \cup Y$ indexes the columns of $\hat{B}$, I is an identity matrix, and where a subset of E is independent in M if and only if the corresponding column vectors of $\hat{B}$ are linearly independent. Note that we may index the rows of $\hat{B}$ by X. Then we can implicitly specify $\hat{B}$ (and thus represent M) by just writing B with its row and column indices, i.e.,

$$\begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y} \\ \cline{2-2} X & B \\ \cline{2-2} \end{array}$$

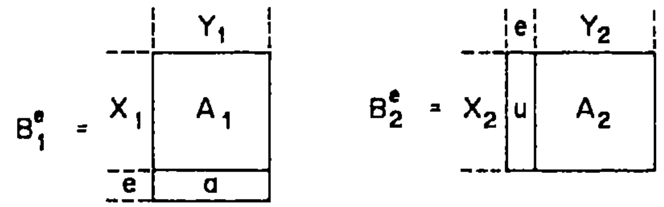
Let $k \geqslant 1$ be an integer and E_1, E_2 be a partition of E. Then the pair (E_1, E_2) is a **Tutte k-separation** of M if $|E_i| \geqslant k$, $i = 1, 2$, and $r(E_1) + r(E_2) \leqslant r(E) + k - 1$. Here $r(\cdot)$ denotes the rank function of M. For $k \geqslant 2$, M is **Tutte k-connected** if it has no l-separation with $l < k$. It is customary to call a Tutte 2-connected matroid just **connected**. We will only deal with Tutte k-separations of M when M is k-connected. Thus for every Tutte k-separation, we know that $r(E_1) + r(E_2) = r(E) + k - 1$ holds. Below, every k-separation or k-connectivity will be of the Tutte kind, so for simplicity we omit "Tutte" from now on when specifying any k-separation or k-connectivity.

Suppose we are given a k-separation (E_1, E_2) of M, $k \geqslant 1$. Let X_2 be a basis of E_2 and X_1 be an independent subset of E_1 such that $X := X_1 \cup X_2$ is a basis of E. Then the submatrix B of the representation matrix $\hat{B}$ produced by X can be partitioned as

$$B = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{Y_1} & \multicolumn{1}{c}{Y_2} \\ \cline{2-3} X_1 & A_1 & 0 \\ \cline{2-3} X_2 & D & A_2 \\ \cline{2-3} \end{array} \quad ; \ \text{rank}(D) = k - 1 \tag{2.1}$$

where $E_1 = X_1 \cup Y_1$ and $E_2 = X_2 \cup Y_2$. Conversely, any matrix B satisfying (2.1) specifies a k-separation $(X_1 \cup Y_1,\ X_2 \cup Y_2)$ if $|X_i \cup Y_i| \geqslant k$, $i = 1, 2$ holds.

In his paper the cases $k = 2$ and 3 are of particular interest. Suppose $k = 2$. Then D has rank 1 and thus all nonzero rows (resp. columns) of D are identical. We construct two matrices B_1^e and B_2^e from B of (2.1) as follows. In the first case we delete all columns indexed by Y_2 and all but one nonzero row, say a, from D. This row receives the new index e. In the second case we delete all rows of B indexed by X_1 and all but one nonzero column, say u, from D. This column is also indexed by e. Thus we obtain the following matrices:



(2.2)

We define M_i to be the binary matroid specified by the B_i^e, $i = 1, 2$, and declare these matroids to be the components of a **2-sum decomposition** of M. The process is clearly reversible since D can be computed as $D = u \cdot a$, and we thus call M a **2-sum** of M_1 and M_2.

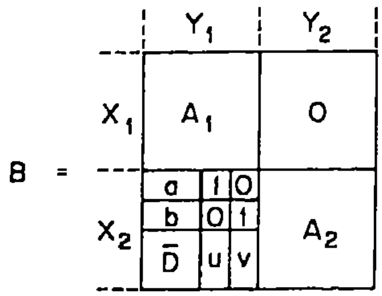
We use the notation

$$M = M_1 \oplus_e M_2$$

to indicate that M is a 2-sum of M_1 and M_2, and that M can be 2-sum decomposed into M_1 and M_2. The index e of $\oplus_e$ refers to the element e along which the 2-sum is performed. At times we will also use the term **e-sum** when we want to explicitly specify that e is the special element of the 2-sum.

Note that each circuit C of M is either a circuit C_i of M_i without e, $i = 1$ or 2, or can be composed from circuits C_i of M_i, $i = 1$ and 2, each containing e, by taking C to be the symmetric difference of C_1 and C_2, i.e., $C = (C_1 \cup C_2) \backslash (C_1 \cap C_2)$. The above statements remain valid when C, C_1, and C_2 are cocircuits instead of circuits.

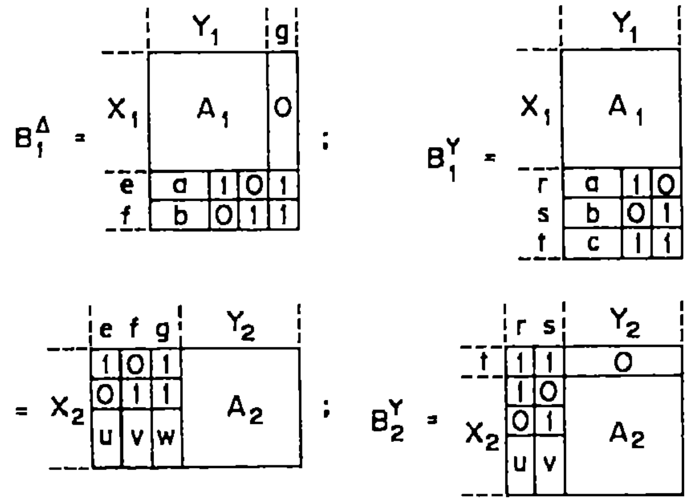
The case $k = 3$ is a bit more complicated. Indeed, several 3-sums are possible, but we will see that some of these are not suitable for the problems studied here. We contemplate a 3-sum decomposition only when a 3-connected M has a 3-separation (E_1, E_2) with $|E_i| \geqslant 4$, $i = 1, 2$. Under this assumption one easily shows—see Truemper [18]—that $X_1 \subseteq E_1$ and $X_2 \subseteq E_2$ exist so that the matrix B of (2.1) is actually of the form



(2.3)

where $|X_i \cup Y_i| \geq 4$, $i = 1, 2$.

From B we derive four matrices denoted by $B_1^{\triangle}$, $B_2^{\triangle}$, B_1^{Y} and B_2^{Y}. To obtain $B_1^{\triangle}$ we delete from B all columns indexed by Y_2 and all rows indexed by X_2 except for the two rows containing a and b, which receive new indices e and f; we then adjoin a new column, indexed by g, which contains only 0's except for two 1's in the rows e and f. To obtain $B_2^{\triangle}$ we delete from B all rows indexed by X_1 and all columns indexed by Y_1 except for the two columns containing u and v, which receive new indices e and f; we then adjoin a new column, indexed by g, which is the sum of the columns indexed by e and f. Below we display $B_1^{\triangle}$ and $B_2^{\triangle}$, and also the two matrices B_1^{Y} and B_2^{Y} which are constructed analogously.



(2.4)

Let $M_{1\triangle}$, $M_{2\triangle}$, M_{1Y}, and M_{2Y} denote the binary matroids specified by the matrices $B_1^{\triangle}$, $B_2^{\triangle}$, B_1^{Y}, and B_2^{Y}. One can extend arguments of Seymour [15] or Truemper [18] to show that the matroids just defined are isomorphic to proper minors of M. For any choice of $M_1 \in \{M_{1\triangle}, M_{1Y}\}$ and of $M_2 \in \{M_{2\triangle}, M_{2Y}\}$, it is possible to define a reversible 3-sum operation that decomposes M into M_1 and M_2. Two of these cases, involving $M_{1\triangle}$ and $M_{2\triangle}$, and $M_{1\triangle}$ and M_{2Y} have been used in Seymour [15] and Truemper [18], respectively. For our purposes the pairs $M_{1\triangle}$ and

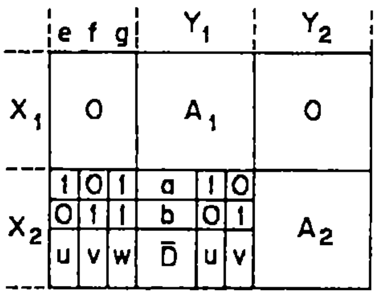
$M_{2\triangle}$, and M_{1Y} and M_{2Y} are of particular interest. For brevity we call them $\triangle$-sum and Y-sum, and denote the two 3-sums by

$$M_{1\triangle} \oplus_{\triangle} M_{2\triangle} \quad \text{and} \quad M_{1Y} \oplus_{Y} M_{2Y}.$$

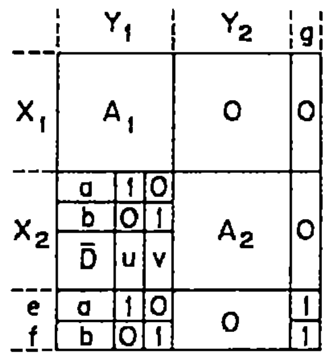
In matrix terms the $\triangle$-sum of $M_{1\triangle}$ and $M_{2\triangle}$ is carried out as follows. From both $B_1^{\triangle}$ and $B_2^{\triangle}$ the column indexed by g is deleted. Then we overlay the reduced $B_1^{\triangle}$ and $B_2^{\triangle}$ so that the order 2 identity matrices explicitly shown in (2.4) are identified. The upper right-hand corner is filled with zeros and the missing matrix $\bar{D}$ of (2.3) is calculated by

$$\bar{D} = [u|v]\left[\frac{a}{b}\right].$$

The matrix operations can be translated into matroid operations in several ways. First we remark that the set $\{e, f, g\}$ forms a triangle in both $M_{1\triangle}$ and $M_{2\triangle}$ (hence the "$\triangle$"). Loosely speaking, the composition of $M_{1\triangle}$ and $M_{2\triangle}$ to M involves identification of the two triangles to a new triangle, which is then removed. We purposely used the nonspecific terms "identification" and "removed" since at least two distinct ways exist to carry out these operations. In one of the two ways the identification produces the matroid $\hat{M}$ represented by



and in the second one a matroid $\tilde{M}$ is generated which is represented by



From $\hat{M}$ the matroid M can be obtained by deleting e, f, g, while $\tilde{M}$ is reduced to M by contracting e, f, and g.

