Hypohamiltonian Digraphs*)

M. Grötschel, Bonn and Y. Wakabayashi, São Paulo

Abstract: *Hypohamiltonian digraphs G are digraphs which are not hamiltonian, i.e. do not contain a hamiltonian circuit, but have the property that for every node v in G the node-deleted digraph G−v is hamiltonian. We show the existence of a simple class of hypohamiltonian digraphs and define three methods with which new hypohamiltonian digraphs can be constructed from known ones. We obtain hypomamiltonian digraphs of all orders $n \geq 6$.*

1. Introduction and Notation

Hypohamiltonian graphs have been intensively studied during the last years, c.f. [1], [2], [3], [6], [7], and a lot of classes and unexpected properties of these quite unusual graphs have been discovered. Although - at first sight - it does not seem that these graphs might be useful for practical applications, it was found out recently, c.f. [4], that hypohamiltonian graphs are one of the reasons why the

*) The authors were partially supported by bilateral GMD-CNP1-contract, IME-USP São Paulo, and SFB 21 Bonn.

symmetric travelling salesman problem is difficult. It was shown that many of these graphs induce facets of the symmetric travelling salesman polytope, a fact that indicates that it is very unlikely that a complete linear characterization of this polytope can ever be found.

Trying to show that the asymmetric travelling salesman polytope is highly complex too, we felt that hypohamiltonian digraphs would be the appropriate objects to start with. Although the concept of hypohamiltonian graphs can be easily carried over to digraphs, this kind of digraphs does not seem to have attracted much attention in the literature. Therefore, we report in this paper about various methods to construct hypohamiltonian digraphs and prove that such digraphs of order n exist if and only if $n \geq 6$. In a subsequent paper [5] we shall use these digraphs in order to show that the asymmetric travelling salesman polytope has very complicated facets.

We use the same graph theoretical notation with respect to undirected graphs as [3]. Graphs are denoted by $G = [V,E]$. A graph G is called <u>hypohamiltonian</u> if G is not hamiltonian but $G-v$ is hamiltonian for all $v \in V$.

A <u>digraph</u> $G=(V,E)$ consists of a finite set V of <u>nodes</u> and a finite set E of ordered pairs of distinct elements of V called <u>arcs.</u> If $e=(u,v) \in E$ then u and v are called <u>endnodes</u> of e; u is said to be the <u>initial node</u> and v the <u>terminal node</u> of e; u and v are called <u>neighbours</u> if (u,v) or $(v,u) \in E$. $|V|$ is the <u>order</u> of G. A nonempty set of arcs $P = \{(v_1,v_2),(v_2,v_3),\dots,(v_{k-1},v_k)\} \subset E$ is called a <u>path</u> of <u>length</u> $k-1$ if $v_i \neq v_j$ for $i \neq j$, and is denoted by $[v_1,v_2,\dots,v_k]$. If $(v_k,v_1) \in E$ then $C := P \cup \{(v_k,v_1)\}$ is called a circuit of lenght k, denoted by $<v_1,v_2,\ldots,v_k>$. A circuit (path) of lenght $|V|$ (length $|V|-1$) is called hamiltonian. Two paths P, Q are called a node covering pair of paths in a digraph G if they are nonempty, node-disjoint and contain all nodes of G. G-v is the digraph with node set $V-\{v\}$ and all arcs in E that do not contain the node v. A digraph G = (V,E) is called hamiltonian (traceable) if G contains a hamiltonian circuit (path). G=(V,E) is called hypohamiltonian if G is not hamiltonian but the digraph G-v is hamiltonian for all $v \in V$. If $(u,v) \in E$ then G-(u,v) is the digraph G without the arc (u,v); if $(u,v) \notin E$ then G + (u,v) denotes the digraph obtained from G by adding (u,v) to the arc set E. If a digraph G=(V,E) has a property π then G is called maximal with respect to π if G+(u,v) does not have property π for all pairs $(u,v) \notin E$; G is called minimal with respect to π if G-(u,v) does not have property π for all $(u,v) \in E$. $d^+(v)$ ($d^-(v)$) is the number of arcs having v as initial node (terminal node), $d(v)=d^+(v)+d^-(v)$; $d^+(v)$ is called outdegree of v, $d^-(v)$ indegree of v, d(v) degree of v. N(v) is the set of neighbours of v.

2. Trivial Properties, Directing Hypohamiltonian Graphs

Considering the definition of hypohamiltonian digraphs we can immediately conclude that such digraphs have to be 3-connected (i.e. at least 3 nodes have to be deleted in order to make the resulting digraphs disconnected), and they have to be strongly connected (for every two nodes u,v there is a path from u to v and from v to u). Furthermore, the outdegree and the indegree of any node are at least two, and every node has to have at least three neighbours. A node x satisfying these minimum requirements, i.e., such that $d^+(x) = d^-(x) = 2$, $|N(x)| = 3$ is called distinguished.

