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ON THE STRUCTURE OF THE MONOTONE ASYMMETRIC TRAVELLING SALESMAN POLYTOPE II: HYPOTRACEABLE FACETS*

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The monotone asymmetric travelling salesman polytope $\tilde{P}^n_T$ is the convex hull of the incidence vectors of all hamiltonian circuits and subsets of hamiltonian circuits of the complete digraph of order n. It is shown in this paper that certain hypotraceable digraphs, $G=(V,E)$, i.e. digraphs which do not contain a hamiltonian path but G-v does for all $v \in V$, defines facets $x(E) \le n-2$ for $\tilde{P}^k_T$ for all $k \ge n$. Since hypotraceable digraphs constitute a very complicated class of digraphs these results show that the asymmetric travelling salesman polytope $\tilde{P}^n_T$ has a large number of highly complex facets.

Key words: Asymmetric Travelling Salesman Problem, Facets, Cutting Planes, Hypotraceable Digraphs, Hypohamiltonian Digraphs.

1. Introduction and notation

The purpose of this note is to demonstrate that there are not only language theoretical but also polyhedral reasons for the intractability of the asymmetric travelling salesman problem (ATSP). This is done by associating a polytope $\tilde{P}^n_T$ with the n-city ATSP in a natural way which has the property that every asymmetric travelling salesman problem can be solved as a linear program over $\tilde{P}^n_T$. We then show that certain hypotraceable digraphs (cf. [4,6]) induce facets of $\tilde{P}^n_T$. Since a good characterization of hypotraceable digraphs seems to be hard to obtain, these results prove that the polytope $\tilde{P}^n_T$ is highly complex and that it is very unlikely that a complete linear characterization of $\tilde{P}^n_T$ can ever be given explicitly.

A *digraph* $G=(V,E)$ consists of a finite set V of *nodes* and a set E of ordered pairs of distinct elements of V called *arcs*.

If $e=(u,v)\in E$ then u and v are called *endnodes* of e, u is said to be the *initial node* and v the *terminal node* of e; u and v are called *neighbours*. $|V|$ is the *order* of G. The set of all arcs in E having both endnodes in a subset W of V

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is denoted by $E(W)$. The set of all nodes which are endnodes of at least one arc of a subset F of E is denoted by $V(F)$.

If $|V| = n$, then E_n is the set of all ordered pairs (u, v), $u \neq v$, of elements of V, and the digraph $K_n = (V, E_n)$ is called *complete*. The set $\omega^+(v)$ resp. $\omega^-(v)$ is the set of all arcs in E having v as its initial resp. terminal node; $\omega(v) = \omega^+(v) \cup \omega^-(v)$. $d^+(v) = |\omega^+(v)|$ is called *outdegree* of v; $d^-(v) = |\omega^-(v)|$ is called *indegree* of v, and $d(v) = d^+(v) + d^-(v)$ is called *degree* of v. A node $v \in V$ with $\omega^-(v) = \emptyset$ resp. $\omega^+(v) = \emptyset$ is called a *source* resp. *sink*.

$N^+(v) = \{w \in V: (v, w) \in E\}$, $N^-(v) = \{w \in V: (w, v) \in E\}$, $N(v) = N^+(v) \cup N^-(v)$. Whenever there is an ambiguity about the digraph with respect to which the above symbols are used, we write a subscript for clarification ($\omega_G(v)$, $N_G(v)$, $E_G(W)$ etc).

A non-empty set of arcs $P = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k)\} \subset E$ where $v_i \neq v_j$ for $i \neq j$, is called a *path* of length $k-1$ and is denoted by $[v_1, v_2, \ldots, v_k]$. If $(v_k, v_1) \in E$, then $C := P \cup \{(v_k, v_1)\}$ is called a *circuit* of length k and is denoted by $\langle v_1, v_2, \ldots, v_k \rangle$. A circuit (path) of length $|V|$ ($|V|-1$) is called *hamiltonian*; such a circuit is also called a *tour*. Given a digraph $G = (V, E)$ then $G - v$ is the digraph with node set $V - \{v\}$ and all arcs in E which do not contain the node v.

Definition 1.1. Let $G = (V, E)$ be a digraph.

(a) G is called *traceable* (*hamiltonian*) if G contains a hamiltonian path (circuit).

(b) G is called *hypotraceable* (*hypohamiltonian*) if

(b_1) G is not traceable (hamiltonian) and

(b_2) G-v is traceable (hamiltonian) for all $v \in V$.

For $u, v \in V$ we define $G - (u, v)$ to be the digraph $(V, E - \{(u, v)\})$ and $G + (u, v)$ the digraph $(V, E \cup \{(u, v)\})$.

If a digraph G has a property π then G is called *maximal* (*minimal*) with respect to π if $G + (u, v)(G - (u, v))$ does not have property π for all pairs $(u, v) \notin E$ $((u, v) \in E)$.

Hypohamiltonian and hypotraceable (undirected) graphs are defined analogously. Given a graph $G = [V, E]$ then the digraph $\vec{G} = (V, \vec{E})$ obtained from G by substituting the two arcs (u, v), (v, u) for every edge $\{u, v\} \in E$ is called the *trivial direction* of G.

Given n cities $1, 2, \ldots, n$ and distances $c_{ij} \in \mathbb{R}$ between every pair of cities $i \neq j$ where the distance c_{ij} from i to j can be different from c_{ji}. The *asymmetric travelling salesman problem* is to find a tour T such that $\sum_{(i,j)\in T} c_{ij}$ is as small as possible. The ATSP can be described in graphical terminology as follows: Let $K_n = (V, E_n)$ be the complete digraph on n nodes and let $c_{ij} \in \mathbb{R}$ be "distances" for all $(i, j) \in E_n$. Find the shortest hamiltonian circuit in K_n.

A polytope having (0, 1)-vertices can be associated with the ATSP in a natural way. Let $\tilde{T}_n$ be the set of arc sets which are hamiltonian circuits or subsets of hamiltonian circuits in K_n. With each arc $e = (i, j) \in E_n$ we associate a variable

$x_e = x_{ij}$, and with each $T \in \tilde{T}_n$ we associate an *incidence vector* $x^T \in \mathbb{R}^m$, where $m := |E_n| = n(n-1)$, by setting $x_e^T = 1$ if $e \in T$ and $x_e^T = 0$ if $e \notin T$. The convex hull $\tilde{P}_T^n$ of all incidence vectors of tours and subsets of tours is called *(monotone) asymmetric travelling salesman polytope* (c.f. [1, 5]), i.e.

$$\tilde{P}_T^n := \mathrm{conv}\{x^T \in \mathbb{R}^m : T \in \tilde{T}_n\}.$$

Clearly, every asymmetric travelling salesman problem can be solved as a linear maximization problem over $\tilde{P}_T^n$.

By definition $\tilde{P}_T^n$ contains the zero vector and all unit vectors, thus it is fully dimensional, i.e. $\dim \tilde{P}_T^n = m$. We call an inequality $ax \le a_0$ *valid* with respect to $\tilde{P}_T^n$, if $a \ne 0$ and $\tilde{P}_T^n \subseteq \{x \in \mathbb{R}^m : ax \le a_0\}$; a valid inequality $ax \le a_0$ is called *maximal*, if the inequality obtained by increasing any component a_{ij} of a by any $\epsilon > 0$ is not valid; a valid inequality $ax \le a_0$ is called a *facet* of $\tilde{P}_T^n$ if

$$\dim(\tilde{P}_T^n \cap \{x \in \mathbb{R}^m : ax = a_0\}) = \dim \tilde{P}_T^n - 1 = m - 1.$$

Clearly, any facet with nonzero right-hand side is a maximal inequality.

As $\tilde{P}_T^n$ is fully dimensional there exists a unique (up to a constant factor) finite non-redundant system of linear inequalities $Ax \le b$ such that $\tilde{P}_T^n = \{x \in \mathbb{R}^m : Ax \le b\}$. This system is given by the set of all facets of $\tilde{P}_T^n$. Thus, in order to characterize $\tilde{P}_T^n$ completely and non-redundantly we have to characterize all the facets of $\tilde{P}_T^n$.

It was shown in [1] that the trivial inequalities, the subtour-elimination constraints, some comb inequalities and various other classes of inequalities are facets of $\tilde{P}_T^n$. All these facets share the property that they are describable in a "combinatorially pleasant way", hence one might think that $\tilde{P}_T^n$ has "nice" facets only. But here we will show that $\tilde{P}_T^n$ also has facets which are highly complex.