It is helpful to visualize the composition process in graphs. Suppose M is the graphic matroid of a graph G. Then the matroids $M_{1\Delta}$ and $M_{2\Delta}$ are also graphic, say produced by $G_{1\Delta}$ and $G_{2\Delta}$. The latter graphs can be constructed from G as follows. Let H_1 and H_2 be the subgraphs of G induced by the edge sets $E_1 = X_1 \cup Y_1$ and $E_2 = X_2 \cup Y_2$. H_1 and H_2 have exactly three nodes in common. For $i = 1, 2$, $G_{i\Delta}$ is obtained from H_i by adding a triangle $\{e, f, g\}$ on these three nodes. The matroid $\hat{M}$ is also graphic. Indeed a graph $\hat{G}$ for $\hat{M}$ is produced from $G_{1\Delta}$ and $G_{2\Delta}$ by identifying the edges e, f, and g of $G_{1\Delta}$ with e, f, and g of $G_{2\Delta}$. Finally, deletion of e, f, g from $\hat{G}$ produces G. It is interesting to note that the second construction of M via $\tilde{M}$ cannot be realized by graph operations since one can show that $\tilde{M}$ is never graphic.

We now explain the Y-sum briefly. The matrix operations producing the representation B of (2.3) for M from B_1^Y and B_2^Y (representing M_{1Y} and M_{2Y}) should be obvious from the above discussion. Note that the elements r, s, t form a triad (a cocircuit of cardinality three) in M_{1Y} and M_{2Y} (hence the "Y"). The composition also has at least two matroidal interpretations. One of the identification processes produces the matroid $\hat{N}$ represented by

	r	s	Y_1			Y_2
t	1	1	0			0
X_1	0		A_1			0
X_2	1	0	a	1	0	A_2
	0	1	b	0	1	
	u	v	$\bar{b}$	u	v	

and the other one creates the matroid $\tilde{N}$ given by

	Y_1			Y_2
X_1	A_1			0
X_2	a	1	0	A_2
	b	0	1	
	$\bar{b}$	u	v	
r	a	1	0	0
s	b	0	1	
t	c	1	1	

Deletion (contraction) of r, s, t in $\hat{N}$ ($\tilde{N}$) results in M. This time both procedures can be realized in graph operations when M is graphic. Using the previous notation, the graph of $\hat{N}$ is obtained from G by adding a new degree 3 node that is linked to the three distinguished nodes of G. The graph of $\tilde{N}$ is obtained from H_1 and H_2 by connecting each of the three distinguished nodes of H_1 with the corresponding node in H_2 with one edge. The three new edges are r, s, t, they form a cut in the new graph, and their contraction produces G.

Essential for the theorems of the next section is the fact that circuits and cocircuits of M can be nicely expressed in terms of circuits and cocircuits of $M_{1\triangle}$, $M_{2\triangle}$, M_{1Y}, and M_{2Y} of the $\triangle$-sum and Y-sum. For a convenient presentation of this circuit/cocircuit result let us call the elements e, f, g, r, s, and t the **connecting elements** of the latter matroids.

If a circuit C (cocircuit C^*) of M is contained in $X_i \cup Y_i$, for $i=1$ or 2, then C (C^*) is a circuit (cocircuit) in both $M_{i\triangle}$ and M_{iY}. Thus we only need to consider the situation where C or C^* intersects both $E_1 = X_1 \cup Y_1$ and $E_2 = X_2 \cup Y_2$, say in $\bar{E}_1$ and $\bar{E}_2$ in the circuit case, and in $\bar{\bar{E}}_1$ and $\bar{\bar{E}}_2$ in the cocircuit case.

We first consider the circuit case. Let D be the submatrix of B of (2.3) whose rows and columns are indexed by X_2 and Y_1, respectively. Define d to be the sum of the columns of D indexed by $\bar{E}_1 \cap Y_1$. Suppose $d=0$. Then from B of (2.3) it is obvious that both $\bar{E}_1$ and $\bar{E}_2$ index dependent column submatrices of $[I \mid B]$, i.e., C is not a circuit, a contradiction. Thus d is equal to the column of D containing u, or v, or to $(1, 1, w^T)^T$. We may suppose the first case since the other two cases are handled in essentially the same manner. One readily confirms that the column submatrix of $[I \mid B_1^{\triangle}]$ indexed by $C_{1\triangle} = \bar{E}_1 \cup \{e\}$ is minimally dependent, so $C_{1\triangle}$ is a circuit of $M_{1\triangle}$. Similarly $C_{2\triangle} = \bar{E}_2 \cup \{e\}$ is a circuit of $M_{2\triangle}$, so C is the symmetric difference of $C_{1\triangle}$ and $C_{2\triangle}$. Note that e, the selected connecting element, is unique for the given d, i.e., we could not have chosen f or g to draw the same conclusions. Still assuming that d is equal to the column of D containing u, one can similarly show that $\bar{E}_i \cup \{r, t\}$ is a circuit C_{iY} in M_{iY}, for $i=1$ and 2, and that no other pair of connecting elements will do. We conclude that C is the symmetric difference of C_{1Y} and C_{2Y} as well.

Entirely analogous results follow from duality for the cocircuit case. Thus there exist unique connecting elements $x, y \in \{e, f, g\}$ and $z \in \{r, s, t\}$ such that $\bar{\bar{E}}_i \cup \{x, y\}$ is a cocircuit $(C^*)_{i\triangle}$, and $\bar{\bar{E}}_i \cup \{z\}$ is a cocircuit $(C^*)_{iY}$ of M_{iY}, for $i=1$ and 2. The cocircuit C^* is then the symmetric difference of $(C^*)_{1\triangle}$ and $(C^*)_{2\triangle}$, and also of $(C^*)_{1Y}$ and $(C^*)_{2Y}$.

The composition of circuits and cocircuits of the components of a $\triangle$- or Y-sum is slightly more complicated. We briefly indicate the circuit relationships and leave their verification and filling in of the cocircuit results to the reader.

If C_1 and C_2 are circuits of $M_{1\triangle}$ and $M_{2\triangle}$ that contain e but not f or g, then $(C_1 \cup C_2)\backslash\{e\}$ is the disjoint union of at most two circuits of M. If instead C_2 contains f and g but not e, then $(C_1 \setminus \{e\}) \cup (C_2 \setminus \{f, g\})$ is a circuit of M. If C_1 and C_2 are circuits of M_{1Y} and M_{2Y} that contain r and t but not s, then $(C_1 \cup C_2) \setminus \{r, t\}$ is the disjoint union of at most two circuits of M.

These results imply that for every cycle C of M there are cycles $C_{1\triangle}$ and $C_{2\triangle}$ of $M_{1\triangle}$ and $M_{2\triangle}$, respectively and also cycles C_{1Y} and C_{2Y} of M_{1Y} and M_{2Y}, respectively, such that C is the symmetric difference of $C_{1\triangle}$ and $C_{2\triangle}$ and also of C_{1Y} and C_{2Y}. The last observation, or equivalently two pivots on the 1's of the explicitly shown 2×2 identity submatrix of B of (2.3), prove that the $\triangle$-sum and Y-sum operations are commutative, i.e.,

$$M_{1\triangle} \oplus_{\triangle} M_{2\triangle} = M_{2\triangle} \oplus_{\triangle} M_{1\triangle}$$

and

$$M_{1Y} \oplus_{Y} M_{2Y} = M_{2Y} \oplus_{Y} M_{1Y}.$$

Furthermore, dualization changes a $\triangle$-sum to a Y-sum, and vice versa, i.e., if $M = M_{1\triangle} \oplus_{\triangle} M_{2\triangle}$, then $M^* = (M_{1\triangle})^* \oplus_Y (M_{2\triangle})^*$, where the triangles $\triangle$ of $M_{1\triangle}$ and $M_{2\triangle}$ have become the triads for the Y-sum.

We now deal with the problem of locating 2- and 3-sums for certain matroid classes. But before we proceed, we simplify the notation for 3-sums to unclutter the exposition. So far we have used the notation $M_{1\triangle} \oplus_{\triangle} M_{2\triangle}$ and $M_{1Y} \oplus_Y M_{2Y}$ for the $\triangle$-sum and Y-sum, but actually $M_1 \oplus_{\triangle} M_2$ and $M_1 \oplus_Y M_2$ suffices once one agrees that in the former case the set $\triangle$ is assumed to be a triangle in both M_1 and M_2, while in the latter case the set Y is assumed to be a triad of M_1 and M_2.

Profound decomposition theorems of Seymour [15] and Wagner [21] give necessary and sufficient conditions for 2- and 3-sum decomposition of certain matroids. The following theorem is based on these results. It will be repeatedly invoked in the subsequent sections. Below, $K_{3,3}$ is the complete bipartite graph with three nodes on each side.

(2.5) THEOREM. (a) *Let $\mathcal{N}_1$ be the class consisting of the following matroids*:

- (i) *all graphic matroids of 2-connected series-parallel graphs,*
- (ii) *all 3-connected graphic matroids,*
- (iii) *all 3-connected cographic matroids, and*
- (iv) *F_7, F_7^*, and R_{10}.*

Then there exists a polynomial-time algorithm that, for any connected binary matroid M without and F_7 or F_7^* minor, either

(1) *declares $M \in \mathcal{N}_1$ or*

(2) *finds a 2-sum or* $\triangle$*-sum decomposition of* M *into* M_1 *and* M_2 *such that*

- — M_1 *is in* $\mathcal{M}_1'$ *and 3-connected,*
- — M_2 *is connected, and*
- — M_1 *and* M_2 *are both isomorphic to minors of* M.