We show below the neighbourhood structure of x:

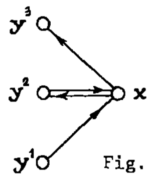

Fig. 2.1

In the sequel distinguished nodes will play an important role, and we will assume that for any distinguished node v the neighbours u^1, u^2, u^3 are numbered as shown above.

Lemma 2.1. *Let $G=(V,E)$ be a hypohamiltonian digraph of order n. Then $d(v) \leq n-1$ for all $v \in V$.*

Proof: Suppose there is a node $v \in V$ such that $d(v) \geq n$. Let $C=\langle v_1, v_2, \dots, v_{n-1}\rangle$ be a hamiltonian circuit in $G-v$. If $(v_i, v) \in E$ then $(v, v_{i+1}) \notin E$, otherwise $\langle v_1, \dots, v_i, v, v_{i+1}, \dots, v_{n-1}\rangle$ would be a hamiltonian circuit in G.

Assume there are $k \geq 1$ nodes which are not neighbours of v, then there have to be at least $k+1$ nodes $w \in V$ which are doubly linked to v (i.e. $(v,w),(w,v) \in E$). In this case there exists a path $[v_p, \dots, v_q]$, $p \neq q$, on C where all nodes v_i, $p \leq i \leq q$, are neighbours of v such that v_p and v_q are doubly linked to v. As $(v_p, v) \in E$ we have $(v, v_{p+1}) \notin E$, i.e. $(v_{p+1}, v) \in E$, hence $(v, v_{p+2}) \notin E$, i.e. $(v_{p+2}, v) \in E$. Continuing this reasoning we obtain $(v, v_q) \notin E$. Contradiction!

If follows that all nodes must be neighbours of v. As $d(v) \geq n$ there is one node, say v_1, which is doubly linked to v. As above $(v_1, v) \in E$ implies $(v, v_2) \notin E$, thus $(v_2, v) \in E$, and finally $(v_{n-1}, v) \in E$. But then $(C-\{(v_{n-1}, v_1)\}) \cup \{(v_{n-1}, v), (v, v_1)\}$ is a hamiltonian circuit in G. Contradiction! □

Later examples show that the bound given above is best possible, i.e. there are indeed nodes with degree $n-1$.

One way to find hypohamiltonian digraphs is to check whether hypohamiltonian graphs can be appropriately directed such that the defining properties (in the directed sense) are maintained. The simplest way to direct a graph $G=[V,E]$ is to substitute the two arcs (u,v) and (v,u) for every edge $\{u,v\} \in E$. Let us call such a digraph trivially directed and denote it by $\vec{G}=(V,\vec{E})$. Then the following obviously holds:

Theorem 2.2.

a) *Every trivially directed hypohamiltonian graph is a hypohamiltonian digraph.*

b) *If* $\vec{G}=(V,\vec{E})$ *is a trivially directed hypohamiltonian graph, then the digraph* $\vec{G}-(u,v)$ *is hypohamiltonian for every arc* $(u,v) \in \vec{E}$. □

As there exist hypohamiltonian graphs of order 10, 13, 15, 16, 18 and larger (c.f. [7]) there are also hypohamiltonian digraphs of these orders.

Statement b) of theorem 2.2 says that none of the trivially directed hypohamiltonian graphs is minimally hypohamiltonian. Also, this statement is not very strong as one can, in general, leave out much more of the arcs of a trivially directed graph such that the resulting digraph is still hypohamiltonian. One example is the following modification of the Petersen graph:

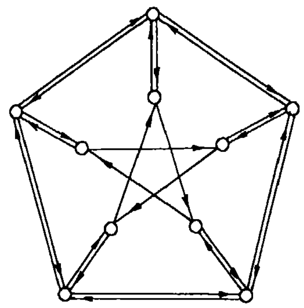

Fig. 2.2

which is a minimal hypohamiltonian digraph.

It is easy to see that for every hypohamiltonian digraph $G=(V,E)$ the reverse digraph obtained from G by replacing each arc (u,v) by the reverse arc (v,u). is also hypohamiltonian, and possibly not isomorphic to G.

As digraphs seem to allow more constructions than graphs with respect to the properties we are looking for, it can be expected that there are hypohamiltonian digraphs which are not trivially directed or simple modifications of these digraphs, hence a richer variety of these strange digraphs seems to be likely to exist. This will be verified in the next sections.

3. Basic Hypohamiltonian Digraphs

Some of the following simply structured digraphs turn out to be minimal hypohamiltonian digraphs and will serve as the building blocks for further constructions.