To show that a certain inequality $ax \le a_0$ is a facet of $\tilde{P}_T^n$ we mainly use the following technique. We take another valid inequality $bx \le b_0$ with the property $\tilde{P}_T^n \cap \{x : ax = a_0\} \subseteq \tilde{P}_T^n \cap \{x : bx = b_0\}$ and show that $b = \pi a$ where $\pi \in \mathbb{R} - \{0\}$; this proves that $\tilde{P}_T^n \cap \{x : ax = a_0\}$ is contained in one hyperplane only, and thus $\dim(\tilde{P}_T^n \cap \{x : ax = a_0\}) = m - 1$. Here, intensive use is made of the structural properties of the arc sets $T \in \tilde{T}_n$, which satisfy $ax^T = a_0$, to conclude, that certain components b_e of b have to satisfy certain equations and therefore are equal to a_e.

To shorten notation we abbreviate the sum $\sum_{(i,j)\in E} x_{ij} = \sum_{e \in E} x_e$ by $x(E)$. For a valid inequality $ax \le a_0$ we define its *face* to be $H_a = \{x \in \tilde{P}_T^n : ax = a_0\}$. The face of a valid inequality $x(E) \le r$ is denoted by H_E.

2. Sufficient conditions

In order to relate a hypotraceable digraph $G = (V, E)$ of order n to the polytope $\tilde{P}_T^k$, $k \ge n$, we consider G as a subdigraph of K_k, thus $V \subseteq \{1, \ldots, k\}$ and $E \subseteq E_k$.

Proposition 2.1. *Let $G=(V,E)$ be a hypotraceable digraph of order n. Then the hypotraceable inequality $x(E)\leq n-2$ is a valid inequality with respect to $\tilde{P}_T^k$ for all $k\geq n$.*

Proof. By definition, the longest path in a hypotraceable digraph contains exactly $n-2$ arcs. Thus for any $T\in\tilde{T}_k$ we have $|T\cap E|\leq n-2$.

Clearly, only those hypotraceable inequalities, which are maximal, are candidates for facets of $\tilde{P}_T^k$. It can be easily checked that a hypotraceable inequality is maximal with respect to $\tilde{P}_T^k$ if and only if the corresponding hypotraceable digraph G is maximal, i.e. if $G+e$ is traceable for all $e\notin E$.

Although this observation suggests a concentration on maximal digraphs we shall see that most of the structural properties of such digraphs are largely determined by the underlying minimal hypotraceable digraphs. Therefore, we are able to carry out most of the proof load on minimal hypotraceable digraphs, which in general are rather sparse and easier to handle, and get the desired results by completion.

To shorten notation we say that a hypotraceable digraph $G=(V,E)$ of order n has the *affine independence property* if the following holds

(a) There exist $m:=|E|$ subsets $F_1,F_2,\dots,F_m$ of E such that $F_i\in\tilde{T}_n$, the incidence vectors x^{F_i}, $i=1,\dots,m$ are affinely independent and satisfy $x(E)\leqq n-2$ with equality.

For such vectors affine and linear independence are of course identical. It is easy to prove that the affine independence property (a) is equivalent to

(b) For every inequality $dx\leqq d_0$ which is valid with respect to $\tilde{P}_T^n$ and satisfies $H_E\subseteq H_d$ there is $\alpha>0$ such that $d_{uv}=\alpha$ for all $(u,v)\in E$.

Proposition 2.2 (Completion). *Let $G'=(V,E')$ be a hypotraceable digraph of order n with the affine independence property. Then for every maximal hypotraceable digraph $G=(V,E)$ with $E'\subseteq E$, the hypotraceable inequality $x(E)\leqq n-2$ is a facet of $\tilde{P}_T^n$.*

The proof is similar to the proof of Lemma 2.2 in [5] and therefore omitted. It was shown in [5] that hypohamiltonian facets are not trivially liftable but—as the following proposition shows—hypotraceable facets are.

Proposition 2.3 (Trivial lifting). *Let $G=(V,E)$ be a hypotraceable digraph of order n such that $x(E)\leq n-2$ is a facet of $\tilde{P}_T^n$. Then $x(E)\leq n-2$ is also a facet of $\tilde{P}_T^k$ for all $k>n$.*

Proof. Since $x(E)\leq n-2$ is a facet of $\tilde{P}_T^n$ there are $|E|$ arc sets $T_i\in\tilde{T}_n$ with $T_i\subseteq E$ such that the vectors $x^{T_i}\in\mathbb{R}^m$, $i=1,\dots,|E|$, are linearly independent and satisfy the hypotraceable inequality with equality. The arc sets T_i are paths or unions of paths, thus $T_i \in \bar{T}_k$, $i = 1, \ldots, |E|$, and the incidence vectors x^{T_i} now taken with respect to $\tilde{P}_T^k$, i.e. $x^{T_i} \in \mathbb{R}^q$, $q = k(k-1)$, are also linearly independent and satisfy $x(E) \le n-2$ with equality.

Let $e \in E_k - E$. If e has both endnodes in V, then $G + e$ contains a hamiltonian path P_e containing e. If e has at most one endnode in V, then let u be this endnode, if it exists, otherwise let u be any node in V. $G - u$ contains a hamiltonian path P, thus $P_e := P \cup \{e\} \in \bar{T}_k$, $|P_e \cap E| = n-2$, and therefore $x^{P_e} \in H_E$.

The set of vectors $\{x^{T_i}: i = 1, \ldots, |E|\} \cup \{x^{P_e}: e \in E_k - E\}$ is a set of q linearly independent vectors in $\tilde{P}_T^k$ which all satisfy $x(E) \le n-2$ with equality. This proves that $\dim(H_E) = q - 1 = \dim \tilde{P}_T^k - 1$, thus $x(E) \le n-2$ is a facet of $\tilde{P}_T^k$.

Proposition 2.3 proves that given a hypotraceable facet $ax \le a_0$ of $\tilde{P}_T^n$ we can lift this facet to any higher dimension by just giving zero coefficients to the new variables. This shows that hypotraceable facets are inherited, i.e., whenever we have shown that one of these digraphs of order n induces a facet of $\tilde{P}_T^n$ this digraph will keep its facet-inducing property for any travelling salesman polytope of larger dimension.

It was shown in [2] that many hypohamiltonian as well as hypotraceable graphs induce facets of the symmetric travelling salesman polytope. A certain technical property which we now translate into "directed language" played an important role in the proofs.

Definition 2.4. Let $G = [V, E]$($D = (V, E)$) be a hypotraceable graph (digraph). The node $v \in V$ is said to have *property* Δ(Δ^+ *resp.* Δ^-) with respect to G(D) if for any two neighbours $v_1, v_2 \in N(v)$($v_1, v_2 \in N^+(v)$ resp. $v_1, v_2 \in N^-(v)$) at least one of the following properties is satisfied:

(1) $G - v_1$($D - v_1$) contains a hamiltonian chain (path) which contains the edge $\{v, v_2\}$ (the arc (v, v_2) resp. the arc (v_2, v)).

(2) $G - v_2$($D - v_2$) contains a hamiltonian chain (path) which contains $\{v, v_1\}$ ((v, v_1) resp. (v_1, v)).

(3) There exists a node $v_3 \in N(v)$($v_3 \in N^+(v)$ resp. $v_3 \in N^-(v)$) such that both $G - v_1$($D - v_1$) and $G - v_2$($D - v_2$) contain a hamiltonian chain (path) which contains $\{v, v_3\}$ ((v, v_3) resp. (v_3, v)).

G(D) has *property* Δ(Δ^+ *resp.* Δ^-) if all nodes of G(D) have property Δ(Δ^+ resp. Δ^-).

If several digraphs are under consideration we use a subscript and write "v has property Δ_G^+" to say that v has property Δ^+ with respect to the digraph G.

It is not hard to see that the trivial direction of a hypotraceable graph is a hypotraceable digraph, cf. [4], furthermore, the following statement is easily verified.

Lemma 2.5. *If $G = [V, E]$ is a hypotraceable graph having property Δ then its* *trivial direction* $\vec{G} = (V, \vec{E})$ *is a hypotraceable digraph having properties* Δ^+ *and* Δ^-.

It was shown in [2] that most of the known classes of hypotraceable graphs have this property, thus, almost all trivially directed hypotraceable graphs have properties Δ^+ and Δ^-. For general hypotraceable digraphs we easily get:

Lemma 2.6. *Let* $G = (V, E)$ *be a hypotraceable digraph. Then every node* $v \in V$ *with* $d^+(v) \le 3 (d^-(v) \le 3)$ *has property* $\Delta^+(\Delta^-)$.