Moreover, graphs are found representing the graphic matroids among M, M^*, M_1, M_1^*, M_2, *and* M_2^*.

(b) *Let* $\mathcal{M}_2'$ *be the class consisting of the following* **graphic** *matroids arising from*

(i) *all 2-connected series-parallel graphs,*

(ii) *all 3-connected planar graphs, and*

(iii) $K_{3,3}$ *and* V_8 *(shown in Fig. 1).*

Then there exists a polynomial-time algorithm that, for any connected graphic binary matroid M *without an* $M(K_5)$ *minor, and for any edge set* L *that is the edge set of a triangle of* M *or is empty, either*

(1) *declares* $M \in \mathcal{M}_2'$, *or*

(2) *finds a 2-sum or* $\triangle$*-sum decomposition of* M *into* M_1 *and* M_2 *such that*

- — $M_1 \in \mathcal{M}_2'$,
- — M_2 *is connected and contains* L, *and*
- — M_1 *and* M_2 *are both isomorphic to minors of* M.

Moreover, graphs are found representing the matroids M, M_1, *and* M_2.

Proof. (a) Assume M to be 2-separable. With a slight modification, the polynomial time algorithm of Truemper [17] finds a 2-sum decomposition $M = M_1 \oplus_e M_{11}$, where M_1 is 3-connected or determines a 2-connected series-parallel graph G such that M is the graphic matroid of G. In the latter case we are done. In the former case we can stop as well if $M_1 \in \mathcal{M}_1'$, except possibly for the determination of the graphs. (We will cover this later.) Otherwise M is regular and has a 3-sum decomposition by Seymour [15]. Indeed, an algorithmic implementation of the proofs of Seymour

V_8 =

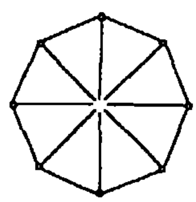


FIGURE 1

[15] or anyone of several algorithms [23, 14, 20] can be adapted to locate a Δ-sum $M_1 = M_2 \oplus_{\Delta_2} M_{21}$, where element e, the connecting element of the 2-sum, is in M_{21} and where M_2 is 3-connected. One easily confirms that $M = M_2 \oplus_{\Delta_2} \tilde{M}_{21}$ where $\tilde{M}_{21} = M_{21} \oplus_e M_{11}$. If $M_2 \in \mathcal{N}_1$, we are done. Otherwise, we find in polynomial time, with one of the just-cited algorithms, a Δ-sum decomposition $M_2 = M_3 \oplus_{\Delta_3} M_{31}$, where Δ_2 occurs in M_{31} and where M_3 is 3-connected. That such a Δ-sum indeed exists, is not immediately obvious, but can be readily deduced from special properties of the crucial binary matroid R_{12} of Seymour [15] and 3-connectivity results of Truemper [18]. A check of the matrix operations of Section 2 then validates the claim that $M = M_3 \oplus_{\Delta_3} \tilde{M}_{31}$ with $\tilde{M}_{31} = M_{31} \oplus_{\Delta_2} \tilde{M}_{21}$. Continuing in this fashion we eventually get an $M_s \in \mathcal{N}_1$ and $M = M_s \oplus_{\Delta_s} \tilde{M}_{s1}$, for some connected $\tilde{M}_{s1}$. Reexamining the preceding steps one also establishes that M_s and $\tilde{M}_{s1}$ are isomorphic to minors of M. A shortened version of the procedure handles the case where M is 3-connected and not in $\mathcal{N}_1$. Finally the graphs are produced by any one of several polynomial algorithms, see, e.g., Fujishige [8], Bixby and Wagner [5].

(b) This part is handled in an analogous fashion, except that the 3-sum result of Wagner [21] is invoked instead of Seymour [15]. It seems that the result is a bit easier to derive if one uses Truemper [19], which contains a strengthened and more detailed version of the decomposition theorem of Wagner [21]. ∎

Note that part (b) with the optional triangle L condition allows concatenation of parts (a) and (b) as is evident from the proof of Theorem (2.5). We should also point out that M_2 of part (a) or (b) is strictly smaller than the original M since M_1 has at least 2 (4) elements in case of a 2-sum (Δ-sum). Thus, if one applies Theorem (2.5) recursively, i.e., first to M, then to M_2, etc., then after at most $|E|$ applications a matroid in $\mathcal{N}_1$ or $\mathcal{N}_2$ is obtained.

The next two sections introduce and develop results for polyhedral F-sums, which are the polyhedral counterparts to the matroidal k-sums. The reader mainly interested in applications of Theorem (2.5), may skip ahead to Sections 5 and 6 without loss of continuity.

3. Polyhedral F-Sums

We now define compositions of polyhedra. For certain polytopes associated with binary matroids, these compositions will be closely related to the k-sums defined in the preceding section.

Let $P_1 \subseteq \mathbb{R}^{E_1}$ and $P_2 \subseteq \mathbb{R}^{E_2}$ be polyhedra such that $F := E_1 \cap E_2 \neq \emptyset$. For notational convenience let us write each vector $x \in \mathbb{R}^{E_i}$ in the form $x =$

(x^i, y), $i = 1, 2$, where y is the vector of the components indexed by F. The F-sum of P_1 and P_2 is the polyhedron

$$P_1 \oplus_F P_2 := \{(x^1, x^2) \in \mathbb{R}^{(E_1 \cup E_2) \setminus F} \mid \exists y \text{ such that } (x^i, y) \in P_i, i = 1, 2\}. \tag{3.1}$$

Geometrically, the F-sum of P_1 and P_2 is obtained in two steps. First the polyhedron $P_{12} := \{(x^1, x^2, y) \in \mathbb{R}^{E_1 \cup E_2} \mid (x^i, y) \in P_i,\ i = 1, 2\}$ is formed, and then it is projected into (x^1, x^2)-space. Suppose P_1 and P_2 are given in the form

$$P_i = \{(x^i, y) \mid A^i x^i + D^i y \leqslant a^i\},\ i = 1, 2.$$

Then

$$P_{12} = \{(x^1, x^2, y) \mid A^i x^i + D^i y \leqslant a^i, i = 1, 2\}, \tag{3.2}$$

and the F-sum $P_1 \oplus_F P_2$ can be obtained from this description of P_{12} by Fourier–Motzkin elimination of y.

4. F-Sums of the Polytopes $P(\cdot)$ and $Q(\cdot)$

For given P_1, P_2, and F it generally seems difficult to describe structural properties of the F-sum in terms of structural properties of P_1 and P_2. Here we are interested in F-sums of polytopes of type $P(\cdot)$ and $Q(\cdot)$ defined in (1.1) and (1.6). Specifically, the sets E_i are ground sets of binary matroids and F is either a singleton or a triangle or triad. Indeed the F-sums are motivated directly by the matroid 2-, $\triangle$-, and Y-sums in the following way. Suppose a binary matroid M on E is the e-sum $M_1 \oplus_e M_2$. Then we will compare the e-sums (short for $\{e\}$-sums) $P(M_1) \oplus_e P(M_2)$ and $Q(M_1) \oplus_e Q(M_2)$ with $P(M)$ and $Q(M)$. Similarly, if M is the $\triangle$-sum $M_1 \oplus_\triangle M_2$, then we will relate $P(M_1) \oplus_\triangle P(M_2)$ and $Q(M_1) \oplus_\triangle Q(M_2)$ to $P(M)$ and $Q(M)$. In this notation, the $\triangle$ denotes the triangle used for the composition. Analogously, the Y-sum case is of interest as well.

The first result is easy, and its proof is left to the reader.

(4.1) Theorem. *Let $M_1 \oplus_e M_2$ be the e-sum of two binary matroids M_1 and M_2. Then the following holds.*

(a) $P(M_1 \oplus_e M_2) = P(M_1) \oplus_e P(M_2)$.

(b) $Q(M_1 \oplus_e M_2) = Q(M_1) \oplus_e Q(M_2)$.

The situation becomes much more complicated in the case of 3-sums. Later we shall prove that the analogue of Theorem (4.1) does hold for

Y-sums. However, this is generally not so for $\triangle$-sums, as we now demonstrate by two counterexamples.

Let M_i be the cographic matroid of the graph G_i, $i=1,2$, shown in Fig. 2. So $P(M_i)$ is the cut polytope of G_i. The cut $\triangle = \{r, s, t\}$ forms a triangle in M_1 and M_2. Obviously, $M = M_1 \oplus_\triangle M_2$ is the cographic matroid of a graph $G=(V,E)$ which is K_5 with one edge subdivided. Neither G_1 nor G_2 are contractible to K_5, thus by Theorem (1.7), $P(M_i) = Q(M_i)$, $i=1,2$, holds. It is easily checked that the vector $(x^2, y) = \frac{2}{3}\mathbb{1} \in \mathbb{R}^{10}$, where $\mathbb{1} = (1, 1, ..., 1)$ is contained in $P(M_2)$. Similarly the following vector $(x^1, y) \in \mathbb{R}^7$ is in $P(M_1)$: component h (see Fig. 2) has value 0, and all other components have value $\frac{2}{3}$. Thus the vector $z = (x^1, x^2) \in \mathbb{R}^{11}$ has value $\frac{2}{3}$ in each component except for component h. By definition (3.1), z is contained in $P(M_1) \oplus_\triangle P(M_2)$. The inequality $a^T x = \sum_{e \in E \setminus h} x_e - x_h \leq 6$ is clearly valid for the cut polytope $P(M)$ of G (in fact, it defines a facet). But $a^T z = \frac{20}{3}$, and so z is not contained in $P(M)$. This shows $P(M) \neq P(M_1) \oplus_\triangle P(M_2)$. The same matroids can be used to show that $Q(M)$ need not be equal to $Q(M_1) \oplus_\triangle Q(M_2)$. For M_1, let (x^1, y) be the vector in $\mathbb{R}^7$ that has value $\frac{1}{2}$ in each component except for a 1 as entry of the component corresponding to edge g of G_1. For M_2, let (x^2, y) be the vector in $\mathbb{R}^{10}$ that contains only $\frac{1}{2}$'s except for two 1's in the components corresponding to the edges i and j of G_2. Then $z = (x^1, x^2)$ is in $Q(M_1) \oplus_\triangle Q(M_2)$, but z cannot be in $Q(M)$ since z has three 1's as entries for a triad of M.

