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ON THE STRUCTURE OF THE MONOTONE ASYMMETRIC TRAVELLING SALESMAN POLYTOPE I: HYPOHAMILTONIAN FACETS*

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The monotone asymmetric travelling salesman polytope $\tilde{P}^n_T$ is defined to be the convex hull of the incidence vectors of all hamiltonian circuits and all subsets of these in a complete digraph of order n. We prove that certain hypohamiltonian digraphs $G=(V, E)$, i.e. digraphs which are not hamiltonian but such that $G-v$ is hamiltonian for all $v \in V$, induce facets $x(E) \leq n-1$ of $\tilde{P}^n_T$. This result indicates that $\tilde{P}^n_T$ has very complicated facets and that it is very unlikely that an explicit complete characterization of $\tilde{P}^n_T$ can ever be given.

1. Introduction and notation

In this paper we show that the intractability of the asymmetric travelling salesman problem (ATSP) is closely related with the difficulty of characterizing hypohamiltonian digraphs. We associate with the n-city ATSP a polytope $\tilde{P}^n_T$ which has the property that every ATSP can be solved as a linear maximization problem over $\tilde{P}^n_T$. Then we show that certain hypohamiltonian digraphs (cf. [8]) induce facets of $\tilde{P}^n_T$.

These results indicate that the polytope $\tilde{P}^n_T$ is very complex, and that it is most unlikely that a complete explicit characterization of $\tilde{P}^n_T$ can ever be found.

A *digraph* $G=(V, E)$ consists of a finite set V of *nodes* and a set E of ordered pairs of distinct elements of V called *arcs*. If $e=(u, v) \in E$ then u and v are called *endnodes* of e, u is said to be the *initial node* and v the *terminal node* of e; u and v are called *neighbours*. $|V|$ is the order of G. The set of all arcs in E having both endnodes in a subset W of V is denoted by $E(W)$. If $|V|=n$, then E_n is the set of all ordered pairs (u, v), $u \neq v$, of elements of V, and the digraph $K_n=(V, E_n)$ is called *complete*.

A non-empty set of arcs

$$P=\{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \subset E$$

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where $v_i \neq v_j$ for $i \neq j$, is called a *path* of length $k-1$ and is denoted by $[v_1, v_2, \ldots, v_k]$. If $(v_k, v_1) \in E$ then $C := P \cup \{(v_k, v_1)\}$ is called a *circuit* of length k and is denoted by $(v_1, v_2, \ldots, v_k)$. A circuit (path) of length $|V|$ ($|V|-1$) is called *hamiltonian;* such a circuit is also called a *tour.* Given a digraph $G=(V, E)$ then $G-v$ is the digraph with node set $V-\{v\}$ and all arcs in E which do not contain the node v.

Definition 1.1. Let $G=(V, E)$ be a digraph.
(a) G is called *hamiltonian* if G contains a hamiltonian circuit.
(b) G is called *hypohamiltonian* if
(b_1) G is not hamiltonian and
(b_2) $G-v$ is hamiltonian for all $v \in V$.

For $u, v \in V$ we define $G-(u, v)$ to be the digraph $(V, E-\{(u, v)\}$ and $G+(u, v)$ the digraph $(V, E \cup \{(u, v)\})$. If a digraph G has a property π, then G is called *maximal* (*minimal*) with respect to π if $G+(u,v)$ $(G-(u,v))$ does not have property π for all pairs $(u, v) \notin E$ $((u, v) \in E)$. $\omega^+(v)$ resp. $\omega^-(v)$ is the set of all arcs having v as its initial resp. terminal node; $\omega(v) = \omega^+(v) \cup \omega^-(v)$. $d^+(v) = |\omega^+(v)|$ is called *outdegree* of v; $d^-(v) = |\omega^-(v)|$ is called *indegree* of v, and $d(v) = d^+(v) + d^-(v)$ is called *degree* of v.

The asymmetric travelling salesman problem (ATSP) is the problem of finding the shortest hamiltonian circuit in a weighted complete digraph. The nodes of the digraph can be interpreted as cities and the weights as the distances between the cities. We can associate a polytope having (0, 1)-vertices with the ATSP in the following way:

Let $K_n = (V, E_n)$ be the complete digraph on n nodes. Let $c_{ij} \in \mathbb{R}$ be the weights associated with the arcs (i, j) and let $\tilde{T}_n$ be the set of arc sets which are hamiltonian circuits or subsets of hamiltonian circuits in K_n. With each arc $e = (i, j) \in E_n$ we associate a variable $x_e = x_{ij}$, and with each $T \in \tilde{T}_n$ we associate an *incidence vector* x^T, i.e. a vector such that $x_e^T = 1$ if $e \in T$ and $x_e^T = 0$ otherwise. As $|E_n| = n(n-1) =: m$ we have $x^T \in R^m$. The convex hull $\tilde{P}_T^n$ of the incidence vectors of all tours and subsets of tours is called (*monotone*) *asymmetric travelling salesman polytope* (cf. [4]), i.e.

$$\tilde{P}_T^n = \operatorname{conv}\{x^T \in \mathbb{R}^m : T \in \tilde{T}_n\}.$$

It is clear that every ATSP can be solved as a linear maximization problem over $\tilde{P}_T^n$. As $\tilde{P}_T^n$ contains the zero vector and all unit vectors, it is fully dimensional, i.e. $\dim \tilde{P}_T^n = m$.

An inequality $ax \leq a_0$ is called *valid* with respect to $\tilde{P}_T^n$ if $\tilde{P}_T^n \subset \{x \in \mathbb{R}^m : ax \leq a_0\}$. A valid inequality $ax \leq a_0$ is called *maximal* if the inequality obtained by increasing any component a_e of a by any $\varepsilon > 0$ is not valid. A valid inequality $ax \leqslant a_0$ is called a *facet* of $\tilde{P}^n_T$ if

$$\dim(\tilde{P}^n_T \cap \{x \in \mathbb{R}^m : ax = a_0\}) = \dim \tilde{P}^n_T - 1 = m - 1.$$

Clearly, a facet is a maximal inequality.

A theorem of polyhedral theory states that fully dimensional polytopes have a unique linear characterization; this implies that there exists a unique (up to a constant factor) finite non-redundant system of linear inequalities $Ax \leqslant b$ such that $\tilde{P}^n_T = \{x \in \mathbb{R}^m : Ax \leqslant b\}$ and that this system is given by the set of all facets of $\tilde{P}^n_T$. In order to characterize $\tilde{P}^n_T$ completely and non-redundantly we have to characterize all facets of $\tilde{P}^n_T$.