This lemma is best possible since there are hypotraceable digraphs with nodes v satisfying $d^+(v) = d^-(v) = 4$ which neither have property Δ^+ nor Δ^-. Unfortunately the properties Δ^+ and Δ^- are not sufficient conditions for a hypotraceable inequality to be a facet, but nevertheless they allow some helpful constructions.

Proposition 2.7. *Let* $G = (V, E)$ *be a hypotraceable digraph of order* n, *and let* $x(E) \le n-2$ *be the corresponding inequality. Let* $bx \le b_0$ *be any valid inequality for* $\tilde{P}^k_T$, $k \ge n$, *which satisfies* $H_E \subseteq H_b$. *Then the following holds.*

(a) *If* $v \in V$ *has property* Δ^+_G *then there exists* $\pi^+_v \in \mathbb{R}$ *such that* $b_e = \pi^+_v$ *for all* $e \in \omega^+_G(v)$.

(b) *If* $v \in V$ *has property* Δ^-_G *then there exists* $\pi^-_v \in \mathbb{R}$ *such that* $b_e = \pi^-_v$ *for all* $e \in \omega^-_G(v)$.

Proof. We prove (a); (b) follows analogously.

Let $v \in V$ have property Δ^+_G, let $e = (v, v_1)$, $f = (v, v_2) \in \omega^+_G(v)$. We have to show that $b_e = b_f$ holds. By Definition 2.4 one of the following cases must be satisfied.

(1) $G - v_1$ contains a hamiltonian path P which contains the arc (v, v_2). In this case let $Q := (P - \{(v, v_2)\}) \cup \{(v, v_1)\}$. Q is the union of two disjoint paths and all its $n-2$ arcs are contained in E, thus the incidence vectors of P and Q satisfy $x(E) \le n-2$ with equality, i.e. x^P, $x^Q \in H_E \subseteq H_b$. Therefore, $0 = b_0 - b_0 = bx^P - bx^Q = b_{vv_2} - b_{vv_1}$, which proves the assertion.

(2) $G - v_2$ contains a hamiltonian path P which contains the arc (v, v_1). The proof is the same as in case (1).

(3) There is a node $v_3 \in N^+_G(v)$ such that both $G - v_1$ and $G - v_2$ contain a hamiltonian path P_1, resp. P_2 which contains (v, v_3). Using (1) we first show $b_{vv_1} = b_{vv_3}$, (2) gives $b_{vv_2} = b_{vv_3}$ which proves our claim.

Corollary 2.8. *Let the general assumptions of Proposition 2.7 be satisfied, and let* G *have properties* Δ^+ *and* Δ^-. *Then for all* $v \in V$ *there exist* $\pi^+_v, \pi^-_v \in \mathbb{R}$ *such that* $b_e = \pi^+_v$ *for all* $e \in \omega^+_G(v)$ *and* $b_e = \pi^-_v$ *for all* $e \in \omega^-_G(v)$.

The following property will also turn out to be useful in subsequent proofs.

Definition 2.9. Let $G=(V, E)$ be a digraph and $v \in V$. A sequence of arcs (v_1, v_2), (v_3, v_2), (v_3, v_4), (v_5, v_4), $(v_5, v_6), \ldots, (v_{r-1}, v_{r-2})$, (v_{r-1}, v_r) in G, such that $v=v_1=v_r$, is called an *alternating v-trail.* A node v is said to have the *alternating trail property* (with respect to G) if there exists an alternating v-trail in G.

Note that by definition the number of arcs of an alternating v-trail is always odd and at least three.

Remark 2.10. If $v \in V$ has the alternating trail property then every node in $N^-(v) \cap N^+(v)$ has the alternating trail property.

The next proposition gives an important class of digraphs with the above property.

Proposition 2.11. *Let $G=[V, E]$ be a connected non-bipartite graph and $\vec{G}=(V, \vec{E})$ its trivial direction. Then every node $v \in V$ has the alternating trail property with respect to $\vec{G}$.*

Proof. By definition G contains a cycle $C=\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_r, v_1\}\}$ with $r \geq 3$ and odd. Then $(v_1, v_2), (v_3, v_2), \ldots, (v_r, v_{r-1}), (v_r, v_1)$ is an alternating v_1-trail in $\vec{G}$. Since G is connected we get the assertion by iteratively applying Remark 2.10.

Remark 2.12. No hypotraceable graph of odd order is bipartite.

The proof of 2.12 is obvious. Thus, the trivial directions of hypotraceable graphs of odd order are hypotraceable digraphs where all nodes have the alternating trail property. All hypotraceable graphs of even order that are known in the literature (cf. [7, 8]), are non-bipartite, since they result from constructions where hypohamiltonian graphs are involved and these always contain odd cycles. We conjecture that all hypotraceable graphs are non-bipartite but were not able to prove it for the case of even order.

The alternating trail property is very useful to obtain the following information.

Proposition 2.13. *Let $bx \leq b_0$ be a valid inequality with respect to $\tilde{P}_T^k$. Let $D=(W, F)$ be a subdigraph of K_k with the property that for all $v \in W$ there exist $\pi_v^+, \pi_v^- \in \mathbb{R}$ with $b_e=\pi_v^+$ for all $e \in \omega_D^+(v)$ and $b_e=\pi_v^-$ for all $e \in \omega_D^-(v)$. Then the following holds.*

If $v \in W$ has the alternating trail property with respect to the digraph D then $\pi_v^+=\pi_v^-$, i.e. there is $\pi_v \in \mathbb{R}$ with $b_e=\pi_v$ for all $e \in \omega_D(v)$.

Proof. If there exists an alternating v-trail in D, we have $\pi_v^+ = b_{vv_2} = \pi_{v_2}^- = b_{v_3v_2} = \pi_{v_3}^+ = b_{v_3v_4} = \pi_{v_4}^- = \cdots = \pi_{v_{r-2}}^- = b_{v_{r-1}v_{r-2}} = \pi_{v_{r-1}}^+ = b_{v_{r-1}v_r} = \pi_{v_r}^- = \pi_v^-$.

To make notation easier we introduce a special class of hypotraceable digraphs.

Definition 2.14. Let $G = (V, E)$ be a hypotraceable digraph.

(1) Let $\emptyset \neq W \subseteq V$, then G is called DT-*hypotraceable in* W if the following conditions are satisfied

(a) Each node $v \in W$ has properties Δ_G^+ and Δ_G^-.

(b) Each node $v \in W$ either has the alternating trail property with respect to $(W, E_G(W))$, or is a source or a sink of G.

(c) $(W, E_G(W))$ is connected.

(2) G is called DT-*hypotraceable* if there exists a node set $\emptyset \neq W \subseteq V$ such that G is DT-hypotraceable in W and if $V - W$ is a stable node set in G (no two nodes in $V - W$ are neighbours).

The next proposition shows the usefulness of the concepts introduced above.

Proposition 2.15. *Let $G = (V, E)$ be a hypotraceable digraph of order n which is DT-hypotraceable in $W \subseteq V$. Let $bx \leq b_0$ be a valid inequality for $\tilde{P}_T^k$, $k \geq n$ satisfying $H_E = \{x \in \tilde{P}_T^k : x(E) = n - 2\} \subseteq H_b$. Then there exists $\pi > 0$ such that $b_e = \pi$ for all $e \in E_G(W)$.*

Proof. Since G is DT-hypotraceable in W every node $v \in W$ has properties Δ^+ and Δ^-. By assumption $H_E \subseteq H_b$, thus Proposition 2.7 implies that for all $v \in W$ there exist π_v^+, $\pi_v^- \in \mathbb{R}$ such that $b_e = \pi_v^+$ for all $e \in \omega_G^+(v)$ and $b_e = \pi_v^-$ for all $e \in \omega_G^-(v)$. Therefore $D = (W, E_G(W))$ is a subdigraph of K_k that satisfies the assumptions of Proposition 2.13. Since G is DT-hypotraceable in W every node $v \in W = V_G(E_G(W))$ has the alternating trail property with respect to D by definition. Proposition 2.13 now implies that for all $v \in W$ there is $\pi_v \in \mathbb{R}$ such that $b_e = \pi_v$ for all $e \in \omega_D(v) = E_G(W) \cap \omega_G(v)$. As G is DT-hypotraceable in W we know that $(W, E_G(W))$ is connected. Thus for any two nodes $u, v \in W$ there is a sequence of arcs $e_1, \ldots, e_r$ in $E_G(W)$ connecting u and v, this implies $\pi_u = b_{e_1} = \cdots = b_{e_r} = \pi_v$, and we have shown that there is $\pi \in \mathbb{R}$ such that $b_e = \pi$ for all $e \in \bigcup_{v \in W} \omega_D(v) = E_G(W)$. Since $bx \leqq b_0$ is valid, π must be positive.