The remainder of this section is devoted to Y-sums. First we prove the analogue of Theorem (4.1) for Y-sums, and then use this result to establish equivalence of (i) and (iii) of Theorem (1.7).

(4.2) THEOREM. *Let $M = M_1 \oplus_Y M_2$ be the Y-sum of two binary matroids M_1 and M_2. Then*

$$P(M_1 \oplus_Y M_2) = P(M_1) \oplus_Y P(M_2).$$

Proof. We use a proof technique due to Cornuéjols, Naddef, and

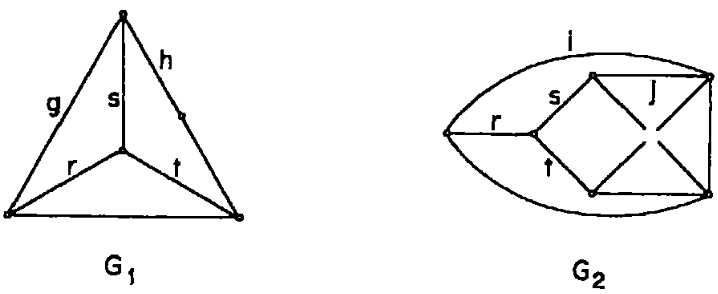


FIGURE 2

Pulleyblank [6]. The proof closely follows the arguments of Barahona [1]. For $i = 1, 2$, let E_i be the ground set of M_i, and suppose $\{r, s, t\}$ is the triad Y of M_1 and M_2. Thus the ground set of M is $E = (E_1 \cup E_2) \setminus Y$.

Recall that for every cycle C of M, there are cycles C_i of M_i, $i = 1, 2$, such that C is the symmetric difference of C_1 and C_2. This immediately implies that for every vertex χ^C of $P(M)$ there are vertices $\chi^{C_1} = (x^1, y)$ and $\chi^{C_2} = (x^2, y)$ of $P(M_1)$ and $P(M_2)$, respectively, such that $(x^1, x^2) = \chi^C$. Here $y = (y_r, y_s, y_t)$ denotes the 3-vector corresponding to the triad Y. Hence by (3.1), $P(M) \subseteq P(M_1) \oplus_Y P(M_2)$.

To show the converse we prove that every point $x \in P(M_1) \oplus_Y P(M_2)$ is a convex combination of points in $P(M)$. So suppose $x = (x^1, x^2)$ with $x^1 \in \mathbb{R}^{E_1 \setminus Y}$ and $x^2 \in \mathbb{R}^{E_2 \setminus Y}$ is an element of $P(M_1) \oplus_Y P(M_2)$. By definition (3.1) there is a vector $y \in \mathbb{R}^Y$ such that $\bar{x}^1 = (x^1, y) \in P(M_1)$ and $\bar{x}^2 = (x^2, y) \in P(M_2)$. The vectors $\bar{x}^1$ and $\bar{x}^2$ are convex combinations of incidence vectors of cycles of M_1 and M_2, respectively; i.e., there exist vertices $p^1, p^2, ..., p^k$ of $P(M_1)$ and $q^1, q^2, ..., q^l$ of $P(M_2)$ and $\lambda_1, ..., \lambda_k \geqslant 0$, $\mu_1, ..., \mu_l \geqslant 0$ with $\lambda_1 + \cdots + \lambda_k = 1$, $\mu_1 + \cdots + \mu_l = 1$ such that

$$(x^1, y) = \sum_{i=1}^{k} \lambda_i p^i, \qquad (x^2, y) = \sum_{i=1}^{l} \mu_i q^i.$$

Set

$$\alpha_{rs} := \sum_{i \in K(r,s)} \lambda_i; \qquad \text{where} \quad K(r, s) := \{i \in \{1, ..., k\} \mid p^i_r = p^i_s = 1\},$$

$$\beta_{rs} := \sum_{i \in L(r,s)} \lambda_i; \qquad \text{where} \quad L(r, s) := \{i \in \{1, ..., l\} \mid q^i_r = q^i_s = 1\}.$$

Define α_{rt}, α_{st}, β_{rt}, β_{st} analogously. Note that a cycle in M_1 or M_2 meets the triad Y either in two elements or in none. So by the above construction we have

$$\alpha_{rs} + \alpha_{rt} = y_r = \beta_{rs} + \beta_{rt}$$
$$\alpha_{rs} + \alpha_{st} = y_s = \beta_{rs} + \beta_{st}$$
$$\alpha_{rt} + \alpha_{st} = y_t = \beta_{rt} + \beta_{st}.$$

These two systems of equations determine $\alpha_{rs}, ..., \beta_{st}$ uniquely; so $\alpha_{rs} = \beta_{rs}$, $\alpha_{rt} = \beta_{rt}$, and $\alpha_{st} = \beta_{st}$. This allows us to match incidence vectors of cycles in M_1 containing r and s with incidence vectors of cycles in M_2 containing r and s, and to perform analogous matchings for the index pairs r, t and s, t. We can also match incidence vectors of cycles in M_1 containing no element of Y with incidence vectors of cycles of M_2 containing no element of Y to obtain incidence vectors χ^{C_i}, $i=1, \ldots, k'$ of cycles of M and scalars $\rho_1, \ldots, \rho_{k'} \geqslant 0$ such that

$$x=\sum_{i=1}^{k'} \rho_i \chi^{C_i}; \qquad \sum_{i=1}^{k'} \rho_i = 1.$$

This finishes the proof. ∎

(4.3) THEOREM. *Let $M=M_1 \oplus_Y M_2$ be the* Y*-sum of two binary matroids M_1 and M_2. Then*

$$Q(M_1 \oplus_Y M_2)=Q(M_1) \oplus_Y Q(M_2).$$

Proof. For convenient reference we first list the inequality systems defining $Q(M)$ and $Q(M_i)$, $i=1, 2$, where we employ the same conventions as in the proof of Theorem (4.2). We also rewrite any inequality of the form $x(F)-x(C\setminus F) \leqslant |F|-1$ of (1.5) as $\hat{x}(F)+x(C\setminus F) \geqslant 1$, where $\hat{x}(F)=\sum_{e \in F}(1-x_e)$. We deduce from (1.4) and (1.5) for $Q(M)$ the inequalities

$$0 \leqslant (x^1, x^2) \leqslant 1, \tag{4.4.1}$$

$$\hat{x}^1(F_1)+x^1(C_1\setminus F_1)+\hat{x}^2(F_2)+x^2(C_2\setminus F_2) \geqslant 1,$$
for all cocircuits $C=C_1 \cup C_2$ of M with $C_1 \subseteq E_1$ and $C_2 \subseteq E_2$, and for all $F=F_1 \cup F_2$ with $F_1 \subseteq C_1$, $F_2 \subseteq C_2$, and $|F_1|+|F_2|$ odd. (4.4.2)

For $Q(M_i)$, $i=1, 2$, we obtain

$$0 \leqslant x^i \leqslant 1, \tag{4.5.1}$$

$$\hat{x}^i(F)+x^i(C\setminus F) \geqslant 1$$
for all cocircuits $C \subseteq E_i$ of M_i with $C \cap Y=\varnothing$, and for all $F \subseteq C$ with $|F|$ odd, (4.5.2)

$$\hat{x}^i(F)+x^i(C\setminus F)+y_h \geqslant 1$$
for all cocircuits $C \cup \{h\}$ of M_i with $C \cap Y=\varnothing$ and $h \in Y$, and for all $F \subseteq C$ with $|F|$ odd, (4.5.3)

$$\hat{x}^i(F)+x^i(C\setminus F)+(1-y_h) \geqslant 1$$
for all cocircuits $C \cup \{h\}$ of M_i with $C \cap Y=\varnothing$ and $h \in Y$, and for all $F \subseteq C$ with $|F|$ even, (4.5.4)

$$(1-y_r)+(1-y_s)+(1-y_t) \geqslant 1, \tag{4.5.5}$$

$$(1-y_r)+y_s+y_t \geqslant 1, \tag{4.5.6}$$

$$y_r+(1-y_s)+y_t \geqslant 1, \tag{4.5.7}$$

$$y_r+y_s+(1-y_t) \geqslant 1. \tag{4.5.8}$$

We have chosen the above elaborate descriptions of $Q(M)$ and $Q(M_i)$ since they simplify the subsequent explanations. The reader may have noticed that we have omitted cocircuit inequalities from (4.5) that involve a cocircuit of the form $C \cup \{e, f\}$, where $e, f \in Y$ and $C \cap Y = \emptyset$. But any such inequality is implied by those of (4.5.3)–(4.5.8), and thus can be eliminated.

In the discussion below we frequently rely on the cocircuit results for Y-sums of Section 2, without explicitly referencing them. We also make repeated use of the fact that the symmetric difference of two cocircuits of a binary matroid is a cocycle.

First we show that $Q(M) \supseteq Q(M_1) \oplus_Y Q(M_2)$. Let $C_1 \cup C_2$ be a cocircuit of M specified in (4.4.2). If C_1, say, is empty, then C_2 is a cocircuit of M_2, and the inequalities of (4.4.2) involving C_2 are also listed in (4.5.2) for M_2. If both C_1 and C_2 are nonempty, then there is a unique element $h \in Y$ such that $C_i \cup \{h\}$ is a cocircuit of M_i, $i = 1, 2$. Thus all constraints (4.4.1) and (4.4.2) can be produced by Fourier–Motzkin elimination from (4.5.1)–(4.5.4) with $i = 1$ and 2, and hence $Q(M) \supseteq Q(M_1) \oplus_Y Q(M_2)$.