Definition 3.1. *Let* $p \geq 2$, *then the digraph having node set*

$$V=\{a_1,a_2,\ldots,a_p,b_1,b_2,\ldots,b_p\},$$

and arc set

$$E=\{(a_i,a_{i+1}),(b_i,b_{i+1}),i=1,\dots,p-1\}\cup\{(a_p,a_1),(b_p,b_1)\}$$
$$\cup\{(a_i,b_i),(b_i,a_i),\ i=1,\dots,p\},$$

is called Marguerite *and is denoted by* M_p *(c.f. Fig. 3.1). If* p *is odd then it is called odd Marguerite, otherwise even Marguerite.* □

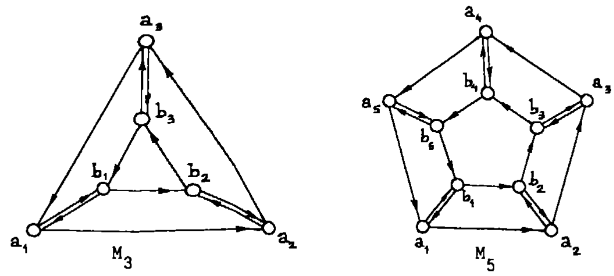

Fig. 3.1

Theorem 3.2.

a) Even Marguerites are hamiltonian.

b) Odd Marguerites are minimally hypohamiltonian.

c) All Marguerites are traceable.

Proof:

a) A hamiltonian circuit is $\langle a_1,b_1,b_2,a_2,a_3,b_3,\dots,a_{p-1},b_p,a_p\rangle$.

b) Suppose there is a hamiltonian circuit C in M_p, p odd. Let $[a_i,a_{i+1},\dots,a_k]$ be the longest path in C containing nodes a_j only. Then C contains the arcs (b_i,a_i), (a_k,b_k). The node b_{i+1} can be reached only through the arcs (b_i,b_{i+1}) or (a_{i+1},b_{i+1}). If $k>i+1$, then C cannot contain either of these arcs, and hence not the node b_{i+1}, contradicting the hamiltonicity. Therefore, if there is a hamiltonian circuit in M_p it is composed of paths of the type $[a_i,b_i,b_{i+1},a_{i+1},a_{i+2}]$. As p is odd a concatenation of such paths cannot form a hamiltonian circuit.

$M_p - a_1$ contains the hamiltonian circuit $\langle b_1,b_2,a_2,a_3,b_3,b_4,\ldots,a_p,b_p\rangle$.
By symmetry M_p-v contains a hamiltonian circuit for any $v \in V$.

If we drop any arc (u,v) in M_p then $d^+(u) = 1$ holds in the resulting digraph, and therefore $M_p - (u,v)$ cannot be hypohamiltonian.

c) Follows from a) and b). □

As we may add any of the reverse arcs (b_{i+1},b_i) or (a_{i+1},a_i) to an odd Marguerite M_p without losing the defining properties, odd Marguerites are not maximal hypohamiltonian digraphs. Adding for instance the arcs $(a_1,a_3),(b_1,b_3)$ to M_3 we obtain a maximal hypohamiltonian digraph the nodes a_1,a_3,b_1,b_3 of which have the maximal possible degree 5.
It is, however, not possible to add one of the reverse circuits $\langle b_p,b_{p-1},\ldots,b_1\rangle$ or $\langle a_p,a_{p-1},\ldots,a_1\rangle$ because then a hamiltonian circuit would be created. This shows that odd Marguerites are not modifications of trivially directed hypohamiltonian graphs.

Corollary 3.3. *There are hypohamiltonian digraphs of order* $4k+2$, $k \geq 1$. □

In order to construct new hypohamiltonian digraphs from known ones it is necessary to have such digraphs of small order available. The following digraphs of order 8,9,12 are hypohamiltonian, yet do not have as nice generalizations as the digraph M_3 from which the Marguerites were obtained.

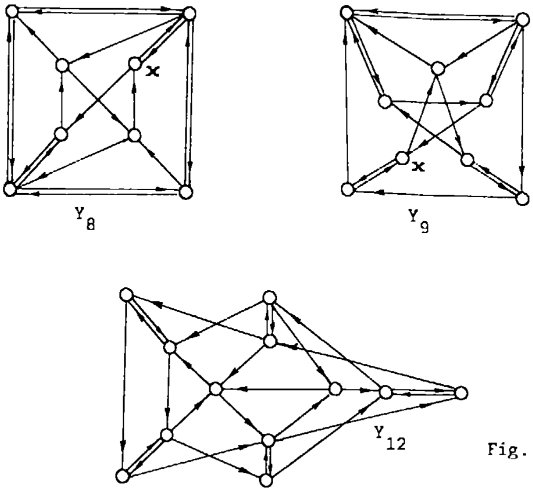

Fig. 3.2

The two defining properties can be checked by simple enumeration.

