Mathematical Programming 47 (1990) 367-387
North-Holland

FACETS OF THE CLIQUE PARTITIONING POLYTOPE

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Received 16 November 1987
Revised manuscript received 14 November 1988

A subset A of the edge set of a graph $G=(V,E)$ is called a clique partitioning of G is there is a partition of the node set V into disjoint sets $W_1,\ldots,W_k$ such that each W_i induces a clique, i.e., a complete (but not necessarily maximal) subgraph of G, and such that $A=\bigcup_{i=1}^{k}\{uv \mid u,v\in W_i, u\neq v\}$. Given weights $w_e\in\mathbb{R}$ for all $e\in E$, the clique partitioning problem is to find a clique partitioning A of G such that $\sum_{e\in A} w_e$ is as small as possible. This problem—known to be $\mathcal{NP}$-hard, see Wakabayashi (1986)—comes up, for instance, in data analysis, and here, the underlying graph G is typically a complete graph. In this paper we study the clique partitioning polytope $\mathcal{P}_n$ of the complete graph K_n, i.e., $\mathcal{P}_n$ is the convex hull of the incidence vectors of the clique partitionings of K_n. We show that triangles, 2-chorded odd cycles, 2-chorded even wheels and other subgraphs of K_n induce facets of $\mathcal{P}_n$. The theoretical results described here have been used to design an (empirically) efficient cutting plane algorithm with which large (real-world) instances of the clique partitioning problem could be solved. These computational results can be found in Grötschel and Wakabayashi (1989).

Key words: Polyhedral combinatorics, clique partitioning, data analysis.

1. Introduction and notation

In this paper we study the facial structure of the clique partitioning polytope $\mathcal{P}_n$ of the complete graph K_n. This section contains some introductory material. In Section 2 the clique partitioning problem is described along with some of its applications. The polytope $\mathcal{P}_n$ associated with this problem is introduced in Section 3. Facet-defining inequalities for $\mathcal{P}_n$ are studied in Sections 4 and 5. Section 6 describes further issues related to the polytope $\mathcal{P}_n$.

We expect the reader to be familiar with the basic concepts of graph theory. All definitions not given here can be found in [2]. All graphs we consider are simple. We denote a graph G with node set V and edge set E by $G=(V,E)$.

We usually denote an edge e with endnodes u and v by uv. If this may cause confusion we will write $e=\{u,v\}$. If an edge is used as a subscript, the braces are always omitted and the comma is used only if needed for clarity. Thus if edges $\{u, v\}$ or $\{i, j+1\}$ are subscripts, say of a vector b, then the notation will be b_{uv} or $b_{i,j+1}$.

If v is a node of $G=(V, E)$ then the set of edges in G incident to v is denoted by $\delta(v)$. For $F \subseteq E$, $V(F)$ denotes the set of nodes in G consisting of the endnodes of the edges in F, and if $S \subseteq V$ then we denote the set of edges in G with both endnodes in S by $E(S)$, that is,

$$E(S)=\{uv \in E \mid u, v \in S\}.$$

Moreover, if $S_1, \ldots, S_k$ are subsets of V then

$$E(S_1, \ldots, S_k) := \bigcup_{i=1}^{k} E(S_i).$$

If S, $T \subseteq V$ and $S \cap T = \emptyset$ then

$$[S:T] := \{uv \mid u \in S, v \in T\}$$

denotes the set of edges with one endnode in S and the other in T.

If the graphs $H=(W, F)$ and $G=(V, E)$ are such that $W \subseteq V$ and $F \subseteq E$ then H is called a *subgraph* of G. In this case we write $H \subseteq G$. If $W \subseteq V$ then the subgraph $H=(W, E(W))$ is said to be *induced by* W and is also denoted by $G[W]$. For $F \subseteq E$, $H=(V(F), F)$ is the subgraph of G *induced by* F.

A *matching* M in a graph $G=(V, E)$ is a set of edges such that no two edges of M have a common endnode. If $|M|=p$ then we say that M is a *p-matching*. This is not a standard terminology but it will be convenient for our purposes. If a node v is the endnode of an edge in a matching M, then we say that v is *covered by* M or M *covers* v. A matching M in a graph $G=(V, E)$ is called *perfect* if every node in V is covered by M.