It was shown in [4] that the trivial inequalities, the subtour elimination constraints, some comb inequalities and other classes of inequalities define facets of $\tilde{P}^n_T$. But these inequalities are not sufficient to characterize $\tilde{P}^n_T$. In this paper we show the existence of a new class of facets which in contrast to the comb- and subtour-elimination constraints are highly complex.

For an inequality $ax \leqslant a_0$ valid with respect to $\tilde{P}^n_T$ we define its *face* to be $H_a := \{x \in \tilde{P}^n_T : ax = a_0\}$. In order to prove that an inequality $ax \leqslant a_0$ is a facet of $\tilde{P}^n_T$ we use the following technique. We consider any other valid inequality $bx \leqslant b_0$ which satisfies $H_a \subset H_b$ and show that $b = \pi a$ where $\pi \in \mathbb{R} - \{0\}$, which proves that $\dim H_a = m - 1$. The sum $\sum_{(i,j) \in E} x_{ij}$ will be abbreviated by $x(E)$. The face of a valid inequality $x(E) \leqslant r$ is denoted by H_E.

2. Trivially directed hypohamiltonian graphs

We consider hypohamiltonian digraphs $G = (V, E)$ of order n as subdigraphs of the complete digraph K_k, $k \geqslant n$, thus every node carries a label $1, 2, \ldots, k$ and every arc $e \in E$ is an arc in K_k. Now, hypohamiltonian digraphs can be related to the asymmetric travelling salesman problem in the following way.

Proposition 2.1. *Let $G = (V, E)$ be a hypohamiltonian digraph of order n. Then the hypohamiltonian inequality*

$$x(E) \leqslant n - 1$$

is a valid inequality with respect to $\tilde{P}^k_T$ for all $k \geqslant n$. $x(E) \leqslant n - 1$ is not maximal if $k > n$.

Proof. Since G is not hamiltonian, every tour in K_k contains at most $n - 1$ arcs of E, thus the hypohamiltonian inequality is valid. If $k > n$ then the subtour-elimination constraint $x(E(V)) \leqslant n - 1$ is valid with respect to $\tilde{P}^k_T$, in fact it is a facet (cf. [4]), and since $E \subsetneq E(V)$ the hypohamiltonian inequality is not maximal.

We are going to prove later that certain hypohamiltonian digraphs actually induce facets, but Proposition 2.1 already suggests an unusual property of asymmetric travelling salesman polytopes. Since the polytopes $\tilde{P}^n_T$ and $\tilde{P}^{n+1}_T$ look rather similar—at first sight—it seems reasonable to expect that all facets of $\tilde{P}^n_T$ will also be facets of $\tilde{P}^{n+1}_T$ after trivial lifting, i.e. giving zero coefficients to the new variables. This anticipation is backed by the fact that all known facets of the asymmetric travelling salesman polytope have this property, see [4]. Proposition 2.1, however, shows that this is not the case for hypohamiltonian inequalities, i.e., if a hypohamiltonian inequality is a facet of $\tilde{P}^n_T$ then this facet is a particular property of the polytope of order n which is not shared by any other polytope $\tilde{P}^k_T$, $k \neq n$.

Lemma 2.2. *Let $G'=(V,E')$ be a hypohamiltonian digraph of order n such that any inequality $dx \leq d_0$ which is valid with respect to $\tilde{P}^n_T$ and satisfies $H_{E'} \subset H_d$ has the property that there exists $\alpha > 0$ such that*

$$d_{uv} = \alpha \quad \text{for all } (u,v) \in E'.$$

Then $d_0 = (n-1)\alpha$ and the hypohamiltonian inequality induced by any maximal hypohamiltonian digraph $G=(V,E)$ with $E' \subset E$ is a facet of $\tilde{P}^n_T$.

Proof. Let $dx \leq d_0$ be an inequality that satisfies the requirements above. By taking a hamiltonian path in G' and summing up the coefficients we get $d_0 = (n-1)\alpha$.

Now let $dx \leq d_0$ be any valid inequality satisfying $H_E \subset H_d$. Since $H_{E'} \subset H_E$, the conditions of the lemma imply that $d_{uv} = \alpha$ for all $(u,v) \in E'$. Let $(v,w) \in E - E'$. $G' - v$ contains a hamiltonian circuit C that contains some arc (u,w). $C_1 := (C - \{(u,w)\}) \cup \{(v,w)\}$ is a hamiltonian path in G, and hence $d_0 = (n-1)\alpha = dx^{C_1} = (n-2)\alpha + d_{vw}$. This implies

$$d_{uv} = \alpha \quad \text{for all } (u,v) \in E.$$

Let $(u,v) \notin E$, then because of maximality $G + (u,v)$ contains a hamiltonian circuit C, that necessarily contains (u,v). $C_1 := C - \{(u,v)\}$ is a hamiltonian path in G and therefore $dx^{C_1} = d_0$. Since $dx^C \leq d_0$ we have $d_{uv} = 0$. This implies $d_{uv} = 0$ for all $(u,v) \in E_n - E$.

Altogether we have shown that $d = \alpha x(E)$ and $d_0 = \alpha(n-1)$ which proves that $x(E) \leq n-1$ is a facet of $\tilde{P}^n_T$.

The simplest way to get a hypohamiltonian digraph is to take a hypohamiltonian graph G (cf. [1, 2, 11]) and substitute the two arcs $(u,v), (v,u)$ for each edge $\{u,v\}$, cf. [8]. Hypohamiltonian digraphs obtained this way are called *trivially directed hypohamiltonian graphs* and are denoted by $\vec{G}$. Recall that a graph is called cubic if all nodes have degree three. For these we can show the following result

Theorem 2.3. *Let $G' = [V, E']$ be a cubic hypohamiltonian graph of order n and $\vec{G}' = (V, \vec{E}')$ its trivial direction. Let $G = (V, E)$ be any maximal hypohamiltonian* *digraph such that $\vec{E}' \subset E$. Then the hypohamiltonian inequality*

$$x(E) \leqslant n-1$$

is a facet of $\tilde{P}^n_T$.