4. Constructing Hypohamiltonian Digraphs from Known Ones.

Construction HH1: *Let* G_1 *and* G_2 *be two disjoint hypohamiltonian digraphs which have the following "local" structures*

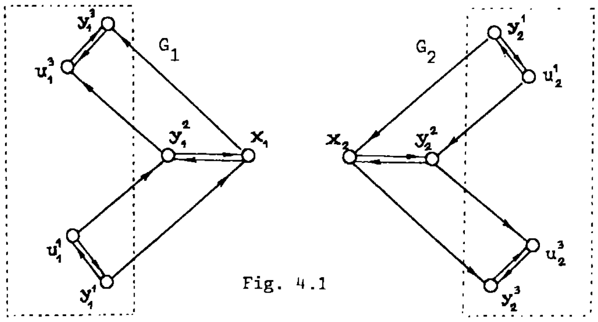

Fig. 4.1

where x_1, y_1^2, *and* x_2, y_2^2 *are distinguished nodes in* G_1, G_2 *resp.* *(The nodes* y_1^1, u_1^1, y_1^3, u_1^3 *and* y_2^1, u_2^1, y_2^3, u_2^3 *may (and must) have other neighbours in* G_1, G_2 *resp.) Define* H_i *to be* $G_i - x_i$, $i=1,2$. *We add the* *digraphs* H_1 *and* H_2 *and identify the nodes* y_1^2 *and* y_2^2 *calling this node* z. *Furthermore, we add the arcs* (u_2^1,y_1^3), (y_2^1,u_1^3), (y_1^1,u_2^3), (u_1^1,y_2^3) *and call this digraph* G. *This construction results in the following local structure of* G *(new arcs are denoted by thick lines):*

G

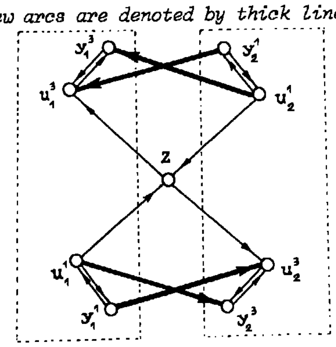

Fig. 4.2

Theorem 4.1. *Given two digraphs* G_1, G_2 *with the above properties then the digraph* G *obtained by Construction HH1 is hypohamiltonian.*

Proof:

a) Suppose G contains a hamiltonian circuit C. As C has to contain the node z (which is equal to y_1^2 in G_1 and y_2^2 in G_2) one of the following cases must be satisfied.

a_1) C contains the path $[u_2^1,z,u_1^3]$. Then C cannot contain either of the arcs (u_2^1,y_1^3) and (y_2^1,u_1^3). Hence C contains a hamiltonian path $P=[z,\ldots,v]$ in H_1 where $v=u_1^1$ or $v=y_1^1$.

a_{11}) If $v=y_1^1$ then $P\cup[y_1^1, x_1, y_1^2]$ is a hamiltonian circuit in G_1. Contradiction.

a_{12}) If $v=u_1^1$ then there is a hamiltonian path $P'=[y_2^3,\ldots,z]$ in H_2. But then $P'\cup[y_2^2, x_2, y_2^3]$ is a hamiltonian circuit in G_2. Contradiction.

a_2) If C contains the path $[u_1^1,z,u_2^3]$ a contradiction follows similarly to case a_1).

a_3) C contains the path $[u_2^1,z,u_2^3]$. Then C contains the arcs (y_2^1,u_1^3) and (u_1^1,y_2^3). Hence there is a hamiltonian path $P=[y_2^3,\ldots,y_2^1]$ in H_2. But then $P\cup[y_2^1,x_2,y_2^3]$ is a hamiltonian circuit in G_2. Contradiction.

a_4) If C contains the path $[u_1^1,z,u_1^3]$ a contradiction follows like in case a_3).

b) We have to show that $G-v$ is hamiltonian for all nodes v.

b_1) $v=z$. The hamiltonian circuit C_2 in G_2-x_2 contains the path $[u_2^1,y_2^2,u_2^3]$. The hamiltonian circuit C_1 in $G_1 - y_1^2$ contains the path $[y_1^1,x_1,y_1^3]$. We delete these paths in C_1 and C_2 and add the arcs (u_2^1,y_1^3) and (y_1^1,u_2^3) obtaining a hamiltonian circuit in $G-z$.

b_2) $v=y_1^1$. The hamiltonian circuit C_2 in $G_2 - x_2$ contains the arc (u_2^1,y_2^2). The hamiltonian circuit C_1 in $G_1 - y_1^1$ contains the path $[y_1^2,x_1,y_1^3]$. We delete this arc and the path and add the arc (u_2^1,y_1^3) obtaining a hamiltonian circuit in $G-y_1^1$.

b_3) $v=u_1^1$. The hamiltonian circuit C_1 in $C_1-u_1^1$ contains the path $[y_1^1,x_1,y_1^2]$. The hamiltonian circuit C_2 in G_2-x_2 contains the path $[u_2^1,y_2^2,u_2^3]$. Delete these paths and add (y_1^1,u_2^3), (u_2^1,z) which gives a hamiltonian circuit in $G - u_1^1$.

b_4) For $v=u_1^3$ and $v=y_1^3$ we get hamiltonian circuits in a similar way.