Theorem 2.16. *Let $G' = (V, E')$ be a* DT-*hypotraceable digraph of order n, and let $G = (V, E)$ be any maximal hypotraceable digraph with $E' \subseteq E$. Then $x(E) \leq n - 2$ is a facet of $\tilde{P}_T^k$ for all $k \geq n$.*

Proof. If we show that G' has the affine independence property, then by Proposition 2.2 $x(E) \leq n - 2$ is a facet of $\tilde{P}_T^n$. If this is true, then by Proposition

2.3, $x(E) \le n-2$ is a facet of $\bar{P}_T^k$ for all $k \ge n$. Therefore all that remains to prove is: Let $dx \leqq d_0$ be any inequality valid with respect to $\bar{P}_T^n$ such that $H_{E'} \subseteq H_d$ then there is $\alpha > 0$ such that $d_{uv} = \alpha$ for all $(u, v) \in E'$.

By assumption there is a node set $W \subseteq V$ such that G' is DT-hypotraceable in W and $V - W$ is stable. Under these conditions Proposition 2.15 implies that there is $\alpha > 0$ such that $d_e = \alpha$ for all $e \in E_{G'}(W)$.

Let $f = (u, v) \in E' - E_{G'}(W)$. Since $V - W$ is stable exactly one of the nodes u, v is in W, say $v \in W$. G' is DT-hypotraceable in W, thus v has property $\Delta_{\bar{G}'}$ and by Proposition 2.7(b) there is $\pi_v^- \in \mathbf{R}$ such that $d_e = \pi_v^-$ for all $e \in \omega_{\bar{G}'}^-(v)$. If the node v has the alternating trail property with respect to $(W, E_{\bar{G}}'(W))$ then there is an arc $g \in E_{G'}(W) \cap \omega_{\bar{G}'}^-(v)$. This implies

$$d_f = \pi_v^- = d_g = \alpha,$$

and we have shown that $d_e = \alpha$ holds for all $e \in E'$. If v is a sink, then—as $(W, E_{G'}(W))$ is connected and obviously $W \ne \{v\}$—there is an arc g as in the previous case and the same result follows. In case $u \in W$, the proof is analogous.

We know several generalizations of Theorem 2.16 which however are rather complicated. We have restricted ourselves to the statement in 2.16 since this is the formulation we need in subsequent applications.

3. Basic hypotraceable facets

It was shown in [2] that all maximal hypotraceable graphs which contain a hypotraceable graph having property Δ induce facets of the monotone symmetric travelling salesman polytope. The analogous result for the trivial direction of these graphs is the following.

Theorem 3.1. *Let $G = [V, E']$ be a non-bipartite hypotraceable graph of order n having property Δ, let $\vec{G} = (V, \vec{E}')$ be the trivial direction of G, and let $D = (V, E)$ be any maximal hypotraceable digraph with $\vec{E}' \subseteq E$. Then $x(E) \le n-2$ is a facet of $\bar{P}_T^k$ for all $k \ge n$.*

Proof. G is connected and non-bipartite by assumption, thus Proposition 2.11 implies that every node $v \in V$ has the alternating trail property with respect to $\vec{G}$; furthermore $\vec{G}$ has properties Δ^+ and Δ^- by Lemma 2.5. Therefore $\vec{G}$ is DT-hypotraceable (in V), cf. 2.14, and the assertion follows from Theorem 2.16.

The smallest known hypotraceable graph has order 34, cf. [7], is non-bipartite and has property Δ. Hence Theorem 3.1 implies:

Corollary 3.2. *For all $k \ge 34$ there exists a hypotraceable digraph $D = (V, E)$ of order $n \le k$ such that $x(E) \le n-2$ is a facet of $\bar{P}_T^k$.*

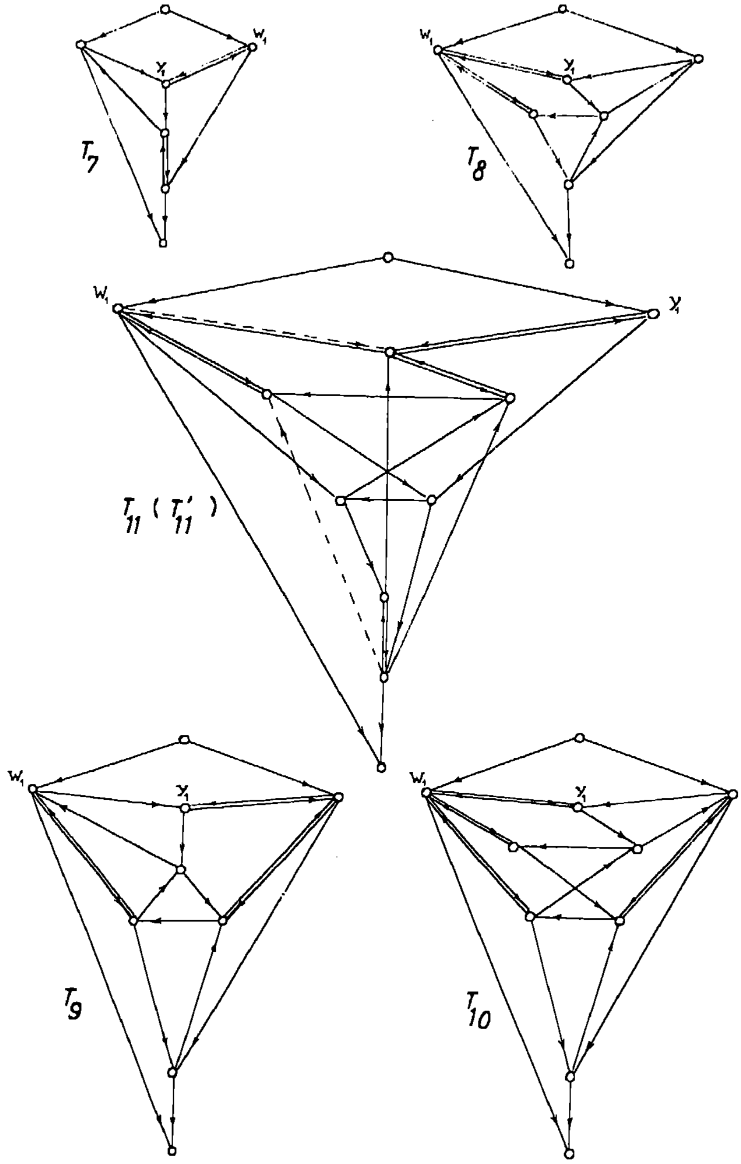

Fig. 3.1.

In general there are many ways to make a hypotraceable digraph maximal, therefore every hypotraceable graph of order n that satisfies the assumptions of 3.1 (these are almost all known ones) will generate many facets of the polytope $\bar{P}^n_T$. Since by Proposition 2.3 a polytope $\bar{P}^k_T$ inherits all hypotraceable facets of all lower dimensional polytopes $\bar{P}^n_T$, $n \leq k$, this shows that the number of hypotraceable facets of $\bar{P}^k_T$ will be huge for large k.

Trivially directed hypotraceable graphs are not at all the only hypotraceable digraphs. It was shown in [4] and [5] that hypotraceable digraphs of order n exist if and only if $n \geqq 7$. Small hypotraceable digraphs of orders $n = 7, 8, \ldots, 13$ were given explicitly in [4] while the others were obtained by means of various constructions. For ease of reference these small hypotraceable digraphs of [4] are shown in Figs. 3.1 and 3.2. The digraphs are denoted by T_7, T_8, T_9, T_{10}, T_{11}, T'_{11}, T_{12}, T'_{12} and T_{13}. The digraphs T_{12} resp. T'_{11} consist of all arcs drawn solidly, while the digraphs T'_{12} resp. T_{11} contain the two additional arcs drawn with dashed lines. The nodes in the digraphs of Figs 3.1 and 3.2 labeled w_1 and y_1 shall play a role in a subsequent construction, cf. Theorem 4.4. It is quite easy to see that all nodes in any of the hypotraceable digraphs T_7, T_8, T_9, T_{10}, T_{13} which are not a source or a sink, have the alternating trail property. All the nodes in the digraphs T_7, T_8 and T_9 and almost all nodes in the digraphs T_{10} and