Proof. Let $bx \leqslant b_0$ be a valid inequality with respect to $\tilde{P}^n_T$ such that $H_E \subset H_b$. Since G' is cubic, every node $z \in V$ has exactly three neighbours in G', say u, v, w. $G'-u$ contains a hamiltonian cycle C which necessarily contains the chain (v, z, w), thus $\vec{G}'-u$ contains a hamiltonian circuit C_1 containing the path $[v, z, w]$. The $n-1$ arcs of the paths $P_1 := (C_1 - \{(v, z)\}) \cup \{(u, z)\}$, $P_2 := (C_1 - \{(z, w)\}) \cup \{(z, u)\}$ are in $\vec{E}'$ and therefore $x^{P_1}, x^{P_2} \in H_E$. This implies

$$0 = b_0 - b_0 = bx^{P_1} - bx^{P_2} = b_{uz} + b_{zw} - b_{vz} - b_{zu}.$$

By reverting the direction of C_1 we get a circuit C_2 containing the path $[w, z, v]$. Similarly we obtain

$$0 = b_{uz} + b_{zv} - b_{wz} - b_{zu}.$$

By considering the other cases $\vec{G}'-v$, $\vec{G}'-w$ in the same way we get the following system of six equations in six unknowns

$$\begin{array}{rrrrrrl}
b_{uz} & -b_{zu} & -b_{vz} & & & +b_{zw} & =0 \\
b_{uz} & -b_{zu} & & +b_{zv} & -b_{wz} & & =0 \\
& +b_{zu} & +b_{vz} & -b_{zv} & -b_{wz} & & =0 \\
-b_{uz} & & +b_{vz} & -b_{zv} & & +b_{zw} & =0 \\
& +b_{zu} & -b_{vz} & & +b_{wz} & -b_{zw} & =0 \\
-b_{uz} & & & +b_{zv} & +b_{wz} & -b_{zw} & =0
\end{array}$$

The set of all solutions of this system is given by $b_{uz} = b_{vz} = b_{wz} = \pi^-$, $b_{zu} = b_{zv} = b_{zw} = \pi^+$. Thus we can conclude that for every node $v \in V$ there are π^+_v, $\pi^-_v \in \mathbb{R}$ such that $b_e = \pi^+_v$ for all $e \in \omega^+(v)$, and $b_e = \pi^-_v$ for all $e \in \omega^-(v)$.

G' is hypohamiltonian implies that G' is not bipartite; i.e. there is a cycle of odd length k in G', and thus in $\vec{G}'$ there exists an odd length sequence of arcs of the following type $(v_1, v_2), (v_3, v_2), (v_3, v_4), (v_5, v_4), \ldots, (v_k, v_{k-1}), (v_k, v_1)$. Hence we can conclude from the above result

$$\pi^+_{v_1} = b_{v_1, v_2} = \pi^-_{v_2} = \cdots = \pi^-_{v_{k-1}} = b_{v_k, v_{k-1}} = \pi^+_{v_k} = b_{v_k, v_1} = \pi^-_{v_1},$$

and have shown that there is a node $v_1 \in V$ with $b_e = \pi_v$ for all $e \in \omega(v)$. This fact and the connectedness of G imply $b_e = \pi$ for all $e \in \vec{E}'$.

The theorem now follows from Lemma 2.2.

3. Marguerites

Trivially directed hypohamiltonian graphs are by far not the only hypohamiltonian digraphs. It was shown in [3, 8, 10] that hypohamiltonian digraphs of order

n exist if and only if $n \geqslant 6$, and that for all these n there are hypohamiltonian digraphs which cannot be derived from trivially directed graphs. The "nicest" class of hypohamiltonian digraphs obtained in [8] are the odd Marguerites which we will consider in this section.

Let $p \geqslant 3$ be odd, $V = \{a_1, a_2, \ldots, a_p, b_1, \ldots, b_p\}$ and $E = A \cup B \cup D$, where

$$A = \{(a_i, a_{i+1}) : i = 1, \ldots, p-1\} \cup \{(a_p, a_1)\} = \langle a_1, a_2, \ldots, a_p \rangle,$$
$$B = \{(b_i, b_{i+1}) : i = 1, \ldots, p-1\} \cup \{(b_p, b_1)\} = \langle b_1, b_2, \ldots, b_p \rangle,$$
$$D = \{(a_i, b_i), (b_i, a_i) : i = 1, \ldots, p\}.$$

Then the digraph $M_p = (V, E)$ is called *odd Marguerite.*

It was shown in [8] that odd Marguerites are hypohamiltonian digraphs, and that the Marguerite M_3, which has six nodes, is the smallest hypohamiltonian digraph, see also [3, 10].

Definition 3.1. Let $G = (V, E)$ be a hypohamiltonian digraph. We say that two nodes $u_i, v_i \in V$ satisfy *structure* 1 (cf. Fig. 1) if the following properties are

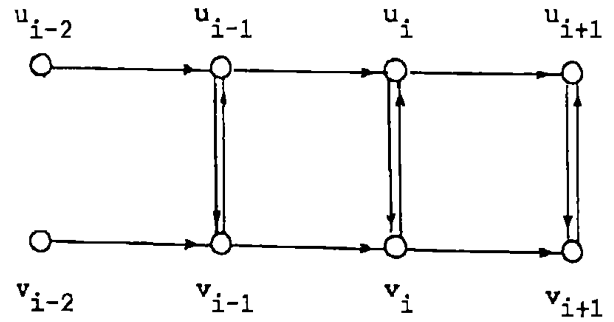

Fig. 1.

satisfied:

(1) $(u_i, v_i), (v_i, u_i) \in E.$

(2) There exist different nodes $u_{i-2}, u_{i-1}, u_{i+1}, v_{i-2}, v_{i-1}, v_{i+1} \in V$ such that

(2.1) there is a hamiltonian circuit C in $G - u_i$ which contains the path $[u_{i-2}, u_{i-1}, v_{i-1}, v_i, v_{i+1}, u_{i+1}]$,

(2.2) there is a hamiltonian circuit K in $G - v_i$ which contains the path $[v_{i-2}, v_{i-1}, u_{i-1}, u_i, u_{i+1}, v_{i+1}]$.