b_5) v in G_1 but different from the nodes considered above. G_1-v contains a hamiltonian circuit C_1, G_2-x_2 contains a hamiltonian circuit C_2. There are three ways to pass x_1 in G_1-v:

b_{51}) C_1 contains $[y_1^1,x_1,y_1^3]$. Delete this path from C_1 and the path $[u_2^1,y_1^2,u_2^3]$ from C_2, add the arcs (u_2^1,y_1^3) and (y_1^1,u_2^3) to obtain a hamiltonian circuit in $G-v$.

b_{52}) C_1 contains $[y_1^1, x_1, y_1^2]$. Delete this path from C_1 and the arc (y_2^2, u_2^3) from C_2. Adding (y_1^1, u_2^3) we obtain a hamiltonian circuit in G-v.

b_{53}) C_1 contains $[y_1^2, x_1, y_1^3]$. Delete this path from C_1 and the arc (u_2^1, y_2^2) from C_2, add the arc (u_2^1, y_1^3) to obtain a hamiltonian circuit in G-v.

We have shown the second property for all nodes v in H_1, by symmetry the same follows for the nodes in H_2. □

Remark 4.2. *Every odd Marguerite satisfies the conditions required for the digraphs G_1, G_2 in Construction HH1, but none of the special hypohamiltonian digraphs Y_8, Y_9, Y_{12} does.* □

Corollary 4.3. *There are hypohamiltonian digraphs of order $4k+1$, $k \geq 2$.* □

The smallest hypohamiltonian digraph obtainable by Construction HH1 is derived from two Marguerites M_3 and has 9 nodes but is different from Y_9. The digraph of order 13 obtained from M_3 and M_5 is shown below.

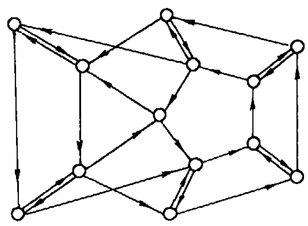

Fig. 4.3

<u>Construction HH2:</u> *Let* G_1 *and* G_2 *be two disjoint hypohamiltonian digraphs having distinguished nodes* x_1, x_2 *respectively. Let* C_1, C_2 *be hamiltonian circuits in* $G_1 - x_1$, $G_2 - x_2$ *respectively.* C_1 *contains arcs* (y_1^1, u_1^1) *and* (u_1^3, y_1^3); C_2 *contains arcs* (y_2^1, u_2^1), (u_2^3, y_2^3), *where* y_1^1, y_1^3 *and* y_2^1, y_2^3 *are neighbours of the distinguished nodes* x_1, x_2 *resp. with the usual numbering. Let* G *be the digraph obtained by adding* G_1 *and* G_2, *identifying the nodes* x_1 *and* x_2 *into a node* x, *and by adding the arcs* $N=\{(y_1^1, u_2^1), (y_2^1, u_1^1), (u_1^3, y_2^3), (u_2^3, y_1^3)\}$ *(see Figure 4.4 below).*

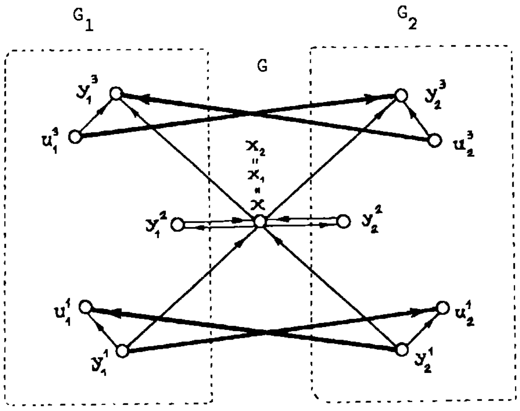

Fig. 4.4

Furthermore, we assume that

$a_1)$ G_1 *does neither contain a node covering pair of paths*
$P_1 = [x_1, y_1^2, \ldots, y_1^1]$, $P_2 = [y_1^3, \ldots\ldots, u_1^3]$ *nor a node covering pair*
$P_1' = [u_1^1, \ldots\ldots, y_1^1]$, $P_2' = [y_1^3, \ldots, y_1^2, x_1]$

$a_2)$ G_2 *does not contain a node covering pair of paths*
$Q_1 = [x_2, y_2^2, \ldots, y_2^1]$, $Q_2 = [y_2^3, \ldots\ldots, u_2^3]$ *nor a node covering pair*
$Q_1' = [u_2^1, \ldots\ldots, y_2^1]$, $Q_2' = [y_2^3, \ldots, y_2^2, x_2]$. $\square$

Theorem 4.4. *Given two digraphs G_1 and G_2 with the above properties then the digraph G obtained by Construction HH2 is hypohamiltonian.*

Proof:

a) We show that $G-v$ is hamiltonian for all v in G.

a_1) $v=x$. In this case delete the arc (y_1^1,u_1^1) from C_1 and the arc (y_2^1,u_2^1) from C_2 and add the arcs (y_2^1,u_1^1), (y_1^1,u_2^1) to get a hamiltonian circuit in $G-x$.