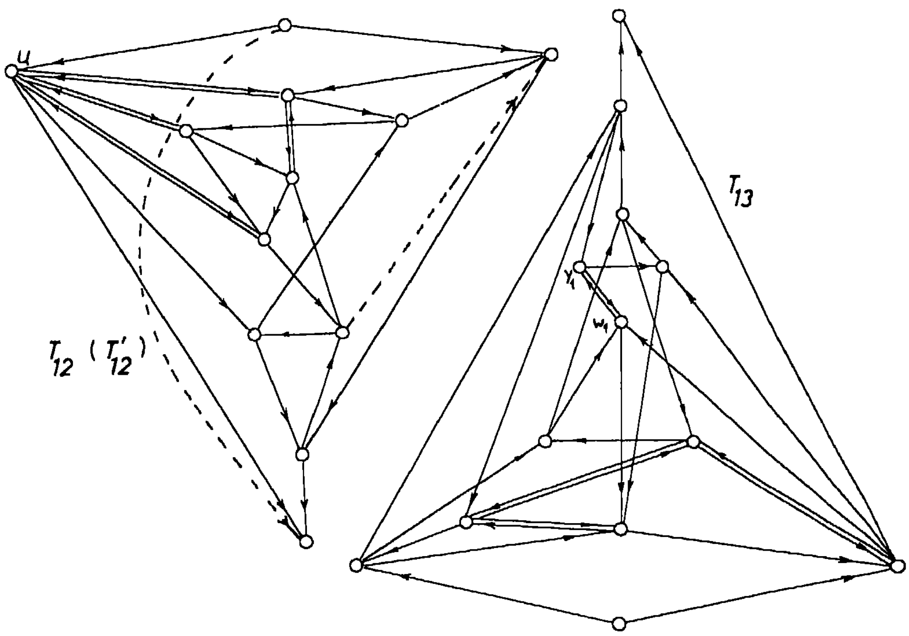

Fig. 3.2.

T_{13} have in- and outdegree less than or equal to three, thus by Lemma 2.6 all these nodes have properties Δ^+ and Δ^-. There is one node in T_{10} with in- and outdegree equal to four and in T_{13} there is one node with indegree equal to four and one with outdegree equal to four, but it is easily checked that these nodes also have properties Δ^+ and Δ^-. Therefore T_7, T_8, T_9, T_{10} and T_{13} are DT-hypotraceable digraphs (in the full node set V).

The hypotraceable digraph T_{11} is not minimal. We can remove the two arcs drawn with dashed lines in Figure 3.1 to obtain the digraph T'_{11} which is also hypotraceable and in which all nodes have in- and outdegree less than four (thus have property Δ^+ and Δ^-) and the alternating trail property in case they are not a source or a sink. The hypotraceable digraph T'_{11} is therefore DT-hypotraceable.

The node labeled u in Fig. 3.2 does neither have property Δ^+ nor Δ^- with respect to the digraph T_{12}. Furthermore, some of the nodes of T_{12} do not have the alternating trail property with respect to $T_{12}-u$. The addition of the two dashed arcs which results in the hypotraceable digraph T'_{12} changes the situation. Let $T'_{12}=(V, E)$ and set $W := V-\{u\}$. Then by Lemma 2.6 all nodes in W have properties Δ^+ and Δ^-, moreover, by simple enumeration one can see that all nodes in W which are not a source or a sink, have the alternating trail property with respect to $(W, E(W))$. This implies that T'_{12} is DT-hypotraceable in W. As $V-W=\{u\}$ is stable, T'_{12} is a DT-hypotraceable digraph.

Thus, summing up the above remarks, Theorem 2.16 implies:

Theorem 3.3. *Every maximal hypotraceable digraph $G=(V,E)$ of order n containing one of the digraphs T_7, T_8, T_9, T_{10}, T'_{11}, T'_{12}, T_{13} induces a facet $x(E)\le n-2$ for all $\tilde{P}^k_T$, $k\ge n$.*

Using a result of [4] which states that hypotraceable digraphs of order n exist if and only if $n\ge 7$ we obtain:

Corollary 3.4. *A monotone asymmetric travelling salesman polytope $\tilde{P}^k_T$ has facets induced by maximal hypotraceable digraphs if and only if $k\ge 7$.*

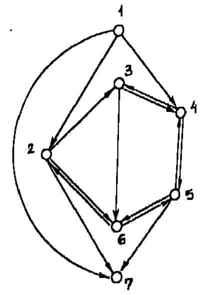

Fig. 3.3

Example 3.5. A maximal hypotraceable digraph containing T_7 is as shown in Fig. 3.3. The corresponding hypotraceable inequality (facet) is

$$x_{12}+x_{14}+x_{17}+x_{23}+x_{26}+x_{27}+x_{34}+x_{36}+x_{43}+x_{45}+x_{54}+x_{56}+x_{57}+x_{62}+x_{65}\leq 5$$

All labellings of this digraph induce different facets of $\tilde{P}^k_T$ for all $k\geq 7$. This implies that $\tilde{P}^k_T$, $k\geqq 7$, has $\binom{k}{7}\cdot 7!=\binom{k}{7}\cdot 5040$ facets of this type.

4. Facets from Constructions HT1, HT2, HT3

In [4] we proved that for all $n\geq 7$ there are hypotraceable digraphs of order n which are not trivial directions of hypotraceable graphs by giving three techniques to obtain hypotraceable digraphs of high order. In this section we will show how the hypotraceable digraphs constructed these ways are related to the asymmetric travelling salesman polytope.

In Construction HT1, cf. [4], a "supertraceable" digraph and a reverse of a supertraceable digraph were combined to obtain a hypotraceable digraph. We reformulate (and slightly specialize) this construction in the following way:

Theorem 4.1 (Construction HT1). *Let $G_1=(V_1,E_1)$ resp. $G_2=(V_2,E_2)$ be two disjoint hypotraceable digraphs with source s_1 resp. sink s_2. Let $T_1=N^+_{G_1}(s_1)$, $T_2=N^-_{G_2}(s_2)$, and let S_1 resp. S_2 be two non-empty subsets of those nodes in V_1 resp. V_2 which are initial resp. terminal nodes of at least one hamiltonian path in G_1-s_1 resp. G_2-s_2. (Note that $T_1\cap S_1=T_2\cap S_2=\emptyset$.) Let $V_1'=V_1-\{s_1\}$, $V_2'=V_2-\{s_2\}$, $E_1'=E_1-\omega^+_{G_1}(s_1)$, $E_2'=E_2-\omega^-_{G_2}(s_2)$, $A=\{(s,t)$: $s\in S_2$, $t\in T_1\}$, $B=\{(t,s)$: $s\in S_1$, $t\in T_2\}$. Then $G=(W,F)$ is a hypotraceable digraph where $W:=V_1'\cup V_2'$ and $F:=E_1'\cup E_2'\cup A\cup B$.*

See [4] for a proof.

Theorem 4.2. *Let $G_1=(V_1,E_2)$ resp. $G_2=(V_2,E_2)$ be disjoint hypotraceable digraphs of order n_1 resp. n_2 with a source s_1 resp. sink s_2. Let S_1 resp. S_2 be nonempty subsets of those nodes of V_1 resp. V_2 which are initial resp. terminal nodes of hamiltonian paths in G_1-s_1 resp. G_2-s_2. Furthermore, assume that G_1 and G_2 have the affine independence property. Let $G=(W,F)$ be the hypotraceable digraph obtained from G_1 and G_2 by construction HT1. Then every maximal hypotraceable digraph $H=(W,E)$ with $F\subseteq E$ induces a facet $X(E)\leqq n_1+n_2-4$ of $\tilde{P}^k_T$ for all $k\geqq n_1+n_2-2$.*

Proof. Because of Proposition 2.2 and 2.3 it suffices to show that G has the affine independence property, i.e. that there exist $m:=|F|$ affinely independent vectors of $\tilde{P}^n_T$, $n:=n_1+n_2-2$, satisfying $x(F)\leqq n-2$ with equality.

By assumption there are $m_1:=|E_1|$ resp. $m_2:=|E_2|$ arc sets $P_1, P_2, \ldots, P_{m_1} \subseteq E_1$ resp. $Q_1, Q_2, \ldots, Q_{m_2} \subseteq E_2$ whose incidence vectors are in $\tilde{P}_T^{n_1}$ resp. $\tilde{P}_T^{n_2}$, are affinely independent and satisfy $x(E_1) \leqq n_1 - 2$ resp. $x(E_2) \leqq n_2 - 2$ with equality. Choose two nodes $s' \in S_1$ and $s'' \in S_2$ and let H' resp. H'' be a hamiltonian path in $G_1 - s_1$ resp. $G_2 - s_2$ with initial resp. terminal node s' resp. s''. Identify the nodes s' and s_2 resp. s'' and s_1. Then the arc sets $H'' \cup P_i$, $i = 1, \ldots, m_1$ and $H' \cup Q_j$, $j = 1, \ldots, m_2$ are by construction elements of $\tilde{T}_n$ and their incidence vectors with respect to $\tilde{P}_T^n$ satisfy $x(F) \leqq n-2$ with equality. Obviously, $m_1 + m_2 - 1$ of these incidence vectors are affinely independent.