Lemma 3.2. *Let $G = (V, E)$ be a hypohamiltonian digraph of order n with nodes u_i, v_i satisfying structure* 1. *Let $u_{i-2}, u_{i-1}, u_{i+1}, v_{i-2}, v_{i-1}, v_{i+1} \in V$ and C, K be as in Definition* 3.1. *Let $dx \leqslant d_0$ be a valid inequality with respect to $\tilde{P}^n_T$ such that*

$H_E \subset H_d$. Then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(a) $d_{u_{i-2},u_{i-1}} = d_{u_i,u_{i+1}} = \alpha$,
(b) $d_{v_{i-2},v_{i-1}} = d_{v_i,v_{i+1}} = \beta$,
(c) $d_{v_{i-1},u_{i-1}} = d_{v_{i+1},u_{i+1}} = \gamma$,
(d) $d_{u_{i-1},v_{i-1}} = d_{u_{i+1},v_{i+1}} = \delta$,
(e) $d_{v_{i-1},v_i} = \beta + \gamma - \delta$,
(f) $d_{u_{i-1},u_i} = \alpha + \delta - \gamma$,
(g) $d_{v_i,u_i} = \alpha + \beta - \gamma$,
(h) $d_{u_i,v_i} = \alpha + \beta - \delta$,

Proof. We construct several hamiltonian paths T_j, S_j in G the incidence vectors of which are contained in H_E and hence satisfy $dx \leqslant d_0$ with equality. For simpler index calculations we set $1 = u_{i-2}$, $2 = u_{i-1}$, $3 = v_{i-1}$, $4 = v_i$, $5 = v_{i+1}$, $6 = u_{i+1}$, $7 = u_i$, $8 = v_{i-2}$, thus—by assumption—the nodes 4, 7 satisfy structure 1, and the path in C is $[1, 2, 3, 4, 5, 6]$, the path in K is $[8, 3, 2, 7, 6, 5]$. (See Fig. 2.)

Define:

$$T_1 := (C - [2,3]) \cup [2,7], \qquad T_2 := (C - [1,2,3,4]) \cup [3,2,7,4],$$
$$T_3 := (C - [3,4]) \cup [7,4], \qquad T_4 := (C - [4,5]) \cup [4,7],$$
$$S_1 := (K - [3,2]) \cup [3,4], \qquad S_2 := (K - [8,3,2,7]) \cup [2,3,4,7],$$
$$S_3 := (K - [2,7]) \cup [4,7], \qquad S_4 := (K - [7,6]) \cup [7,4].$$

C contains an arc $(8, v)$ where v is some node $v \in V$, while K contains some arc $(1, u)$ $u \in V$. We define

$$T_5 := (C - ([8,v] \cup [2,3] \cup [5,6])) \cup ([8,3] \cup [2,7,6]),$$
$$T_6 := (C - ([8,v] \cup [2,3,4])) \cup ([8,3] \cup [2,7,4]),$$
$$S_5 := (K - ([1,u] \cup [3,2] \cup [6,5])) \cup ([1,2] \cup [3,4,5]),$$
$$S_6 := (K - ([1,u] \cup [3,2,7])) \cup ([1,2] \cup [3,4,7]).$$

The matrix corresponding to the following 8 equations

$$dx^{T_1} - dx^{T_2} = dx^{T_1} - dx^{T_3} = dx^{T_1} - dx^{T_4} = dx^{T_5} - dx^{T_6} = 0$$
$$dx^{S_1} - dx^{S_2} = dx^{S_1} - dx^{S_3} = dx^{S_1} - dx^{S_4} = dx^{S_5} - dx^{S_6} = 0$$

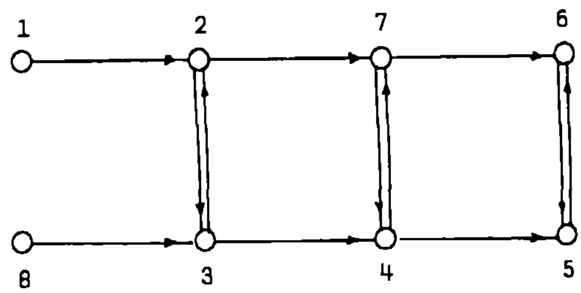

Fig. 2.

has by construction the following form:

$$\begin{array}{cccccccccccc}
12 & 23 & 27 & 32 & 34 & 45 & 47 & 56 & 65 & 74 & 76 & 83 \\
\hline
1 & & & -1 & 1 & & & & & -1 & & \\
& -1 & 1 & & 1 & & & & & -1 & & \\
& -1 & 1 & & & 1 & -1 & & & & & \\
& & & & 1 & & & -1 & & -1 & 1 & \\
& -1 & 1 & & & & -1 & & & & & 1 \\
& & 1 & -1 & 1 & & -1 & & & & & \\
& & & -1 & 1 & & & & & -1 & 1 & \\
& & 1 & & & 1 & -1 & & -1 & & &
\end{array}$$

This matrix contains an $(8, 8)$ upper triangular matrix with ± 1 on the main diagonal, hence it is of rank 8. It is easy to calculate that the null-space of this matrix can be represented in the way stated in the lemma.

Proposition 3.3. *Let $M_p = (V, E)$ be an odd Marguerite with $p \geqslant 5$, $E = A \cup B \cup D$, and let $dx \leqslant d_0$ be a valid inequality with respect to $\tilde{P}^n_T$ such that $H_E \subset H_d$. Then there exist $\alpha, \beta \in \mathbb{R}$ with*

$$d_{uv} = \alpha \quad \forall (u, v) \in A,$$

$$d_{uv} = \beta \quad \forall (u, v) \in B,$$

$$d_{uv} = \tfrac{1}{2}(\alpha + \beta) \quad \forall (u, v) \in D,$$

$$d_0 = \tfrac{1}{2}(n-1)(\alpha + \beta).$$

Proof. Clearly every pair a_i, b_i of nodes $i = 1, \ldots, p$ satisfies structure 1. By applying Lemma 3.2 to a_3, b_3 we first get from 3.2(a) that $d_{a_1a_2} = d_{a_3a_4} = \alpha$ and from 3.2(b) that $d_{b_1b_2} = d_{b_3b_4} = \beta$. Now we apply Lemma 3.2 consecutively to the node pairs a_5, b_5; a_7, b_7; $\ldots$; a_p, b_p; a_2, b_2; a_4, b_4; $\ldots$; a_{p-1}, b_{p-1} obtaining

$$\alpha = d_{a_3a_4} = d_{a_5a_6} = \cdots = d_{a_{p-2}a_{p-1}} = d_{a_pa_1} = d_{a_2a_3} = \cdots = d_{a_{p-1}a_p}.$$

This proves that $d_{uv} = \alpha$ for all $(u, v) \in A$, and similarly we get $d_{uv} = \beta$ for all $(u, v) \in B$.

The application of Lemma 3.2 as described above also gives the following identities via (b) and (e): $\beta = \beta + \gamma - \delta$ i.e. $\gamma = \delta$, and via (c) and (g): $\gamma = \alpha + \beta - \gamma$ i.e. $2\gamma = \alpha + \beta$. Therefore we have $d_{uv} = \frac{1}{2}(\alpha + \beta)$ for all $(u, v) \in D$.