The proof of the reverse containment is more difficult. We will extend an arbitrary $(x^1, x^2) \in Q(M)$ to $(x^i, y) \in Q(M_i)$, for $i = 1, 2$. Derivation and justification of such a y is accomplished in several steps. First we calculate for each $h \in Y$ from the given $(x^1, x^2) \in Q(M)$,

$$\begin{aligned} a_h &:= \min\{1, \min\{\hat{x}^i(F_i) + x^i(C_i \setminus F_i)\}\}, \\ b_h &:= \max\{0, \max\{1 - [\hat{x}^i(F_i) + x^i(C_i \setminus F_i)]\}\}, \end{aligned} \tag{4.6}$$

where the inner minimization for a_h (inner maximization for b_h) is over $i = 1$ and 2, over all $C_i \subseteq E_i$ that form a cocircuit together with h in M_i—note that by our definition of 3-sums at least one such cocircuit $C_i \cup \{h\}$ must exist—and over all $F_i \subseteq C_i$ of even (odd) cardinality. For $h \in Y$ and $a_h < 1$ ($b_h > 0$), let $C^{a,h}$ and $F^{a,h}$ ($C^{b,h}$ and $F^{b,h}$) be a cocircuit $C_i \cup \{h\}$ and a set F_i producing the minimum (maximum). Define $i(a, h)$ ($i(b, h)$) to be the index i of the matroid M_i containing $C^{a,h}$ ($C^{b,h}$). Thus we have for $j = i(a, h)$ and $k = i(b, h)$,

$$\begin{aligned} &a_h = 1 \text{ or } a_h = \hat{x}^j(F^{a,h}) + x^j(C^{a,h} \setminus F^{a,h}) \quad \text{with even } |F^{a,h}|, \\ &b_h = 0 \text{ or } b_h = 1 - [\hat{x}^k(F^{b,h}) + x^k(C^{b,h} \setminus F^{b,h})] \text{ with odd } |F^{b,h}|. \end{aligned} \tag{4.7}$$

We next list and then prove several useful inequalities about the a_h and b_h, $h \in Y$.

$$1 \geqslant a_h \geqslant b_h \geqslant 0 \qquad \text{for all} \quad h \in Y; \tag{4.8.1}$$

$$\begin{aligned} a_r &\geqslant |b_s - b_t|, \\ a_s &\geqslant |b_r - b_t|, \\ a_t &\geqslant |b_r - b_s|; \end{aligned} \tag{4.8.2}$$

$$a_r + b_s + b_t \leqslant 2,$$
$$b_r + a_s + b_t \leqslant 2, \tag{4.8.3}$$
$$b_r + b_s + a_t \leqslant 2.$$

Proof of (4.8.1). $0 \leqslant a_h,\ b_h \leqslant 1$ obviously holds, so we may assume $a_h < 1$ and $b_h > 0$. Then by (4.7) $a_h \geqslant b_h$ if and only if for $j = i(a, h)$ and $k = i(b, h)$,

$$\hat{x}^j(F^{a,h}) + x^j(C^{a,h} \setminus F^{a,h}) + \hat{x}^k(F^{b,h}) + x^k(C^{b,h} \setminus F^{b,h}) \geqslant 1. \tag{4.9}$$

Now (4.9) holds trivially if an element of $C^{a,h} \cap C^{b,h}$ occurs in just one of $F^{a,h}$ and $F^{b,h}$, since then the left-hand side is at least 1. Otherwise $|F^{a,h}| + |F^{b,h}| \equiv 1 \pmod 2$. This fact plus symmetric differences and Section 2 results confirm that (4.9) is implied by one of the inequalities (4.4.2) for $Q(M)$.

Proof of (4.8.2). By symmetry we only need to show $a_r + b_t \geqslant b_s$, or equivalently $a_r + (1 - b_s) \geqslant (1 - b_t)$. In the nontrivial case $a_r < 1$ and $b_s > 0$, so by (4.7) we need to show that

$$\hat{x}^j(F^{a,r}) + x^j(C^{a,r} \setminus F^{a,r}) + \hat{x}^k(F^{b,s}) + x^k(C^{b,s} \setminus F^{b,s})$$
$$\geqslant \begin{cases} \hat{x}^l(F^{b,t}) + x^l(C^{b,t} \setminus F^{b,t}) & \text{if } b_t > 0, \\ 1 & \text{if } b_t = 0, \end{cases} \tag{4.10}$$

where $j = i(a, r)$, $k = i(b, s)$, and $l = i(b, t)$. The inequality clearly holds if an element $h \in C^{a,r} \cap C^{b,s}$ occurs only in one of $F^{a,r}$ and $F^{b,s}$, since then the left-hand side is at least 1. Otherwise the symmetric difference of $C^{a,r}$ and $C^{b,s}$ on one hand and of $F^{a,r}$ and $F^{b,s}$ on the other hand contain a cocircuit $C_i \cup \{t\}$ of M_i, $i = 1$ or 2, and a set $F_i \subseteq C_i$ such that (C_i, F_i) is a candidate pair for the maximization problem for b_t; or an inequality of (4.4.2) for $Q(M)$ implies that the left-hand side of (4.10) is at least 1. In either situation (4.10) holds.

Proof of (4.8.3). By symmetry we only need to consider $a_r + b_s + b_t \leqslant 2$, or equivalently

$$a_r \leqslant (1 - b_s) + (1 - b_t). \tag{4.11}$$

In the nontrivial case $b_s,\ b_t > 0$ and with $k = i(b, s)$ and $l = i(b, t)$ we have by (4.7),

$$(1 - b_s) + (1 - b_t) = \hat{x}^k(F^{b,s}) + x^k(C^{b,s} \setminus F^{b,s})$$
$$+ \hat{x}^l(F^{b,t}) + x^l(C^{b,t} \setminus F^{b,t}), \tag{4.12}$$

which is at least 1 (and hence (4.11) holds) if $C^{h,s}$ and $C^{h,t}$ contain an element that occurs in just one of $F^{h,s}$ and $F^{h,t}$. Otherwise the symmetric difference of $C^{h,s}$ and $C^{h,t}$ on one hand and of $F^{h,s}$ and $F^{h,t}$ on the other hand give a cocircuit $C_i \cup \{r\}$ of M_i, $i=1$ or 2, and a set $F_i \subseteq C_i$ such that (C_i, F_i) is a candidate for the minimization problem for a_r; or an inequality of (4.4.2) for $Q(M)$ implies that the right-hand side of (4.12) is at least 1. We are done in either case.

With (4.8.1)–(4.8.3) established, we now proceed with the derivation of y such that $(x^i, y) \in Q(M_i)$, $i=1, 2$. Without loss of generality we may suppose $b_r \leqslant b_s \leqslant b_t$. Define

$$\begin{aligned} y_r &= \min\{a_r, b_t\}, \\ y_s &= b_s, \\ y_t &= b_t. \end{aligned} \tag{4.13}$$

We claim that $y=(y_r, y_s, y_t)$ so specified will do. Clearly (4.5.2) holds by (4.4.2). By the definition of a_h and b_h and the fact that $a_h \geqslant y_h \geqslant b_h$, $h \in Y$, (4.5.1), (4.5.3), and (4.5.4) are satisfied as well. Note that the latter conclusion is nothing but the statement that Fourier–Motzkin elimination works. Thus (4.5.5)–(4.5.8) remain.

(4.5.5). $(1-y_r)+(1-y_s)+(1-y_t) \geqslant 1$ becomes $\min\{a_r, b_t\}+b_s+b_t \leqslant 2$, which holds by (4.8.3).

(4.5.6). $(1-y_r)+y_s+y_t \geqslant 1$ holds since $\min\{a_r, b_t\} \leqslant b_s+b_t$ trivially.

(4.5.7). $y_r+(1-y_s)+y_t \geqslant 1$ also holds trivially since $y_t=b_t \geqslant b_s=y_s$.

(4.5.8). $y_r+y_s+(1-y_t) \geqslant 1$ translates to $\min\{a_r, b_t\} \geqslant b_t-b_s$, which is satisfied by (4.8.2).

Thus $(x^i, y) \in Q(M_i)$, $i=1, 2$, and $Q(M)=Q(M_1) \oplus_Y Q(M_2)$. ∎

Theorems (4.2) and (4.3) can be combined with Theorem (2.5) (which is an algorithmic version of two decomposition theorems of Seymour [15] and Wagner [21]) and with the Edmonds and Johnson [7] characterization of the Euler subgraph polytope, to a direct proof of the equivalence of (i) and (iii) of Theorem (1.7). We first restate this equivalence for convenient reference.

(4.14) THEOREM. *Let M be a connected binary matroid. Then $Q(M)=P(M)$ if and only if M has no F_7^*, $M(K_5)^*$, or R_{10} minor.*

Proof. First we show that the listed minors cannot be present. Fourier–Motzkin elimination applied to one variable x_e of (1.4) and (1.5) reduces that defining system for $Q(M)$ to one for $Q(M/e)$. Setting $x_e=0$ in (1.4) and (1.5) produces a system for $Q(M\setminus e)$. Analogous statements obviously hold for $P(M)$. Thus $P(M)=Q(M)$ implies $P(N)=Q(N)$ for every minor of M, and our first claim follows once we prove that $P(N) \neq Q(N)$, for $N=F_7^*$, $M(K_5)^*$, or R_{10}.

$N=F_7^*$. The vector $x=\frac{2}{3}\cdot\mathbb{1}\in\mathbb{R}^7$ is in $Q(F_7^*)$ but not in $P(F_7^*)$ since it violates the facet defining inequality $\sum_{j=1}^{7} x_j \leq 4$ of $P(F_7^*)$.

$N=M(K_5)^*$. The vector $x=\frac{2}{3}\cdot\mathbb{1}\in\mathbb{R}^{10}$ is in $Q(M(K_5)^*)$ but not in $P(M(K_5)^*)$ as it violates the facet defining inequality $\sum_{j=1}^{10} x_j \leq 6$ of $P(M(K_5)^*)$.