Observe that an arc set $H'' \cup P_i$ resp. $H' \cup Q_j$ contains an arc of $A \cup B$ if and only if P_i resp. Q_j contains an arc incident from s_1 resp. incident to s_2. This implies that these arc sets contain at most one arc of $A \cup B$ and that none of these arc sets contains an arc of

$$C := A \cup B - (\{(t, s') \mid t \in T_2\} \cup \{(s'', t) \mid t \in T_1\}).$$

For every arc $(u, v) \in C$ we construct an arc set $R_{uv} \in \tilde{T}_n$ containing (u, v) with $x^{R_{uv}} \in H_F$ as follows: If $(u, v) \in A \cap C$, then $u \in S_2$ and we can take R_{uv} to be the union of a hamiltonian path in $G_2 - s_2$ ending in u, the arc (u, v), and a hamiltonian path in $G_1 - v - s_1$ (such a hamiltonian path clearly exists). If $(u, v) \in B \cap C$, then $v \in S_1$ and we let R_{uv} be the union of a hamiltonian path in $G_2 - u - s_2$, the arc (u, v), and a hamiltonian path in $G_1 - s_1$ starting in v. The incidence vectors $x^{R_{uv}} \in \tilde{P}_T^n$ are mutually affinely independent as well as affinely independent from those constructed previously because each $x^{R_{uv}}$ is the only vector with a nonzero value in component (u, v). Moreover, each arc set R_{uv} contains exactly one arc of $A \cup B$.

Finally, we construct an additional arc set R as follows: Choose any $t_1 \in T_1$ and $t_2 \in T_2$, then H' contains a unique arc $e_1 \in \omega^-_{G_1}(t_1)$ and H'' contains a unique arc $e_2 \in \omega^+_{G_2}(t_2)$. Define

$$R := (H' - \{e_1\}) \cup (H'' - \{e_2\}) \cup \{(s'', t_1), (t_2, s')\}.$$

Clearly, $x^R \in H_F$ and R is the only among the arc sets $H'' \cup P_i$, $i = 1, \ldots, m_1$, $H' \cup Q_j$, $j = 1, \ldots, m_2$, R_{uv}, $(u, v) \in C$, and R with two elements from $A \cup B$. Thus, x^R is affinely independent from the other vectors. This shows that G contains $|F| = |E_1| + |E_2| - 1 + (|S_1| - 1)|T_2| + (|S_2| - 1)|T_1| + 1$ arc sets of $\tilde{T}_n$ whose incidence vectors are in H_F and are affinely independent, hence the theorem is proved.

If the digraphs G_1 and G_2 used in construction HT1 both have a source and a sink, then the resulting hypotraceable digraph also has a source and a sink. Since the basic hypotraceable digraphs $T_7, \ldots, T_{13}$ have a source and sink we can get for any order $n \geqq 12$ hypotraceable digraphs having a source and a sink by iteratively applying construction HT1. We have shown that the digraphs $T_7, \ldots, T_{10}, T'_{11}, T'_{12}, T_{13}$ are DT-hypotraceable. Hence by Proposition 2.15 all these digraphs (and all maximal hypotraceable digraphs containing these) have the affine independence property. Therefore, we can use these digraphs in construction HT1 to generate via Theorem 4.2 new facet inducing hypotraceable digraphs which obviously also have the affine independence property (and a source and a sink). By iteratively applying construction HT1, Theorem 4.2 yields:

Corollary 4.3. *For every $n \geqq 7$ there exist hypotraceable digraphs $G=(V,E)$ of order n which induce facets $x(E) \leqq n-2$ of $\tilde{P}_T^k$ for all $k \geqq n$.*

Theorem 4.4 (Construction HT2). *Let G_1 and G_2 be two disjoint hypotraceable digraphs both with source and sink. Let u_1, v_1 be the source resp. sink of G_1 and y_1 the terminal node of a hamiltonian path in $G_1 - v_1$. Let u_2, v_2 be the source resp. sink of G_2 and x_2 the initial node of a hamiltonian path in $G_2 - u_2$.*

Furthermore, assume that G_1 has a node w_1, $w_1 \notin \{u_1, v_1, y_1\}$, such that the following conditions are satisfied:

$(c_1) G_1$ does not contain two node-disjoint paths $Q=[w_1,\dots,y_1]$ and $Q'=[u_1,\dots,v_1]$ which contain all nodes of G_1, and

$(c_2) G_1$ does not contain two node-disjoint paths $R=[u_1,\dots,y_1]$ and $R'=[w_1,\dots,v_1]$ which contain all nodes of G_1.

Let G be the digraph obtained by adding the digraphs G_1 and G_2 identifying the nodes v_1 and x_2 into a node z and by adding the arcs $A=\{(v_2,u_2),(y_1,u_2),(v_2,w_1)\}$. Then G is hypotraceable.

A proof of Theorem 4.5 can be found in [4].

Theorem 4.5. *Let $G_1=(V_1,E_1)$ resp. $G_2=(V_2,E_2)$ be hypotraceable digraphs of order n_1 resp. n_2 satisfying the assumptions of construction HT2 and assume that G_1 and G_2 have the affine independence property. Let $G=(V,E)$ be the digraph obtained from G_1 and G_2 by construction HT2. Choose any node $t \in N^-_{G_1}(v_1)$ and let $G'=(V,E')$ where $E' := E \cup \{(t,v_2)\}$. Then every maximal hypotraceable digraph $H=(V,F)$ with $E' \subseteq F$ induces a facet $x(F) \leqq n_1+n_2-3$ of $\tilde{P}_T^k$ for all $k \geqq n_1+n_2-1$.*

Proof. The digraph G is hypotraceable by Theorem 4.4; it is easy to verify that G' is also hypotraceable because G_1 satisfies the condition (c_1).

Because of Propositions 2.2 and 2.3 it is sufficient to prove that for any inequality $bx \leq b_0$ valid with respect to $\tilde{P}_T^n$, $n := n_1+n_2-1$, with $H_{E'} := \{x \in \tilde{P}_T^n : x(E')=n-2\} \subseteq \{x \in \tilde{P}_T^n : bx=b_0\} =: H_b$ there exists $\alpha > 0$ such that $b_e = \alpha$ for all $e \in E'$.

Define vectors $c, d, r \in \mathbf{R}^{|E_n|}$ in the following way $c_e = b_e$ if $e \in E_1$, $d_e = b_e$ if $e \in E_2$, $r_e = b_e$ if $e \in E_n - (E_1 \cup E_2)$ set all other components of c, d, r to zero, thus $b = c+d+r$. Let R_2' be a hamiltonian path in $G_2 - u_2$ starting in x_2 and $R_2 = R_2' \cup \{(v_2,u_2)\}$ i.e. $|R_2| = n_2-1$, let R_1' be a hamiltonian path in $G_1 - v_1$ ending in y_1 and $R_1 = R_1' \cup \{(y_1,u_2)\}$, i.e. $|R_1| = n_1-1$.

Now let T_1 be any subset of E_1 containing n_1-2 arcs such that $T_1 \in \bar{T}_{n_1}$ (i.e. T_1 is a path of length n_1-2, or consists of two disjoint paths that contain all nodes of V_1). Since $\omega^+_{\bar{G}_1}(v_1)=\emptyset$ the node v_1 is either not in T_1 or an endpoint of T_1. Thus $T_1 \cup R_2 \in \bar{T}_n$, $|T_1 \cup R_2| = n-2$ and $x^{T_1 \cup R_2} \in H_{E'}$. Let $c_0 := b_0 - bx^{R_2}$ then $cx \leq c_0$ is clearly a valid inequality with respect to $\bar{P}^n_T$ and

$$H_{E_1} := \{x \in \bar{P}^n_T : x(E_1) = n_1 - 2\} \subseteq \{x \in \bar{P}^n_T : cx = c_0\} =: H_c$$

holds.

The affine independence property of G_1 now implies the existence of an $\alpha > 0$ such that $c_0 = \alpha$ for all $e \in E_1$.

With the same line of reasoning using arc sets $T_2 \subseteq E_2$ with n_2-2 arcs and $T_2 \in \bar{T}_{n_2}$, combining them with R_1, and setting $d_0 = b_0 - bx^{R_1}$ we obtain $H_{E_2} \subseteq H_d$ and $d_e = \beta$ for all $e \in E_2$ and some $\beta > 0$.