The incidence vector of the hamiltonian path $P = [b_1, a_1, a_2, b_2, b_3, \ldots, b_p, a_p]$ in M_p satisfies $dx^P = d_0$. By the results above

$$dx^P = \tfrac{1}{2}(p-1)\alpha + \tfrac{1}{2}(p-1)\beta + \tfrac{1}{2}p(\alpha + \beta) = \tfrac{1}{2}(n-1)(\alpha + \beta)$$

which proves the assertion.

Proposition 3.3 shows that odd Marguerites almost induce facets since any face containing the face generated by Marguerites can be parameterized by two parameters only.

Theorem 3.4. *Let $n=2p$ where $p\geqslant 5$ and odd. Let $M_p'=(V,E')$ be an odd Marguerite where two arcs of the form (a_{i+1},a_i), (a_i,a_{i-1}) or (b_{i+1},b_i), (b_i,b_{i-1}) are added. Let $G=(V,E)$ be any maximal hypohamiltonian digraph of order n with $E'\subset E$. Then*

$$x(E)\leqslant n-1$$

is a facet of $\tilde{P}_T^n$.

Proof. It should be clear that M_P' is also a hypohamiltonian digraph.

Let $dx\leqslant d_0$ be a valid inequality with respect to $\tilde{P}_T^n$ such that $H_E\subset H_d$. The coefficients d_{uv} for (u,v) in M_p are given by Proposition 3.3.

W.l.o.g. we may assume that the arcs added are (a_3,a_2) (a_2,a_1). M_p-a_2 contains a hamiltonian circuit C containing the arcs (a_p,a_1) and (b_1,b_2). Then $C_1:=(C-[b_1,b_2])\cup[a_2,b_2]$ and $C_2:=(C-[a_p,a_1])\cup[a_2,a_1]$ are hamiltonian paths in M_p resp. M_p'. We obtain

$$d_0=dx^{C_1}=\tfrac{1}{2}(p(\alpha+\beta)+(p-1)(\alpha+\beta))=\tfrac{1}{2}(n-1)(\alpha+\beta)$$

and

$$\begin{aligned}0=d_0-d_0=dx^{C_1}-dx^{C_2}&=d_{a_2b_2}+d_{a_pa_1}-d_{a_2a_1}-d_{b_1b_2}\\&=\tfrac{1}{2}(\alpha+\beta)+\alpha-d_{a_2a_1}-\beta\end{aligned}$$

and thus $d_{a_2a_1}=\tfrac{1}{2}(3\alpha-\beta)$. Similarly, we can show $d_{a_3a_2}=\tfrac{1}{2}(3\alpha-\beta)$.

On the other hand M_p-a_1 contains a hamiltonian circuit K which contains the path $[a_p,b_p,b_1,b_2,a_2,a_3,b_3]$, thus the path

$$K_1:=(K-([b_p,b_1]\cup[b_2,a_2,a_3,b_3]))\cup([a_3,a_2,a_1,b_1]\cup[b_2,b_3])$$

is hamiltonian in M_P' and therefore

$$\begin{aligned}\tfrac{1}{2}(n-1)(\alpha+\beta)=dx^{K_1}&\\&=3\alpha-\beta+\tfrac{1}{2}((p-2)(\alpha+\beta)+(p+1)\beta+(p-3)\alpha)\\&=(p-1)(\alpha+\beta)+\tfrac{1}{2}(3\alpha-\beta),\end{aligned}$$

which shows $\alpha=\beta$, and hence

$$d_{uv}=\alpha\quad\text{for all }(u,v)\in E'.$$

Now Lemma 2.2 implies that $x(E)\leqslant n-1$ is a facet of $\tilde{P}_T^n$.

Remark 3.5. By similar arguments as in the proofs of 3.2 and 3.3 one can show that the statement of Proposition 3.3 also holds for the smallest Marguerite $M_3=(V,E)$, i.e. that any valid inequality $dx\leqslant d_0$ such that $H_E\subset H_d\subset\tilde{P}_T^6$ satisfies

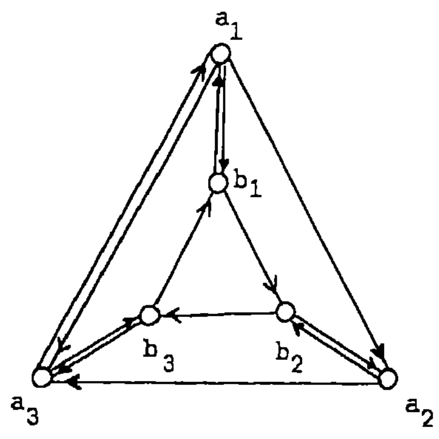

Fig. 3.

$d_{uv}=\alpha$ $\forall(u,v)\in A$, $d_{uv}=\beta$ $\forall(u,v)\in B$, $d_{uv}=\frac{1}{2}(\alpha+\beta)$ $\forall(u,v)\in D$. However, the digraph obtained from M_3 by adding two arcs (a_{i+1},a_i), (a_i,a_{i-1}) or (b_{i+1},b_i), (b_i,b_{i-1}) is not hypohamiltonian, thus Theorem 3.4 is not applicable.

The maximal hypohamiltonian digraphs containing M_3 are those digraphs where two arcs of the form (a_{i+1},a_i), (b_{j+1},b_j) are added. Unfortunately these digraphs do not induce facets since one can show that $d_{a_{i+1},a_i}=\frac{3}{2}\alpha-\frac{1}{2}\beta$ and $d_{b_{i+1},b_i}=\frac{3}{2}\beta-\frac{1}{2}\alpha$ holds. The inequalities induced by these digraphs are subfacets, i.e. are the intersection of two facets. Since $\tilde{P}_T^6$ is a monotone polytope all coefficients of d have to be nonnegative and the right hand side positive. Thus, by taking the extremal values for α and β we get the two desired facets by setting $\alpha=3\beta$ or $\beta=3\alpha$. Therefore, the facets that are generated by the Marguerites M_3 are of the form

$$x(B)+2x(D)+3x(A)+4x_{a_{i+1},a_i}\leqslant 10$$

or

$$x(A)+2x(D)+3x(B)+4x_{b_{i+1},b_i}\leqslant 10.$$

where we fixed the parameter β to be 1 in the first and 3 in the second case.

The digraph induced by the positive coefficients of the first inequality is also a hypohamiltonian digraph, cf. Fig. 3. Therefore we have an example of a hypohamiltonian digraph which induces a facet of $\tilde{P}_T^6$ in an unexpectedly complicated way in that the arcs have to be weighted and are not just summed up in the usual manner, cf. Proposition 2.1.