a_2) $v \neq x$. Assume w.l.o.g. that v is in G_1. Then there is a hamiltonian circuit C in $G_1 - v$ such that C contains one of the arcs (x_1,y_1^3) or (y_1^1,x_1).

a_{21}) If (x_1,y_1^3) is in C then delete it; delete the arc (u_2^3,y_2^3) from C_2 and add (x,y_2^3), (u_2^3,y_1^3) to get a hamiltonian circuit in $G-v$.

a_{22}) If (y_1^1,x_1) is in C then delete it; delete the arc (y_2^1,u_2^1) from C_2 and add (y_1^1,u_2^1), (y_2^1,x) to get a hamiltonian circuit in $G-v$.

b) Suppose G contains a hamiltonian circuit C. Then, at least one of the arcs of N has to be in C.

b_1) $|N \cap C|=1$. Suppose (y_1^1,u_2^1) is in C. In this case C contains a hamiltonian path P in G_1 from x_1 to y_1^1. But then, $P \cup \{(y_1^1,x_1)\}$ is a hamiltonian circuit in G_1, which is a contradiction. Similarly, a contradiction follows in the other cases.

b_2) $|N \cap C|=2$. Suppose (y_1^1,u_2^1) and (y_2^1,u_1^1) are in C. Then, it follows from this assumption that the path $[w,x,v]$ in C is either in G_1 or in G_2, say G_1. Hence, G_1 contains a hamiltonian path P from u_1^1 to y_1^1. But then, $P \cup \{(y_1^1,u_1^1)\}$ is a hamiltonian circuit in

G_1. Contradiction. Similarly, a contradiction follows if the arcs (u_1^3,y_2^3) and (u_2^3,y_1^3) are in C. Suppose (u_2^3,y_1^3) and (y_1^1,u_2^1) are in C. In this case the arcs (y_1^1,x) and (x,y_1^3) cannot be in C and therefore C contains a hamiltonian path P in G_1-x_1 from y_1^3 to y_1^1. But then, $P\cup\{(y_1^1,x_1),\ (x_1,y_1^3)\}$ is a hamiltonian circuit in G_1, which is a contradiction. By symmetry, a contradiction follows if the arcs (u_1^3,y_2^3) and (y_2^1,u_1^1) are in C.

b_3) $|N\cap C|=4$. As C is a hamiltonian circuit it contains a path $[w,x,v]$. Since $N \subset C$, this path cannot contain one of the arcs (x,y_1^3), (x,y_2^3), (y_1^1,x), (y_2^1,x); on the other hand, $[w,x,v]$ has to be either in G_1 or in G_2 which is impossible.

b_4) $|N\cap C|=3$. Suppose $(y_1^1,u_2^1) \notin C$. Then C necessarily contains the arc (x,y_2^2), and one of the arcs (y_1^1,x) or (y_1^2,x).

If $(y_1^1,x) \in C$, then either C contains two disjoint paths $P_1 = [x,y_2^2,\ldots,u_2^3]$, $P_2=[y_2^3,\ldots,y_2^1]$ containing all nodes of G_2, in which case $P_1\cup\{(u_2^3,y_2^3)\}\cup P_2\cup\{(y_2^1,x_2)\}$ is a hamiltonian circuit in G_2, a contradiction, or G_2 contains a node covering pair of paths $Q_1 = [x_2,y_2^2,\ldots,y_1^1]$, $Q_2 = [y_2^3,\ldots,u_2^3]$ which by assumption a_2) is impossible.

If $(y_1^2,x) \in C$ a similar reasoning gives a contradiction.

The three other cases can be proved in a similar manner. □

Because of the assumptions a_1) and a_2) Construction HH2 is not universal and it seems to be hard to check whether a digraph G satisfies condition a_1). However, the following is easily verified.

Remark 4.5.

a) If the digraphs G_1, G_2 *with distinguished nodes* x_1, x_2 *have the local structures required for Construction HH1 and if the nodes* u_j^i *in Construction HH1 are those used in HH2 then* G_1 *and* G_2 *satisfy conditions* $a_1)$ *and* $a_2)$ *resp. of Construction HH2.*

b) All odd Marguerites M_p, $p \geq 3$, *satisfy conditions* $a_1)$ *and* $a_2)$ *of Construction HH2.* □

Corollary 4.6. *There are hypohamiltonian digraphs of order* $4k+3$, $k \geq 2$. □

The hypohamiltonian digraph obtained by Construction HH2 from Marguerites M_5 has 19 nodes and is shown below.

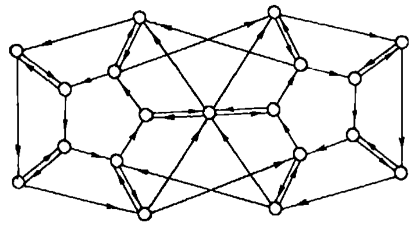

Fig. 4.5

In [5] Thomassen gave a construction to obtain a hypohamiltonian graph from two hypohamiltonian graphs by dropping two nodes and identifying some others. We modify this construction using hypohamiltonian digraphs with distinguished nodes losing, however, the universality of Thomassen's construction.