By definition $R_1 \cup R_2'$, $R_2 \cup R_1' \in \bar{T}_n$ and $x^{R_1 \cup R_2'}$, $x^{R_2 \cup R_1'} \in H_{E'}$, hence $0 = bx^{R_1 \cup R_2'} - bx^{R_2 \cup R_1'} = b_{y_1 u_2} - b_{v_2 u_2}$. If P is a hamiltonian path in $G_1 - w_1$ then P ends in v_1, therefore $Q_1 := P \cup R_2$ and $Q_2 := P \cup R_2' \cup \{(v_2, w_1)\}$ are in $\bar{T}_n$ and x^{Q_1}, $x^{Q_2} \in H_{E'}$. Since $H_{E'} \subseteq H_b$ we obtain $0 = bx^{Q_1} - bx^{Q_2} = b_{v_2 u_2} - b_{v_2 w_1}$. Thus there is $\gamma \in \mathbb{R}$ such that $b_{y_1 u_2} = b_{v_2 w_1} = b_{v_2 u_2} = \gamma$.

The hamiltonian path P in $G_1 - w_1$ contains an arc (y_1, w) for some $w \in V_1$. Let $Q_3 := (P - \{(y_1, w)\}) \cup \{(y_1, u_2)\} \cup R_2' \cup \{(v_2, w_1)\}$ then $Q_3 \in \bar{T}_n$ and $x^{Q_3} \in H_{E'}$, therefore $0 = bx^{Q_2} - bx^{Q_3} = b_{y_1 w} - b_{y_1 u_2} = \alpha - \gamma$, i.e. $\alpha = \gamma$.

It remains to show that $\alpha = \beta$ holds. Let $t \in N^-_{\bar{G}_1}(v_1)$ be the node such that $(t, v_2) \in E'$. $G_1 - t$ contains a hamiltonian path S ending in v_1. Thus $Q_4 := S \cup R_2 \in \bar{T}_n$ and $x^{Q_4} \in H_{E'}$. R_2 contains an arc (w, v_2) for some $w \in N^-_{\bar{G}_2}(v_2)$. Let $Q_5 := S \cup (R_2 - \{(w, v_2)\}) \cup \{(t, v_2)\}$, clearly $Q_5 \in \bar{T}_n$ and $x^{Q_5} \in H_{E'}$. Thus $0 = bx^{Q_4} - bx^{Q_5} = b_{wv_2} - b_{tv_2}$, i.e. $b_{tv_2} = \beta$. On the other hand, since G is hypotraceable, there is a hamiltonian path U in $G - v_2$; U cannot end in t because of the structure of G, therefore U contains an arc $(t, t') \in E$. Let $U_1 = (U - \{(t, t')\}) \cup \{(t, v_2)\}$, then $U, U_1 \in \bar{T}_n$; x^U, $x^{U_1} \in H_{E'}$ and therefore $0 = bx^U - bx^{U_1} = b_{tt'} - b_{tv_2} = \alpha - \beta$.

Altogether we have shown that $b_e = \alpha > 0$ for all $e \in E'$ and are done.

Remark 4.6. Let $\mathcal{T}$ be the set of hypotraceable digraphs T_7, T_8, T_9, T_{10}, T'_{11} and T_{13} and $\mathcal{S}$ be the set of facet inducing maximal hypotraceable digraphs which are completions of hypotraceable digraphs obtained by construction HT1, cf. Theorem 4.2. Since the digraphs in $\mathcal{S}$ induce facets, they have the affine independence property. The digraphs in T are DT-hypotraceable and therefore have the affine independence property by Theorem 2.16. All digraphs in $\mathcal{S} \cup \mathcal{T}$ have a source and a sink. Furthermore, the digraphs in $\mathcal{T}$ satisfy the conditions required for G_1 in construction HT2, the special nodes w_1 and y_1 are indicated in Figs. 3.1 and 3.2. Thus, using in construction HT2 any of the digraphs of $\mathcal{T}$ as G_1 and any of the digraphs of $\mathcal{S} \cup \mathcal{T}$ as G_2 we can generate hypotraceable digraphs of all orders $n \geq 13$ such that each of their completions induces a facet of $\tilde{P}_T^k$ for all $k \geqq n$.

Example 4.7. The digraph G' shown in Fig. 4.1 is a hypotraceable digraph obtained by construction HT2 using T_8 as G_1, T_7 as G_2 and by adding an arc (t, v_2) as required in Theorem 4.5. Every different labeling of every maximal hypotraceable digraph containing G' induces a facet of $\tilde{P}_T^k$ for all $k \geqq 14$.

In [4] we gave a further construction called HT3 to obtain hypotraceable digraphs. This method is derived from one defined by Thomassen [7] for the case of undirected graphs and uses four hypohamiltonian digraphs with special properties to obtain a new strongly connected hypotraceable digraph. The hypohamiltonian digraphs Y_8, Y_9 and the odd Marguerites M_p, $p \geq 5$, cf. [3] and [4], have the properties required by HT3, thus this method produces hypotraceable digraphs of all orders $n \geq 26$, cf. [4].

With methods similar to the ones used in the previous proofs we can show:

Theorem 4.8. *Let $G' = (V, E')$ be any hypotraceable digraph of order $n \geq 26$ constructed from any four of the hypohamiltonian digraphs Y_8, Y_9 and M_p, $p \geq 5$ and odd, with Construction HT3, and let $G = (V, E)$ be any maximal hypotraceable digraph with $E' \subseteq E$, then $x(E) \leq n - 2$ is a facet of $\tilde{P}_T^k$ for all $k \geq n$.*

To prove Theorem 4.8 it suffices to show that G' has the affine independence property. We shortly outline the proof: If we use odd Marguerites M_p in HT3 only, then all nodes v in G' satisfy $d^+_{G'}(v) \leq 3$ and $d^-_{G'}(v) \leq 3$ thus have properties $\Delta^+_{G'}$ and $\Delta^-_{G'}$ by Lemma 2.6. If the digraphs Y_8 and Y_9 are used, some nodes have higher in- and outdegrees, however, the proof of Theorem 4.11 of [4] shows how to construct the hamiltonian paths that are needed to satisfy one of the conditions of Definition 2.4. Therefore, the properties $\Delta^+_{G'}$ and $\Delta^-_{G'}$ are quite easily established for all nodes $v \in V$.

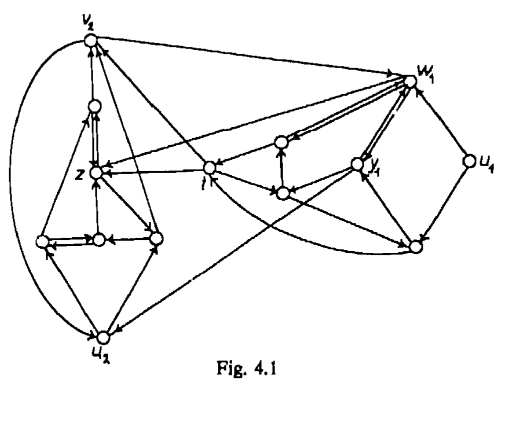

Fig. 4.1

To obtain a hypotraceable digraph with Construction HT3 a so called distinguished node has to be deleted in any of the digraphs used. Any of the nodes x of a Marguerite is distinguished, while the distinguished nodes of Y_8 and Y_9 are labeled x in Fig. 3.2 of [3]. The digraph $M_p - x$, $Y_8 - x$ and $Y_9 - x$ are by construction subdigraphs of G'. It is obvious that any node in $M_p - x$ has the alternating trail property with respect to $M_p - x$, similarly one can show by enumeration that any node of $Y_8 - x$ ($Y_9 - x$) has the alternating trail property with respect to $Y_8 - x$ ($Y_9 - x$).

Knowing these facts we can apply Proposition 2.7 and Proposition 2.13 to show that any valid inequality $bx \leqq b_0$ for $\tilde{P}^n_T$ satisfies $b_e = \alpha$ for all $e \in E'$ and some $\alpha > 0$. This shows that G' has the affine independence property, and by using Propositions 2.2 and 2.3 (completion and lifting) we can complete the proof.

5. Lifting hypohamiltonian facets

In [6] a very simple way was found to construct a hypotraceable digraph from a hypohamiltonian one. This method works as follows:

Theorem 5.1 (Construction 5.1). *Take a hypohamiltonian digraph $G = (V, E)$ of order n and let v be any node in V. Split v into a source s and a sink t and call the new digraph $G_v = (V_v, E_v)$, i.e.*

$$V_v = (V - \{v\}) \cup \{s, t\},$$

$$E_v = (E - \omega(v)) \cup \{(s, w)\colon w \in N^+_G(v)\} \cup \{(w, t)\colon w \in N^-_G(v)\}.$$

Then G_v is a hypotraceable digraph of order $n + 1$.