This example also shows that even the small polytope $\tilde{P}_T^6$ has facets which are quite complex and far away from having (0, 1)-coefficients only.

4. Other hypohamiltonian facets

In addition to trivially directed hypohamiltonian graphs and Marguerites several other digraphs obtained by means of three constructions were shown to be hypohamiltonian in [8]. In this section we will consider those hypohamiltonian digraphs which are produced by using Marguerites in these constructions. By describing the constructions we restrict ourselves to Marguerites, the general techniques are described in [8]. Until the end of this paragraph the nodes of an odd Marguerite M_p are labeled $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ and the nodes of M_q are labeled $a'_1, a'_2, \ldots, a'_q, b'_1, b'_2, \ldots, b'_q$.

Construction HH2. Let M_p, M_q, $p, q \geq 5$, be two odd Marguerites. Let G' be the digraph obtained by adding M_p and M_q, identifying one node of M_p and one node of M_q, say a_1 and a'_1, into a node x and by adding the arcs $N = \{(b_2, a'_2), (b'_2, a_2), (a_p, b'_q), (a'_q, b_p)\}$, cf. Fig. 4. Then G' is a hypohamiltonian digraph.

Remark 4.1. (a) It is easily seen that all pairs of nodes a_i, b_i, $i = 3, \ldots, p-1$, and a'_j, b'_j, $j = 3, \ldots, q-1$, satisfy structure 1, cf. Definition 3.1.

(b) Furthermore, it is not hard to check that the pair of nodes a_2, b_2 satisfies structure 1 with respect to the nodes a'_q, x, a_3 (for a_2) and b_p, b_1, b_3 (for b_2), i.e. there is a hamiltonian circuit C in $G' - a_2$ containing the path $[a'_q, x, b_1, b_2, b_3, a_3]$, and in $G' - b_2$ there is a hamiltonian circuit containing the path $[b_p, b_1, x, a_2, a_3, b_3]$. Similarly, the nodes a_p, b_p satisfy structure 1 since there is a hamiltonian circuit C_p in $G' - a_p$ containing the path $[a_{p-2}, a_{p-1}, b_{p-1}, b_p, b_1, x]$, and there is a hamiltonian circuit K_p in $G' - b_p$ containing $[b_{p-2}, b_{p-1}, a_{p-1}, a_p, x, b_1]$.

Theorem 4.2. *Let $G' = (V, E')$ be a hypohamiltonian digraph of order n obtained from two odd Marguerites M_p, M_q be Construction* HH2. *Then the inequality $x(E) \leq n-1$ is a facet of $\tilde{P}^n_T$ for all maximal hypohamiltonian digraphs $G = (V, E)$ with $E' \subset E$.*

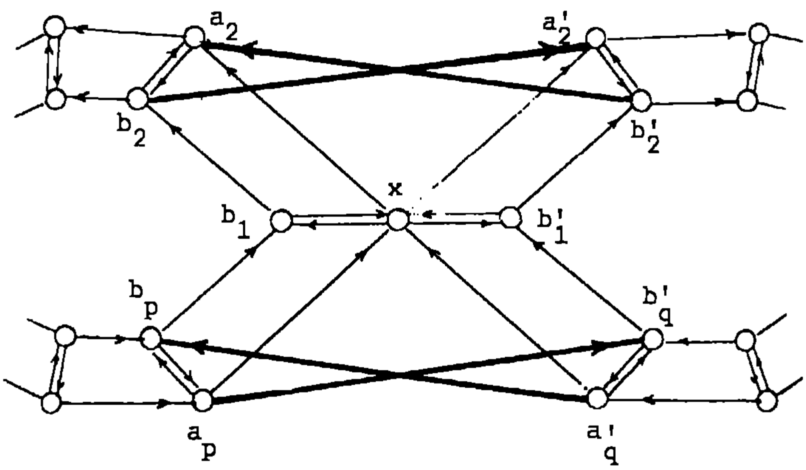

Fig. 4.

Proof. Let $dx \leq d_0$ be a valid inequality such that $H_E \subset H_d$. We apply Lemma 3.2. By Remark 4.1 a_2, b_2 and the node pairs $a_4, b_4; a_6, b_6; \ldots; a_{p-1}, b_{p-1}$ satisfy structure 1, thus we get

$$d_{a_q',x} = d_{a_2,a_3} = d_{a_4,a_5} = \cdots = d_{a_{p-1},a_p} = \alpha, \tag{1}$$

$$d_{b_p,b_1} = d_{b_2,b_3} = d_{b_4,b_5} = \cdots = d_{b_{p-1},b_p} = \beta, \tag{2}$$

$$d_{b_1,x} = d_{b_3,a_3} = d_{b_5,a_5} = \cdots = d_{b_p,a_p} = \gamma, \tag{3}$$

$$d_{x,b_1} = d_{a_3,b_3} = d_{a_5,b_5} = \cdots = d_{a_p,b_p} = \delta, \tag{4}$$

$$d_{b_2,a_2} = \alpha + \beta - \gamma, \tag{5}$$

$$d_{a_2,b_2} = \alpha + \beta - \delta, \tag{6}$$

$$d_{x,a_2} = \alpha + \delta - \gamma, \tag{7}$$

$$d_{b_1,b_2} = \beta + \gamma - \delta. \tag{8}$$

By applying Lemma 3.2 to the pairs of nodes $a_3, b_3;\ a_5, b_5; \ldots; a_p, b_p$ which satisfy structure 1 according to Remark 4.1 we obtain

$$\begin{aligned} \beta + \gamma - \delta &= d_{b_1,b_2} = d_{b_3,b_4} = \cdots = d_{b_p,b_1} = \beta && \text{by (2) and (8)},\\ \alpha + \beta - \gamma &= d_{b_2,a_2} = d_{b_4,a_4} = \cdots = d_{b_1,x} = \gamma && \text{by (3) and (5)},\\ \alpha + \delta - \gamma &= d_{x,a_2} = d_{a_3,a_4} = \cdots = d_{a_p,x} && \text{by (7)}, \end{aligned}$$

i.e. we have $\gamma = \delta$ and $\gamma = \delta = \frac{1}{2}(\alpha + \beta)$. Thus we have shown

$$d_{a_p,x} = d_{x,a_2} = d_{a_i,a_{i+1}} = \alpha, \qquad i = 2, \ldots, p-1, \tag{9}$$