$N=R_{10}$. Let A be the (5, 10) binary (nonstandard) representation matrix of R_{10} where each column has exactly three 1's. Suppose the columns are ordered lexicographically, i.e., for any two columns $A_{.j}$ and $A_{.k}$ with $j<k$, we have $A_{.j}$ lexicolarger than $A_{.k}$. Declare $\{1, 2, ..., 10\}$, the set of column indices of A, to be the ground set of R_{10}. Then straightforward checking shows that $x_2+x_8+x_{10}-\sum_{j\neq 2,8,10} x_j \leq 0$ defines a facet of $P(R_{10})$. Now the point $x\in\mathbb{R}^{10}$ defined by $x_2=x_8=x_{10}=\frac{3}{4}$ and $x_j=\frac{1}{4}$, $j\neq 2$, 8, 10, violates the above inequality, but is clearly in $Q(R_{10})$ since every cocircuit of R_{10} has cardinality of at least 4. Hence $P(R_{10})\neq Q(R_{10})$.

Now we prove that exclusion of F_7^*, $M(K_5)^*$, and R_{10} assures $P(M)=Q(M)$. Straightforward checking (with lengthy and tedious details though) establishes $P(M)=Q(M)$ for $M\in\{F_7, M(K_{3,3})^*, M(V_8)^*\}$. The same conclusion holds for any graphic M, say $M=M(G)$, since then $Q(M)$ is specified by the trivial inequalities and the inequalities of (1.8), and these also define $P(M)$ according to Edmonds and Johnson [7]. Concatenation of parts (a) and (b) of Theorem (2.5), followed by dualization, produces a statement which implies that any connected binary matroid M without the excluded minors F_7^*, $M(K_5)^*$, and R_{10} can be constructed by 2- and Y-sums from graphic matroids and copies of F_7, $M(V_8)^*$, and $M(K_{3,3})^*$. We know $P(N)=Q(N)$ for each matroid N used as a building block, and due to Theorems (4.1), (4.2), and (4.3), we conclude $P(M)=Q(M)$. ∎

The remaining two sections cover optimization and separation aspects of the polytopes $P(\cdot)$ and $Q(\cdot)$. In the next section we show that the separation problem for $Q(M)$ is solvable in polynomial time if by repeated Y-sum decomposition M can be reduced to a collection of matroids for which a special shortest cocircuit problem can be solved in polynomial time.

5. Separation for $Q(\cdot)$

In the separation problem for $Q(M)$ we are given a binary matroid M on a set E and a vector $y\in\mathbb{Q}^E$. We want to decide whether or not y is in $Q(M)$. In the case of a negative answer we also want to find an inequality from those defining $Q(M)$ violated by y. For convenience, we dualize this problem by defining $Q^*(M) := Q(M^*)$, i.e.,

$$Q^*(M) := \{x \in \mathbb{R}^E \mid 0 \leqslant x \leqslant 1;\ x(F) - x(C \setminus F) \leqslant |F| - 1$$
$$\text{for all circuits } C \text{ of } M \text{ and all } F \subseteq C \text{ with } |F| \text{ odd}\}.$$

Since $Q^*(M^*) = Q(M)$, from now on we only examine the separation problem for $Q^*(M)$ without loss of generality. Clearly $y \in Q^*(M)$ if and only if $0 \leqslant y \leqslant 1$ and $\sum_{h \in F} (1 - y_h) + \sum_{h \in C \setminus F} y_h \geqslant 1$, for all circuits C of M and for all $F \subseteq C$ with odd cardinality. The nontrivial part of the separation problem is thus subsumed by either one of the following two problems.

(5.1) Short Odd Circuit Problem $S(M, a, b)$. Given M and a pair (a_h, b_h) for each element h of M, where $a_h, b_h \geqslant 0$ and $a_h + b_h \geqslant 1$, find a circuit C and an odd cardinality subset $F \subseteq C$ such that the length of (C, F), defined as $\sum_{h \in F} a_h + \sum_{h \in C \setminus F} b_h$, is less than 1, or conclude that the length of every pair (C, F) is at least 1.

(5.2) Restricted Shortest Odd Circuit Problem $RS(M, a, b)$. With M, a, b, C, and F as in (5.1), find a pair (C, F) of minimal length, or conclude that the length of every pair (C, F) is at least 1.

Before we go on, we introduce a few conventions to simplify the discussion. First, we will always implicitly assume that any given vector pair (a, b) satisfies $a_h, b_h \geqslant 0$ and $a_h + b_h \geqslant 1$, for all $h \in E$. The **F-set** of a circuit C is the set F in the pair (C, F). Frequently the F-set is implicitly specified; the circuit C is then **odd** if $|F|$ is odd, and the **length** of C is the previously defined length of the pair (C, F), i.e., $\sum_{h \in F} a_h + \sum_{h \in C \setminus F} b_h$.

At times we take the symmetric difference of two circuits, say of C_1 and C_2 with F-sets F_1 and F_2. We invoke this operation only when each element $h \in C_1 \cap C_2$ is either in both sets F_1 and F_2 or in none of them. Thus, we may say that the cycle given by the symmetric difference of C_1 and C_2, has as F-set the symmetric difference of F_1 and F_2.

We now show that $S(M, a, b)$ may be solved if M is decomposable and if certain versions of $S(\cdot)$ and $RS(\cdot)$ can be solved for the components of M.

(5.3) Theorem. *Let M be a binary matroid. If M is a 2-sum $M_1 \oplus_e M_2$ (a Δ-sum $M_1 \oplus_\Delta M_2$), then the short odd circuit problem $S(M, a, b)$ can be solved by calling an algorithm for the restricted shortest odd circuit problem in M_1 three times (seven times) and by solving a short odd circuit problem $S(M_2, a^2, b^2)$ for M_2 once. In each of the cases the encoding lengths of the weight vectors a^i, b^i for M_i, $i = 1, 2$, are bounded by a polynomial in the encoding length of the vectors a and b of $S(M, a, b)$.*

Proof. We only prove the $\triangle$-sum case since the easier 2-sum case follows by analogous arguments. Thus suppose that $M = M_1 \oplus_\triangle M_2$, where $\triangle = \{e, f, g\}$, and that with the given vectors a and b we are to solve the short odd circuit problem $S(M, a, b)$. First we assign to M_1 and M_2 the given a- and b-values except for e, f, and g of the triangle $\triangle$ in M_1 and M_2. Next we solve a problem $RS(M_1, a^1, b^1)$ seven times, where each time we use different values for a_h^1 and b_h^1, $h \in \triangle$. We label the cases 0, $e1$, $e2$, $f1$, $f2$, $g1$, and $g2$, and let the solution triples (shortest odd circuit, F-set, length) be (C^0, F^0, l^0), (C^{e1}, F^{e1}, l^{e1}), (C^{e2}, F^{e2}, l^{e2}), etc.

Case 0. $a_h^1 = b_h^1 = 1$, for all $h \in \triangle$. If $l^0 < 1$, declare (C^0, F^0, l^0) to be the output for M and stop. If $l^0 \geqslant 1$, continue.

Case $h1$. $h = e, f$, or g: $a_h^1 = 0$, all other a^1- and b^1-values for the $\triangle$ of M_1 are 1. If $l^{h1} < 1$, assign l^{h1} as b_h^2 value to h of M_2. Otherwise assign $b_h^2 = 1$ to h of M_2.

Case $h2$. $h = e, f$, or g: $b_h^1 = 0$, all other a^1- and b^1-values for the $\triangle$ of M_1 are 1. If $l^{h2} < 1$, assign l^{h2} as a_h^2 value to h of M_2. Otherwise assign $a_h^2 = 1$ to h of M_2.

With these assignments all a^2- and b^2-values for M_2 are specified. Before we go to the next step of the proof, we want to establish a few inequalities involving the just computed a_h^2 and b_h^2 of $h \in \triangle$ in M_2, where in each case we assume $l^0 \geqslant 1$:

$$
\begin{aligned}
1 \geqslant a_h^2, b_h^2 \geqslant 0 \quad & \text{for all} \quad h \in \triangle, \\
a_h^2 + b_h^2 \geqslant 1 \quad & \text{for all} \quad h \in \triangle,
\end{aligned}
\tag{5.4.1}
$$

$$a_e^2 + a_f^2 + a_g^2 \geqslant 1, \tag{5.4.2}$$

$$a_e^2 + b_f^2 + b_g^2 \geqslant 1, \tag{5.4.3}$$

$$a_e^2 \leqslant a_f^2 + b_g^2, \tag{5.4.4}$$

$$b_e^2 \leqslant a_f^2 + a_g^2, \tag{5.4.5}$$

$$b_e^2 \leqslant b_f^2 + b_g^2. \tag{5.4.6}$$

There are more inequalities due to symmetry, but the listed ones will suffice. We now validate these inequalities.

(5.4.1). The lower and upper bounds on a_h^2 and b_h^2 obviously hold. If $a_h^2 + b_h^2 < 1$, then $l^{h1} + l^{h2} < 1$, $h \in C^{h1} \cap C^{h2}$, $h \in F^{h1}$, and $h \notin F^{h2}$. No other element $r \in C^{h1} \cap C^{h2}$ can be in the symmetric difference of F^{h1} and F^{h2} since then $l^{h1} + l^{h2} \geqslant a_r^1 + b_r^1 \geqslant 1$. But then there exists a short odd circuit in the symmetric difference of C^{h1} and C^{h2}, so $l^0 < 1$, a contradiction.

(5.4.2) and (5.4.3). This is proved similarly to (5.4.1), i.e., the contradicting $l^0 < 1$ is deduced if (5.4.2) or (5.4.3) is violated.

(5.4.4). In the nontrivial case $a_f^2 + b_g^2 < 1$. Then $\{e\}$ plus the symmetric difference of C^{f2} and C^{g1} contains a candidate circuit for case $e2$, and thus $a_e^2 \leqslant a_f^2 + b_g^2$.

(5.4.5) and (5.4.6). Are proved similarly to (5.4.4).