We have shown in [5] that hypohamiltonian facets cannot be lifted trivially to higher dimensions, i.e. if $x(E) \leq n - 1$ is a hypohamiltonian facet of $\tilde{P}^n_T$ then the same inequality is not a facet of $\tilde{P}^k_T$ for $k > n$. Compared with Proposition 2.3 this shows that hypohamiltonian and hypotraceable facets behave differently.

In the following, however, we will prove that Construction 5.1 makes a special way of lifting of hypohamiltonian facets possible.

Theorem 5.2. *Let $G = (V, E)$ be a hypohamiltonian digraph of order n such that $x(E) \leq n - 1$ defines a facet of $\tilde{P}^n_T$. Let $v \in V$ and let $G_v = (V_v, E_v)$ be the hypotraceable digraph of order $n + 1$ obtained by Construction 5.1. Let s resp. t be the source resp. sink of G_v and let $F = E_v \cup \{(s, t)\}$. Then $x(F) \leq n - 1$ is a facet of $\tilde{P}^k_T$ for all $k \geq n + 1$.*

Proof. Since $\dim \tilde{P}_T^n = n(n-1) =: m$ and since $x(E) \le n-1$ is a facet of $\tilde{P}_T^n$ there exist m sets of arcs $T_1, \dots, T_m \in \tilde{T}_n$ such that the incidence vectors $x^{T_1}, \dots, x^{T_m} \in \mathbb{R}^m$ satisfy $x(E) \le n-1$ with equality and are affinely independent; as $0 \not\in H_E$ these vectors are even linearly independent. To satisfy the hypohamiltonian inequality with equality, necessarily $|T_i \cap E| = n-1$ has to hold. Thus the arc sets $P_i = T_i \cap E$, $i = 1, \dots, m$, are hamiltonian paths in G. Therefore G contains r hamiltonian paths $P_1, \dots, P_r$, where $|E| = r$, such that the vectors $x^{P_1}, \dots, x^{P_r}$ are linearly independent. The (r, m)-matrix B' containing these incidence vectors as rows, thus, is of full row rank and can be partitioned into $B' = (A, 0)$ where A is a nonsingular (r, r)-matrix.

Every path P_i in G can be associated with an arc set Q_i in G_v in the following way. If v is neither the initial nor the terminal node of P_i and $u(w)$ is the predecessor (successor) of v in P_i, then set $Q_i := (P_i - \{(u, v), (v, w)\}) \cup \{(u, t), (s, w)\}$. Q_i is a set of $n-1$ arcs in G_v consisting of two disjoint paths, thus $Q_i \in \tilde{T}_{n+1}$. If v is the initial node of P_i, then $Q_i := (P_i - \{(v, w)\}) \cup \{(s, w)\}$; if v is the terminal node, then $Q_i := (P_i - \{(u, v)\}) \cup \{(u, t)\}$; in both cases Q_i is a path in G_v of length $n-1$. Now consider the incidence vectors of the arc sets Q_i, $i = 1, \dots, r$ with respect to the polytope $\tilde{P}_T^{n+1}$, i.e. $x^{Q_i} \in \mathbb{R}^q$, $q = (n+1)n$. Let B be the (r, q)-matrix containing these incidence vectors as rows. Then B can be partitioned in the form $B = (A, 0)$ where A is the nonsingular submatrix of the matrix B' above. Although the sets P_i and Q_i are graph-theoretically not the same, their incidence vectors considered as subsets of $\mathbb{R}^m$ are identical if we interpret the components of x^{P_i} indexed by (u, v) or (v, w) as the components of x^{Q_i} indexed by (u, t) or (s, w). Thus the vectors $x^{Q_1}, \dots, x^{Q_r}$ satisfy $x(F) \le n-1$ with equality and are linearly independent.

Next we prove that for all arcs $e \in E_{n+1} - E_v$ there is an arc set $C_e \in \tilde{T}_{n+1}$ such that $e \in C_e$ and $|C_e \cap F| = n-1$.

First take any $e = (u, w) \in E_{n+1} - F$ with $u, w \in V_v - \{s, t\} = V - \{v\}$. The digraph $G = (V, E)$ is maximally hypohamiltonian, since $x(E) \le n-1$ is a facet of $\tilde{P}_T^n$ (c.f. [5]), thus $G + (u, w)$ contains a hamiltonian circuit C which contains two arcs containing v, say (v_1, v), (v, v_2). Then $C_e := (C - \{(v_1, v), (v, v_2)\}) \cup \{(v_1, t), (s, v_2)\}$ has the desired properties.

If $e = (u, w) \in E_{n+1} - F$ and one of the nodes u, w is equal to s or t, then an arc set C_e with the properties above can be constructed similarly.

Let $e = (s, t)$. $G - v$ contains a hamiltonian circuit C, take any arc $f \in C$ and define $C_e = (C - \{f\}) \cup \{(s, t)\}$.

Altogether we have shown that there exists a set of q vectors $\{x^{Q_1}, \dots, x^{Q_r}\} \cup \{x^{C_e} : e \in E_{n+1} - E_v\}$ in $\tilde{P}_T^{n+1}$ which satisfy $x(F) \le n-1$ with equality and which are obviously linearly independent. This proves that $x(F) \le n-1$ defines a facet of $\tilde{P}_T^{n+1}$, since $\dim \tilde{P}_T^{n+1} = q$. By Proposition 2.3 this hypotraceable inequality is also a facet of $\tilde{P}_T^k$ for all $k \ge n+1$.

Lifting theorems usually have the following form with respect to $\tilde{P}^n_T$, cf. [1].

Let $ax \le a_0$ be a facet of $\tilde{P}^n_T$, and let $a'x' \le a'_0$ be a valid inequality with respect to $\tilde{P}^k_T$, $k > n$ such that $a'_e = a_e$ for all $e \in E_n$. If a'_e has "certain properties" for all $e \in E_k - E_n$, then $a'x' \le a'_0$ is a facet of $\tilde{P}^k_T$.

Theorem 5.2 is a nice variant of these: we have an inequality $\sum_{e\in E} x_e = ax \le n-1$ and we lift it by keeping all the coefficients a_e of $e \in E - \omega(v)$; the coefficients a_e, $e \in \omega(v)$, are kept, but reassociated with the new arcs $\omega^-(t)$ and $\omega^+(s)$ which replace the arcs $\omega^-(v)$ and $\omega^+(v)$; the coefficient of $x_{s,t}$ is set to 1 and all other variables receive a zero coefficient.

In [5] we have shown that for all n (except for some small numbers) there are (many) hypohamiltonian digraphs $G = (V, E)$ of order n such that $x(E) \le n-1$ is a facet of $\tilde{P}^n_T$. We have also shown that these facets are not trivially liftable by adding zero coefficients to the new variables. Theorem 5.1 now proves that all these rather complicated facets can be lifted upwards, thus, the higher dimensional polytopes $\tilde{P}^k_T$ also inherit the bad hypohamiltonian facets of $\tilde{P}^n_T$, $n < k$, however in a peculiar way.

6. Conclusions

We have shown that very large classes of (maximal) hypotraceable digraphs induce facets of the monotone asymmetric travelling salesman polytope. We have accomplished this by proving that these digraphs satisfy certain sufficient conditions (DT-hypotraceable, Δ^+-, Δ^--, alternating trail-, affine independence property). These conditions were introduced to make particular proof methods work, but it is not clear to what extent these conditions are essential. We believe that all maximal hypotraceable digraphs induce facets of $\tilde{P}^n_T$, there may however be some exceptions for small dimensions like in the hypohamiltonian case, cf. [5, Remark 3.5].

The relations to the monotone symmetric travelling salesman polytope have also not been completely revealed. In Theorem 3.1 we have shown that the trivial directions of some facet inducing hypotraceable graphs are facet inducing hypotraceable digraphs. Is this true in general?

The usual polytope studied in connection with the ATSP is

$$P^n_T := \text{conv}\{x^T \in \mathbb{R}^m \mid T \text{ hamiltonian circuit in } K_n\}, \quad m = n(n-1).$$

This polytope P^n_T is a face of $\tilde{P}^n_T$ and has dimension $m-(2n-1) = n(n-3)+1$, cf. [1]. Due to this fact proofs of the type given above become rather massy. By numerical calculation we have verified for several small hypotraceable digraphs, e.g. the one in Example 3.5, that these also induce facets of P^n_T. Therefore it seems rather likely that most of the hypotraceable facets of $\tilde{P}^n_T$ are also facets of P^n_T.

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