$$d_{b_p,b_1} = d_{b_i,b_{i+1}} = \beta, \qquad i = 1, \ldots, p-1, \tag{10}$$

$$d_{a_i,b_i} = d_{b_i,a_i} = \tfrac{1}{2}(\alpha + \beta), \qquad i = 2, \ldots, p, \tag{11}$$

$$d_{x,b_1} = d_{b_1,x} = \tfrac{1}{2}(\alpha + \beta), \tag{12}$$

$$d_{a_q',x} = \alpha. \tag{13}$$

By symmetry we obtain for the other Marguerite M_q using (13):

$$d_{x,a_2'} = d_{a_q',x} = d_{a_j',a_{j+1}'} = \alpha, \qquad j = 2, \ldots, q-1, \tag{14}$$

$$d_{b_j',b_{j+1}'} = d_{b_q',b_1'} = \varepsilon, \qquad j = 1, \ldots, q-1, \tag{15}$$

$$d_{b_i',a_i'} = d_{a_i',b_i'} = \tfrac{1}{2}(\alpha + \varepsilon), \qquad i = 2, \ldots, q, \tag{16}$$

$$d_{x,b_1'} = d_{b_1',x} = \tfrac{1}{2}(\alpha + \varepsilon). \tag{17}$$

Now, $M_p - a_1$ contains a hamiltonian circuit C_1 containing the path $[a_p, b_p, b_1, b_2, a_2, a_3]$ and $M_q - a_1'$ contains a hamiltonian circuit C_2 containing $[a_{q-1}', a_q', b_q', b_1', b_2', a_2']$. The paths

$$C := (C_1 - \{(b_2, a_2)\}) \cup (C_2 - \{(b_2', a_2')\} \cup \{(x, a_2), (b_2, a_2')\}),$$

$$C' := (C_1 - \{(b_2, a_2)\}) \cup (C_2 - \{(b_2', a_2')\} \cup \{(x, a_2'), (b_2', a_2)\}),$$

$$K := (C_1 - \{(b_p, b_1)\}) \cup (C_2 - \{(b_1', b_2')\} \cup \{(b_1', x), (x, b_1)\})$$

are hamiltonian paths in G'. Since $H_E \subset H_d$ and K does not contain any arc of N we can calculate from (9)–(17)

$$\begin{aligned} d_0 = dx^K &= \tfrac{1}{2}p(\alpha+\beta)+\tfrac{1}{2}(p-1)\alpha+\tfrac{1}{2}(p-1)\beta \\ &\quad +\tfrac{1}{2}q(\alpha+\varepsilon)+\tfrac{1}{2}(q-1)\alpha+\tfrac{1}{2}(q-1)\varepsilon \\ &= \tfrac{1}{2}(2p-1)(\alpha+\beta)+\tfrac{1}{2}(2q-1)(\alpha+\varepsilon) \end{aligned} \tag{18}$$

Furthermore

$$\begin{aligned} 0 = d_0 - d_0 = dx^C - dx^{C'} &= d_{x,a_2} + d_{b_2,a_2'} - d_{x,a_2'} - d_{b_2',a_2} \\ &= \alpha + d_{b_2,a_2'} - \alpha - d_{b_2',a_2} \end{aligned}$$

i.e.

$$d_{b_2,a_2'} = d_{b_2',a_2}, \tag{19}$$

and similarly we get by symmetry

$$d_{a_q',b_p} = d_{a_p,b_q'}. \tag{20}$$

Using (18) and

$$\begin{aligned} d_0 = dx^C = \tfrac{1}{2}((p-2)(\alpha+\beta)+(p-1)\alpha+(p+1)\beta+(p-2)(\alpha+\varepsilon) \\ +(q-1)\alpha+(q+1)\varepsilon)+\alpha+d_{b_2,a_2'}, \end{aligned}$$

we obtain $d_{b_2,a_2'}=\alpha$. Now (19) implies

$$d_{b_2,a_2'} = d_{b_2',a_2} = \alpha, \tag{21}$$

and similarly (20) gives

$$d_{a_p',b_p} = d_{a_p,b_q'} = \alpha \tag{22}$$

We define two new hamiltonian paths in G':

$$\begin{aligned} S &:= (C_1-\{(b_2,a_2)\}) \cup (C_2-[b_1',b_2',a_2']) \cup \{(b_1',x),(b_2',a_2)\} \cup [b_1,b_2,a_2'], \\ T &:= (C_1-[b_1,b_2,a_2]) \cup (C_2-\{(b_2',a_2')\}) \cup \{(b_1,x),(b_2,a_2')\} \cup [b_1',b_2',a_2]. \end{aligned}$$

By taking the difference of the incidence vectors we get from (17), (10), (12), (15)

$$\begin{aligned} 0 = dx^S - dx^T &= d_{b_1',x} + d_{b_1,b_2} - d_{b_1,x} - d_{b_1',b_2'} \\ &= \tfrac{1}{2}(\alpha+\varepsilon)+\beta-\tfrac{1}{2}(\alpha+\beta)-\varepsilon \\ &= \tfrac{1}{2}\beta-\tfrac{1}{2}\varepsilon. \end{aligned}$$

Therefore $\beta=\varepsilon$.

Considering one more hamiltonian path in G' we will get the desired result $\alpha=\beta=\varepsilon$. Let

$$\begin{aligned} P := (C_1-[a_p,b_p,b_1,b_2,a_2,a_3]) \cup (C_2-[a_{q-1}',a_q',b_q',b_1',b_2',a_2']) \\ \cup [a_p,b_q',a_q',b_p,b_1,x,b_1',b_2',a_2,b_2,a_2'], \end{aligned}$$

then

$$\begin{aligned} d_0 &= (p+q-1)(\alpha+\beta) = dx^p \\ &= \tfrac{1}{2}(p-1)(\alpha+\beta)+\tfrac{1}{2}(p-3)\alpha+\tfrac{1}{2}(p-1)\beta \\ &\quad +\tfrac{1}{2}(q-1)(\alpha+\beta)+\tfrac{1}{2}(q-3)\alpha+\tfrac{1}{2}(q-1)\beta+4\alpha \\ &= (p+q)\alpha+(p+q-2)\beta \end{aligned}$$

and this implies $\alpha = \beta$.

Altogether we have shown now that

$$d_{uv} = \alpha \quad \text{for all } (u, v) \in E'.$$

Lemma 2.2 completes the proof of the theorem.

By using odd Marguerites in Construction HH1 of [8] we get the following digraphs.