Definition 4.7. *Let* v *be a distinguished node of a hypohamiltonian digraph* G, *and let* u^1, u^2, u^3 *be the neighbours of* v *numbered in the usual way.* (G,v) *is said to have* property β *if there are no hamiltonian paths from* u^1 *to* u^2, *from* u^1 *to* u^3, *and from* u^2 *to* u^3 *in* $G-v$. □

Construction HH3: *Let* G_1, G_2 *be disjoint hypohamiltonian digraphs with distinguished nodes* x_1, x_2 *resp. We assume that both* (G_1, x_1) *and*

(G_2, x_2) have property β. *We add the digraphs $H_1 = G_1 - x_1$ and $H_2 = G_2 - x_2$, and identify the nodes y_1^3 and y_2^1 into z_1, y_1^2 and y_2^2 into z_2, y_1^1 and y_2^3 into z_3, and call this digraph G.* ∎

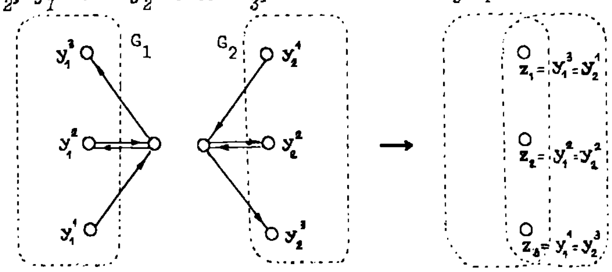

Fig. 4.6

Theorem 4.8. *Given two digraphs G_1, G_2 with the above properties then the digraph G obtained by Construction HH3 is hypohamiltonian.*

Proof:

a) We have to show that $G-v$ is hamiltonian for all nodes v. Assume w.l.o.g. that $v \in H_1$. $G_1 - v$ contains a hamiltonian circuit and hence $G_1 - v - x_1$ contains a hamiltonian path P_1 from y_1^3 to y_1^1, or from y_1^3 to y_1^2, or from y_1^2 to y_1^1, say from y_1^3 to y_1^1. $G_2 - y_2^2$ contains a hamiltonian circuit, and as x_2 is a distinguished node, $G_2 - y_2^2 - x_2$ contains a hamiltonian path P_2 from y_2^3 to y_2^1. $P_1 \cup P_2$ is a hamiltonian circuit in $G-v$.

The other cases are similar.

b) Suppose G contains a hamiltonian circuit C. As C contains the nodes z_1, z_2, z_3 one of the following cases must be satisfied.

1) C contains a path P_1 from z_1 to z_2, a path P_2 from z_2 to z_3 and a path P_3 from z_3 to z_1.

2) C contains a path P_1 from z_1 to z_3, a path P_2 from z_3 to z_2 and a path P_3 from z_2 to z_1.

Suppose 1) holds. Obviously two of the paths P_i are contained in one of the digraphs H_1,H_2. We have to consider several cases.

1.1) P_1, P_2 in H_1; P_3 in H_2. $P_1 \cup P_2$ is a hamiltonian path from $y_1^3 = z_1$ to $y_1^1 = z_3$ in H_1, hence $P_1 \cup P_2 \cup [y_1^1, x_1, y_1^3]$ is a hamiltonian circuit in G_1. Contradiction.

1.2) P_1, P_3 in H_1; P_2 in H_2. Then $P_1 \cup P_3$ is a hamiltonian path in H_1 from y_1^1 to y_1^2 which cannot exist because (G_1,x_1) has property β by assumption.

1.3) P_1 in H_1; P_2, P_3 in H_2. Then $P_2 \cup P_3$ is a hamiltonian path in H_2 from y_2^2 to y_2^1. We add the path $[y_2^1, x_2, y_2^2]$ and obtain a hamiltonian circuit in G_2. Contradiction.

1.4) P_2, P_3 in H_1; P_1 in H_2. Then $P_2 \cup P_3$ is a hamiltonian path in H_1 from y_1^2 to y_1^3 contradicting property β of (G_1,x_1).

1.5) P_2 in H_1; P_1, P_3 in H_2. Then $P_3 \cup P_1$ is a hamiltonian path in H_2 from y_2^3 to y_2^2. We add $[y_2^2, x_2, y_2^3]$ and obtain a hamiltonian circuit in G_2. Contradiction.

1.6) P_3 in H_1; P_1, P_2 in H_2. Then $P_1 \cup P_2$ is a hamiltonian path in H_2 from y_2^1 to y_2^3 contradicting property β of (G_2,x_2).

Case 2) can be contradicted in a similar way. ◻

As usual, our Marguerites behave nicely with respect to the additional properties required for Construction HH3.