We now continue with the proof. By the original assumptions on the vectors a and b and by (5.4.1) we know that $a_r^2, b_r^2 \geqslant 0$ and $a_r^2 + b_r^2 \geqslant 1$, for all elements r of M_2. Now solve the short odd circuit problem $S(M_2, a^2, b^2)$. Five outcomes are possible, to be discussed in detail below. In each case the arguments make extensive use of the circuit results for the $\triangle$-sum of Section 2.

Outcome 1. M_2 has no short odd circuit. We claim that M then has no short odd circuit either. If M does, then any such circuit must contain elements of M_1 and of M_2, i.e., without loss of generality $C = (C_1 \cup C_2)\setminus\{e\}$, where for $i = 1$ and 2, C_i is a circuit of M_i that contains e and no other element of $\triangle$. Now the length of $C_1\setminus\{e\}$ as subset of C is at least $a_e^2 = l^{e2}$ or $b_e^2 = l^{e1}$, depending on whether $C_1\setminus\{e\}$ contains an odd or even number, respectively, of F-elements of C. Then C_2 is a short odd circuit of M_2, provided we declare e to be an F-element, if and only if $C_1\setminus\{e\}$ has an odd number of F-elements of C.

Outcome 2. $\{e, f, g\}$ is a short odd circuit of M_2. This is not possible by (5.4.2) and (5.4.3).

Outcome 3. M_2 has a short odd circuit C that contains e but not f or g of $\triangle$. If e is (is not) an F-element of C, then $(C^{e2} \cup C)\setminus\{e\}$ $((C^{e1} \cup C)\setminus\{e\})$ is a disjoint union of at most two circuits of M with an odd number of F-elements in total. The length of this disjoint union is that of C, so one readily extracts a short odd circuit for M.

Outcome 4. M_2 has a short odd circuit C that contains f and g but not e. If exactly one (none or both) of f, g is (are) F-elements of C, then $\tilde{C} = (C^{e2}\setminus\{e\}) \cup (C\setminus\{f, g\})$ $(\tilde{C} = (C^{e1}\setminus\{e\}) \cup (C\setminus\{f, g\}))$ is a circuit of M with odd number of F-elements. By (5.4.4)–(5.4.6) the length of $\tilde{C}$ cannot exceed that of C, so $\tilde{C}$ is a short odd circuit for M.

Outcome 5. M_2 has a short odd circuit C that does not involve e, f, or g. Then C is also a short odd circuit for M.

Thus for each outcome we either produce a short odd circuit for M, or conclude that none exists. ▌

For Theorem (5.3) to be useful, one must be able to solve the restricted shortest odd circuit problem for interesting classes of binary matroids. The following lemmas demonstrate that this is indeed so. The first result is due to Barahona and Mahjoub [4]. For completeness we sketch the proof.

(5.5) LEMMA. *The restricted shortest odd circuit problem can be solved for graphic matroids in polynomial time.*

Proof. Find a graph $G=(V, E)$ for the given graphic matroid M in polynomial time using any one of several algorithms (see, e.g., Fujishige [8] or Bixby and Wagner [5]). With G and the given vectors a and b at hand, create an (undirected) graph H from two copies of G, say G_1 and G_2, as follows. If edge ij occurs in G, then add an edge from node i of G_1 to node j of G_2, and an edge from node j of G_1 to node i of G_2. To the edges of H assign the following weights. If the edge has both endpoints within G_1 or within G_2, then assign b_h of the corresponding edge of G. If the edge connects a node i of G_1 with a node j of G_2, then assign value a_h, where h is the edge connecting nodes i and j in G.

Clearly a shortest odd circuit C of G, say including node v, has the same length as a shortest path in H from v of G_1 to v of G_2. Conversely, let W_v be a shortest path in H from v in G_1 to v in G_2, say with length l_v. If $\min_{v \in V} l_v = l_{v^*} < 1$, then W_{v^*} immediately yields a shortest odd circuit for G. ▌

The next result for cographic matroids follows from Padberg and Rao [13]. Again we sketch a proof for completeness.

(5.6) LEMMA. *The restricted shortest odd circuit problem can be solved for cographic matroids in polynomial time.*

Proof. First we determine a graph G for the dual matroid of M. We then replace each edge h of G by a series class of two edges, say h_1 and h_2, and assign weight a_h to h_1 and weight b_h to h_2. Let $H=(V, E)$ be the resulting graph, and define T to be the set of nodes i of H which have an odd number of edges of type h_1 (i.e., with weight a_h) incident. Clearly, $|T|$ is even. Then solve the T-**cut problem** for H, i.e., find a cut $\delta(W) \subseteq E$ of minimum total weight such that $|W \cap T|$ is odd. The algorithm of Padberg and Rao [13] produces such a cut $D=\delta(W)$ in polynomial time. If the total weight of D is at least 1, then M has no short odd circuit. Otherwise $C=\{h \in E(G) \mid h_1 \text{ or } h_2 \in D\}$ and $F=\{h \in C \mid h_1 \in D\}$ define a pair (C, F) with minimal length and odd $|F|$. Note that for any given h at most one of h_1 and h_2 can be in D if the total weight of D is less than 1, due to the essential conditions $a_h, b_h \geqslant 0$ and $a_h + b_h \geqslant 1$, for all h. Thus the length of C is simply the sum of the weights of the edges in D. ▌

Theorem (5.3) and Lemmas (5.5) and (5.6) permit the following conclusion.

(5.7) THEOREM. *Let $\mathcal{N}$ be a class consisting of graphic matroids, of cographic matroids, and of a finite number of matroids that are neither* *graphic nor cographic. Let $\mathcal{M}$ be a class of binary matroids such that for each $M \in \mathcal{M} \setminus \mathcal{N}$ one can determine in polynomial time a 2-sum $M = M_1 \oplus_e M_2$ or a $\triangle$-sum $M = M_1 \oplus_{\triangle} M_2$, where $M_1 \in \mathcal{N}$ and $M_2 \in \mathcal{M}$. Then the separation problem for $Q^*(M)$ can be solved in polynomial time for each $M \in \mathcal{M}$.*

Proof. By Lemmas (5.5) and (5.6) the restricted shortest odd circuit problems can be solved for each graphic and cographic $M \in \mathcal{N}$ in polynomial time. For the finite set of additional matroids in $\mathcal{N}$, problem (5.2) can be solved in constant time. Thus by Theorem (5.3) and induction, the short odd circuit problem can be solved in polynomial time for each $M \in \mathcal{M} \setminus \mathcal{N}$. This implies the polynomial time solvability of the separation problem for $Q^*(M)$. ▮

(5.8) COROLLARY. *Let $\mathcal{M}$ be the class of connected binary matroids each of which does not have at least one of F_7, F_7^* as a minor. The for all $M \in \mathcal{M}$ the separation problem for $Q(M)$ and $Q^*(M)$ can be solved in polynomial time.*

Proof. To prove the $Q^*(M)$ case, we apply Theorem (5.7) with $\mathcal{N} = \{$graphic matroids, cographic matroids, R_{10}, $F_7\}$ if M does not contain F_7^*. Exchange the roles of F_7 and F_7^* if M does not contain F_7. By Theorem (2.5) the decomposition condition of Theorem (5.7) can be satisfied, and hence the separation problem for $Q^*(M)$ can be solved in polynomial time for all $M \in \mathcal{M}$. Since $\mathcal{M}$ is closed under dualization the above proof also settles the $Q(M)$ case. ▮

Note that the polynomial time algorithm given in the proof of Corollary (5.8) is combinatorial since this is so for each subroutine and that a suitable implementation produces a practically usable method.

In the final section we describe algorithms to solve optimization problems over $P(\cdot)$.

6. OPTIMIZATION OVER $P(\cdot)$

There are a number of interesting applications that can be viewed as optimization problems of type (1.2). Two examples are the ground-state problem of spin glasses (a problem in the theory of magnetism) and the via minimization problem in VLSI and printed circuit board design. These problems can be phrased as max-cut problems in graphs—see, for instance, the paper Barahona, Grötschel, Jünger, and Reinelt [3], where both applications are outlined. Problem (1.2) contains $\mathcal{NP}$-hard special cases such as the max-cut problem, so there is little hope for a good algorithm in the general case. Two ways of algorithmic attacks on (1.2) are of special interest. In the first approach one restricts the class of binary matroids to a smaller class for which a combinatorial, special purpose polynomial time algorithm can be designed. In the second scheme one solves (1.2) via the problem max $c^T x$, $x \in P(M)$ of (1.3) using linear programming techniques. In the latter method, an LP-relaxation of (1.3) is chosen and solved with a cutting plane procedure with the hope (and in some cases the guarantee) that each optimum vertex solution of this LP is integral and thus a solution of (1.2).

For the max-cut problem both approaches have been successful. For the class of planar graphs, Orlova and Dorfmann [12] and Hadlock [11] have found a reduction of the max-cut problem to at most $|V|^2$ shortest path problems and one perfect matching problem. For the (more general) class of graphs not contractible to K_5, Barahona [1] has designed a combinatorial decomposition algorithm that runs in polynomial time. For a class of toroidal graphs with a universal node (the max-cut problem is $\mathcal{NP}$-hard for this class) Barahona, Grötschel, Jünger, and Reinelt [3] have implemented a cutting plane algorithm that shows very good computational results empirically. In this paper we generalize both approaches, and also unify some of the known algorithms for (1.2).

Let us start with the LP-approach. We consider the LP-relaxation

$$\max\{c^T x \mid x \in Q(M)\} \tag{6.1}$$

of problem (1.3). It follows from Barahona and Grötschel [2] that the number of facets of $Q(M)$ may grow exponentially with the number of elements of M. Thus there is no way to encode the constraints defining $Q(M)$ in polynomial space. Grötschel, Lovász, and Schrijver [10], however, have shown that the number of constraints is not important: (6.1) can be solved in polynomial time if and only if the separation problem for $Q(M)$ can be solved in polynomial time. We do not know whether the separation problem for $Q(M)$ can be solved in polynomial time for all binary matroids M, but we have shown in Section 5 that for a number of interesting classes of binary matroids such algorithms exist. Among them are

— graphic matroids (see Lemma (5.5)),
— cographic matroids (see Lemma (5.6)),
— matroids without F_7 or F_7^* minor (see Corollary (5.8)).