Construction HH1. Let M_p, M_q, $p, q \geq 5$, be two odd Marguerites. Let M_q' be the hypohamiltonian digraph obtained from M_q by adding two arcs (a_{i+1}', a_i'), (a_i', a_{i-1}'). Take any node of M_p, say a_1, and any node of M_q different from a_{i-1}', a_i', a_{i+1}', say a_1'. Let $N_p = M_p - a_1$, $N_q = M_q' - a_1'$. We add the digraphs N_p and N_q and identify the nodes b_1 and b_1' into one node z. Furthermore, we add the arcs (b_q', a_2), (a_q', b_2), (b_p, a_2'), (a_p, b_2') and call this digraph G'. Then G' is a hypohamiltonian digraph.

Theorem 4.3. *Let $G' = (V, E')$ be a hypohamiltonian digraph of order n obtained from two odd Marguerites M_p, M_q' as described in Construction* HH1. *Let $G = (V, E)$ be a maximal hypohamiltonian digraph with $E' \subset E$. Then the hypohamiltonian inequality*

$$x(E) \leq n-1.$$

is a facet of $\tilde{P}_T^n$.

The proof of Theorem 4.3 is technically a little more complicated than that of Theorem 4.2, but does not require any new ideas, therefore it is only outlined. Obviously, most of the node pairs a_i, b_i satisfy structure 1, thus we can apply Lemma 3.2 like in the proof of 4.2. The only difficulty arises with the node pairs a_2, b_2; a_2', b_2'; a_p, b_p; a_q', b_q'. But here similar arguments like in Lemma 3.2 allow the conclusion that certain arcs will carry the same weight π. Putting these observations together, making use of the newly added arcs, and of Lemma 2.2 one can show that any valid inequality $dx \leq d_0$ with $H_E \subset H_d$ is a multiple of the hypohamiltonian inequality $x(E) \leq n-1$ which proves the theorem.

The third way to obtain hypohamiltonian digraphs presented in [8] is the following

Construction HH3. Let $M_p, M_q, p, q \geq 5$ be two odd Marguerites. Take one arbitrary node from each digraph M_p and M_q, say a_1 and a'_1. Let $N_p := M_p - a_1$, $N_q := M_q - a'_1$. Add N_p and N_q identifying the nodes a_p and a'_2 into one node z_1, the nodes b_1 and b'_1 into z_2, and the nodes a_2, a'_q into the node z_3, and call this digraph G'. Then G' is a hypohamiltonian digraph.

Theorem 4.4. *Let $G' = (V, E')$ be a hypohamiltonian digraph obtained from two odd Marguerites as defined in Construction* HH3. *Then for any maximal hypohamiltonian digraph $G = (V, E)$ with $E' \subset E$ the inequality*

$$x(E) \leq n - 1$$

is a facet of $\tilde{P}^n_T$.

The proof of Theorem 4.4 is also similar to the proof of Theorem 4.2 and therefore omitted. Again, use is made of structure 1, Lemma 3.2 and Lemma 2.2 nearly in the same fashion. Complication only arises in considering the new nodes z_1, z_2, z_3 and their neighbours. But here use can be made of the fact that we are able to describe the hamiltonian circuits in a vertex-deleted Marguerite exactly.

5. Conclusions

It is known, cf. [1, 2, 11], that cubic hypohamiltonian graphs of order $n = 10$ and of every even order $n \geq 18$ exist (except possibly for 24 and 32). Since the trivial direction of these graphs gives rise to hypohamiltonian facets of $\tilde{P}^n_T$ by Theorem 2.3, we know that for all these n the asymmetric travelling salesman polytopes have complicated facets of this type. Furthermore, by considering Marguerites and the Constructions HH1, HH2, HH3 using Marguerites, cf. Theorems 4.2, 4.3, 4.4, we obtained hypohamiltonian digraphs of orders 10, 14, 15 and $4k+1$, $4k+2$, $4k+3$, $k \geq 4$, which induce facets of the asymmetric travelling salesman polytope, thus altogether we have hypohamiltonian facets of $\tilde{P}^n_T$ for all $n \geq 14$ (except 16, 24, 32).

We actually know more hypohamiltonian digraphs that induce facets of asymmetric travelling salesman polytopes, but unfortunately the proofs of these results are rather involved and we did not succeed in finding a common theory for all of them like e.g. the sufficient condition in [5] for a hypohamiltonian or hypotraceable graph to induce a facet of the monotone symmetric travelling salesman polytope.

As Remark 3.5 shows, not all (maximal) hypohamiltonian digraphs induce facets in the straightforward way. Maybe this case is just an irregularity of the small dimension 6, since for instance for the next larger order any completion of the hypohamiltonian digraph of order 7 shown in Fig. 4.7 in [8] defines a facet of $\tilde{P}^7_T$.

We have also tried to check whether hypohamiltonian digraphs induce facets of the polytope $P_T^n = \text{conv}\{x^T \in R^m : T \text{ hamiltonian circuit in } K_n\}$, $m = n(n-1)$, considered in [7]. Since the dimension of this polytope is $n(n-3)+1$, cf. [7], proofs of the type given above become quite awkward. By numerical calculation we have shown that the facets of $\tilde{P}_T^6$ given in Remark 3.5 are also facets of P_T^6, and that all completions of the hypohamiltonian digraph of order 7, see above, induce facets of P_T^7, thus it seems likely that most of the hypohamiltonian facets of $\tilde{P}_T^n$ carry over to P_T^n.

It was shown in [6] that a combinatorial optimization problem is solvable in polynomial time if and only if the separation problem for the associated polytope is solvable in polynomial time; in our case, the ATSP is solvable in polynomial time if and only if we can solve the following problem in polynomial time: given $y \in \mathbb{R}^{n(n-1)}$ conclude with one of the following (a) asserting that $y \in \tilde{P}_T^n$ or (b) finding a vector $c \in \mathbb{R}^{n(n-1)}$ such that $cx < cy$ for all $c \in \tilde{P}_T^n$. We believe that the hypohamiltonian facets of $\tilde{P}_T^n$ and the hypotraceable facets given in [9] constitute an obstacle for designing a polynomial separation algorithm. The reason is that to date no nearly satisfactory characterization of hypohamiltonian or hypotraceable digraphs has been found and that it appears to be difficult to recognize whether a digraph is hypohamiltonian or not (the complexity status of this problem is unknown, in the straightforward method one $\overline{\text{NP}}$-complete problem and n NP-complete problems have to be solved). Any separation algorithm must handle some maximal hypohamiltonian and hypotraceable digraphs either implicitly or explicitly, and we have no idea how this could be done. Furthermore, in our opinion the results above indicate that a complete linear characterization of the travelling salesman polytope can never be given explicitly, but this is of course just speculation.

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