<u>Remark 4.9.</u> *Given any odd Marguerite $M_p = (V,E)$, $p \geq 5$, and any $x \in V$, then (M_p,x) has property* β. ◻

Construction HH3 works under more general conditions than those ones given above. As these are quite complicated we have restricted ourselves to the sufficient property β. Although, for example, the Marguerites

M_3 do not have property β the digraph of order 7 obtained by Construction HH3 from two Marguerites M_3 is hypohamiltonian (c.f. Fig. 4.7) and has nodes of maximum degree 6. Similarly, the digraph of order 11 obtained from M_3 and M_5 is hypohamiltonian.

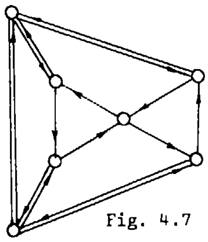

Fig. 4.7

Corollary 4.10. *There are hypohamiltonian digraphs of order* $4k+3$, $k \geq 1$. ◻

So far we have obtained hypohamiltonian digraphs of order 6,7,8,11,12 and 4k+1, 4k+2, 4k+3, for all $k \geq 2$.

In order to construct hypohamiltonian digraphs of order $4k$, $k \geq 4$, we use Construction HH1. Let G_1 be any odd Marguerite M_p, $p \geq 3$ and G_2 be the hypohamiltonian digraph of order 13 shown in Fig. 4.3. Both digraphs obviously contain nodes needed in Construction HH1, hence, this way we obtain hypohamiltonian digraphs of order $4k$, $k \geq 4$.
It is easily shown by enumeration that there are no hypohamiltonian digraphs of order less than 6. Therefore, summarizing the results obtained before we proved:

Theorem 4.11. *There are hypohamiltonian digraphs of order* n *if and only if* $n \geq 6$. ◻

Hence, in contrast to the case of hypohamiltonian graphs where the existence of such graphs of order 14 and 17 is still unknown, we could answer the existence-question in the case of hypohamiltonian digraphs fully. If also follows from the foregoing results that the variety of hypohamiltonian digraphs is richer than those of hypohamiltonian graphs, e.g. even for n=6 there are nonisomorphic hypohamiltonian digraphs, while for n=10 there is up to isomorphism only one hypohamiltonian graph, and for every $n \geq 6$ there are hypohamiltonian digraphs which are not trivially directed hypohamiltonian graphs or simple modifications of these.

Note added in proof: After completion of this paper we were informed by J.A. Bondy and C. Thomassen that the existence problem (Theorem 4.11) was independently but earlier solved by J.-L. Fouquet, J.-L. Jolivet [9] and C. Thomassen [10]. The smallest hypohamiltonian digraph (the Marguerite M_3, c.f. Fig. 3.1) was also found in [9] and [10], and the digraph shown in Fig. 4.7 was first constructed by Thomassen [10], however with a different method. Another hypohamiltonian digraph of order 7 with very interesting properties was found by J.A. Bondy [8].

References

[1] J.A. Bondy, Variations on the hamiltonian theme. Can. Math. Bull. 15(1) (1972) 57-62.

[2] V. Chvâtal, Flip-flops in hypohamiltonian graphs. Can. Math. Bull. 16(1) (1973) 33-41.

[3] J. Doyen and V. van Diest, New families of hypohamiltonian graphs. Discrete Math. 13(1975) 225-236.

[4] M. Grötschel, On the monotone symmetric travelling salesman problem: hypohamiltonian/hypotraceable graphs and facets. Report No. 7788-OR, Institut für Ökonometrie und Operations Research, Universität Bonn, W. Germany.

[5] M. Grötschel and Y. Wakabayashi, On the complexity of the monotone asymmetric travelling salesman polytope I: hypohamiltonian facets. Report No. 78104-OR, Institut für Ökonometrie und Operations Research, Universität Bonn, W. Germany.

[6] C. Thomassen, Hypohamiltonian and hypotraceable graphs. Discrete Math. 9(1974) 91-96.

[7] C. Thomassen, On hypohamiltonian graphs. Discrete Math. 10(1974) 383-390.

[8] J.A. Bondy, Hamilton cycles in graphs and digraphs. Research Report CORR 78-16, Dept. of Combinatorics and Optimization University of Waterloo, Waterloo, Canada, 1978.

[9] J.-L. Fouquet and J.-L. Jolivet, Hypohamiltonian oriented graphs. Cahiers du Centre d'Etudes de Recherche Opérationelle 20(1978) 171-181.

[10] C. Thomassen, Hypohamiltonian graphs and digraphs. In: Y. Alavi and D.R. Lick (eds.), Theory and applications of graphs. Lecture Notes in Mathematics , Springer, Berlin (1978) 557-571

Addresses:

Martin Grötschel
Institut für Ökonometrie
und Operations Research
Universität Bonn
Nassestr. 2
D-5300 Bonn 1
W. Germany

Yoshiko Wakabayashi
Instituto de Matemática
e Estatistica
Universidade de São Paulo
Cidade Universitaria
Caixa Postal 20.570 (Ag. Iguatemi)
BR-01451 São Paulo
Brazil