Indeed, by Theorem (5.7) a polynomial time separation algorithm for $Q(M)$ exists for all matroids M that belong to a class of binary matroids that is built up by taking 2-sums and Δ-sums recursively, where in each step one component matroid is graphic, cographic, or belongs to a finite set of binary matroids—provided such 2- and $\triangle$-decompositions can be detected in polynomial time. By the result of Grötschel, Lovász, and Schrijver [10], any linear function can then be optimized over $Q(M)$ in polynomial time, so Theorem (5.7) has the following corollary.

(6.2) COROLLARY. *For any objective function c, the linear program* $\max\{c^T x \mid x \in Q(M)\}$ *can be solved in polynomial time if*

(a) *M is a matroid without F_7 or F_7^* minor or*

(b) *M has the sum of circuits property or*

(c) *M is cographic or*

(d) *M is graphic.*

By Theorem (1.7), $P(M) = Q(M)$ if and only if M has the sum of circuits property. So in this case, the polynomial time solvability of (6.1) implies the polynomial time solvability of (1.3). Thus we have

(6.3) COROLLARY. *The cycle problem (1.2) can be solved in polynomial time for the matroids with the sum of circuits property.*

Since a cographic matroid has the sum of circuits property if and only if its associated graph is not contractible to K_5, Corollary (6.3) implies Barahona's result that the max-cut problem for graphs not contractible to K_5 is solvable in polynomial time. It also implies that the Eulerian subgraph problem can be solved in polynomial time for any graph. In case M does not contain an F_7^* minor, the facets of $Q(M)$ are also facets of $P(M)$ (this follows from Barahona and Grötschel [2]). Moreover, by Corollary (5.8) the separation problem for $Q(M)$ is polynomially solvable. So for binary matroids without F_7^* minor, the LP (6.1) should furnish a tight and computationally tractable LP-relaxation of (1.3) and should provide a good starting basis for cutting plane algorithms.

The algorithmic results for optimization over $Q(M)$ (resp. $P(M)$) described above have one drawback. They are all based on the ellipsoid method and thus are—in a straightforward implementation—of doubtful practical relevance. However, we can also use decomposition techniques to produce polynomial time combinatorial algorithms for the solution of (1.2) (resp. (1.3)), as shown in the next theorem. Its proof is an adaption of the proof given in Bahahona [1] for the max-cut case.

(6.4) THEOREM. *Let M be a binary matroid that is a 2-sum or Y-sum of two binary matroids M_1 and M_2. If $M = M_1 \oplus_e M_2$ ($M = M_1 \oplus_Y M_2$), then the cycle problem for M can be solved by calling an algorithm for the maximum weight cycle problem in M_1 two times (four times) and by calling once such an algorithm for M_2.*

In both cases, the encoding lengths of objective functions for the maximum weight cycle problems to be solved for M_1 and M_2 are bounded by a polynomial in the encoding length of the objective function of the original problem.

Proof. Let $c_f \in \mathbb{Q}, f \in E$, be the weights of the elements of M and define $B := (|E|+1)\max\{|c_f|+1 \mid f \in E\}$. Suppose $M = M_1 \oplus_e M_2$. Define objective functions c^1 and c^2 for M_1 and M_2 by setting $c^1_f := c_f$ for all f in M_1 different from e and $c^2_f := c_f$ for all f in M_2 different from e. Solve the maximum weight cycle problem for M_1 once with $c^1_e := B$ and once with $c^1_e := -B$. Let v_1 (resp. $\bar{v}_1$) be the optimum values and C_1 (resp. $\bar{C}_1$) be optimum solutions of these problems. Clearly, in the first case the optimum solution must contain e, while it does not in the second one. Now set $c^2_e := v_1 - B - \bar{v}_1$ and solve the maximum weight cycle problem for M_2 with objective function c^2. Let v_2 be the optimum value and C_2 be an optimum solution. Obviously, the optimum value of the maximum weight cycle problem for M is equal to $v_2 + \bar{v}_1$. If $e \in C_2$, then $(C_1 \cup C_2)\backslash\{e\}$ is an optimum solution, and if $e \notin C_2$, then $\bar{C}_1 \cup C_2$ is an optimum solution.

Now suppose $Y = \{r, s, t\}$ and $M = M_1 \oplus_Y M_2$. Define objective functions c^1 and c^2 for M_1 and M_2 as before using the weights in M, except for the elements of Y. For M_1 we consider the following cases:

1. $c^1_r := c^1_s := B,\ c^1_t := -B,$
2. $c^1_r := c^1_t := B,\ c^1_s := -B,$
3. $c^1_s := c^1_t := B,\ c^1_r := -B,$
4. $c^1_r := c^1_s := c^1_t := -B.$

Run the algorithm for M_1 four times with the weights as specified above and denote by v_{rs}, v_{rt}, v_{st}, and $\bar{v}$, respectively, the objective function values. Let C_{rs}, C_{rt}, C_{st}, and $\bar{C}$ be optimum solutions of these problems. By the choice of the weights we have $x, y \in C_{xy}$, $z \notin C_{xy}$ for all choices $x, y, z \in \{r, s, t\}$, and $r, s, t \notin \bar{C}$. To solve the maximum weight cycle problem for M_2 set

$$c^2_r := (v_{rt} + v_{rs} - v_{st} - \bar{v} - 2B)/2,$$

$$c^2_s := (v_{rs} + v_{st} - v_{rt} - \bar{v} - 2B)/2,$$

$$c^2_t := (v_{rt} + v_{st} - v_{rs} - \bar{v} - 2B)/2.$$

Let v_2 be the optimum value and C_2 be an optimum solution for M_2. It is straightforward to verify that the maximum weight of a cycle in M is equal to $v_2 + \bar{v}$ and that an optimum solution for M is given by $(C_{rs} \cup C_2)\backslash\{r, s\}$ if $r, s \in C_2$, by $(C_{rt} \cup C_2)\backslash\{r, t\}$ if $r, t \in C_2$, by $(C_{st} \cup C_2)\backslash\{s, t\}$ if $s, t \in C_2$ and finally by $\bar{C} \cup C_2$ if $Y \cap C_2 = \varnothing$.

The statement in the theorem about the encoding lengths of the objective functions follows from the above construction. ∎

Theorem (6.4) plus decomposition and the matching algorithm give combinatorial optimization algorithms for several interesting classes of binary matroids as follows.

(6.5) THEOREM. *Let $\mathcal{N}$ be a class consisting of graphic matroids and of a finite number of matroids that are non-graphic. Let $\mathcal{M}$ be a class of binary matroids such that for each $M \in \mathcal{M} \setminus \mathcal{N}$ one can determine in polynomial time a 2-sum $M_1 \oplus_e M_2$ or a Y-sum $M_1 \oplus_Y M_2$ for M, where $M_1 \in \mathcal{N}$ and $M_2 \in \mathcal{M}$. Then there is a combinatorial algorithm that solves the maximum weight cycle problem for all matroids in $\mathcal{M}$ in polynomial time.*

Proof. The matching algorithm handles the case of graphic M, while all other situations are processed by Theorem (6.4) and induction. ∎

(6.6) COROLLARY. *For any objective function c, the maximum weight cycle problem can be solved by a combinatorial polynomial time algorithm if*

(a) *M is a matroid without F_7 and $M(K_5)^*$ minor or*

(b) *M is a matroid without F_7^* and $M(K_5)^*$ minor or*

(c) *M has the sum of circuits property, or*

(d) *M is cographic and has no $M(K_5)^*$ minor, or*

(e) *M is graphic.*

Proof. By the dualized version of Theorem (2.5) any M of (a)–(e) is graphic or a 2-sum or Y-sum where M_1 is graphic or equal to F_7, F_7^*, $M(K_{3,3})^*$, $M(V_8)^*$, or R_{10}, and where M_2 is isomorphic to a minor of M. The decomposition can be detected by combinatorial polynomial time algorithms, so the conclusion follows from Theorem (6.6). ∎

We remark that Truemper [19] contains several 2-sum and 3-sum decomposition theorems for graphs, and thus for cographic matroids. These results may be used to prove a number of additional corollaries of Theorem (6.5).

7. CONCLUSIONS AND EXTENSIONS

As shown in Barahona and Grötschel [2] and in this paper, the notion of cycles in binary matroids provides a general conceptual framework for a number of different combinatorial optimization problems. Both structural results (e.g., composition and decomposition, polyhedral descriptions) and algorithmic approaches (combinatorial decomposition techniques, cutting plane methods via separation algorithms) carry over from the known special cases and have been unified under one roof.

There are some extensions of this approach possible. For instance, suppose $\hat{B}$ is any (m, n)-matrix with 0/1-entries and $b \in \{0, 1\}^m$. Let M be the binary matroid associated with $\hat{B}$, and define

$$P(M, b) := \text{conv}\{x \in \{0, 1\}^n \mid \hat{B}x \equiv b \bmod 2\}.$$

Thus $P(M, 0)$ is the cycle polytope $P(M)$ considered before. It was shown in Barahona and Grötschel [2] that any optimization problem $\max\{c^Tx \mid x \in P(M, b)\}$ can be transformed into an optimization problem $\max\{\bar{c}^Tx \mid x \in P(M, 0)\}$ using an arbitrary 0/1-vector $y \in P(M, b)$. One can always find such a vector y easily or prove that no such vector exists. So for all classes of matroids for which the optimization problem for $P(M, 0)$ can be solved in polynomial time, the optimization problem for $P(M, b)$ can also be solved in polynomial time. This latter class of problems contains such interesting special cases as the T-join problem in graphs: Given a graph G, an even cardinality subset T of nodes of G, and weights c_f on the edges f of G, find a subgraph of G of minimal total weight that has odd node degree exactly for the nodes in T.

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