A graph is called *complete* if every pair of its nodes is linked by an edge. A *clique* is a subgraph of a graph that is complete (a clique is not necessarily a maximal complete subgraph). We will frequently have to work with complete subgraphs of complete graphs. In such cases we will use subscripts to distinguish between these graphs. In particular, we will often denote the complete graph of order n by $K_n=(V_n, E_n)$. When we write $K_k \subseteq K_n$, for $k<n$, then we view the complete graph on k nodes as a subgraph of K_n and we assume that the k nodes of K_k are formed by an arbitrarily chosen subset of V_n of cardinality k.

We say that $\Gamma=\{W_1, \ldots, W_k\}$ is a *partition* of V if $W_i \cap W_j=\emptyset$ for $1 \leq i<j \leq k$, $V=W_1 \cup \cdots \cup W_k$, and $W_i \neq \emptyset$ for all i.

A set A of edges in a graph $G=(V, E)$ is called a *clique partitioning* of G if there is a partition $\Gamma=\{W_1, \ldots, W_k\}$ of V such that $A=E(W_1, \ldots, W_k)$ and such that the subgraph $G[W_i]$ induced by W_i is a clique for $i=1, \ldots, k$. Note that every clique partitioning A induces a unique partition $W_1, \ldots, W_k$ of V such that $A=E(W_1, \ldots, W_k)$. In case G is complete, every partition of the node set of G induces a clique partitioning. If the edges of G have weights then the weight of a clique partitioning A is the sum of the weights of the edges in A.

A *path* of length $k-1$ is an edge set of the form $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ where $v_i \neq v_j$ for $1 \leq i < j \leq k$. If $P = \{v_1v_2, \ldots, v_{k-1}v_k\}$ is a path then $P \cup \{v_1v_k\}$ is a *cycle* of length k. A *triangle* is a cycle of length three.

We also assume familiarity with the basic concepts of polyhedral theory. We only define a few terms here. The book [9] by Schrijver contains all the background material needed.

The vector space we work with is $\mathbb{R}^E$, where E is the edge set of a graph G; so the components of a vector are indexed by the edges of G. If $F \subseteq E$ then $\chi^F \in \mathbb{R}^E$ denotes the incidence vector of F, that is, $\chi^F_e = 1$ if $e \in F$, $\chi^F_e = 0$ otherwise. We denote the convex hull of a set $S \subseteq \mathbb{R}^E$ by $\operatorname{conv}(S)$.

A *polytope* P is the convex hull of finitely many points, or equivalently, a bounded set that is the intersection of finitely many halfspaces. An inequality $a^Tx \leq \alpha$ is *valid* with respect to P if $P \subseteq \{x \mid a^Tx \leq \alpha\}$. If $a^Tx \leq \alpha$ is valid with respect to P then $F_a := \{x \in P \mid a^Tx = \alpha\}$ is the *face* induced by $a^Tx \leq \alpha$. A *facet* of P is a nonempty face of P that is contained in no other face of P different from P. Equivalently, a facet F of P is a nonempty face with $\dim(F) = \dim(P) - 1$, where $\dim(S)$ denotes the *dimension* of a set S, i.e., the maximum number of affinely independent points in S minus 1. Note that, if the affine space spanned by S does not contain the zero vector (for example, if $S \subseteq \{x \mid a^Tx = \alpha\}$ where $\alpha \neq 0$), then a set of points in S is affinely independent if and only if it is linearly independent.

If $P \subseteq \mathbb{R}^E$ has dimension $|E|$ then every facet of P is induced by a valid inequality $a^Tx \leq \alpha$ that is unique up to multiplication by a positive constant. If $a^Tx \leq \alpha$ is valid for P and $F_a := P \cap \{x \mid a^Tx = \alpha\}$ is a facet of P we say that $a^Tx \leq \alpha$ is *facet-defining*.

For two sets M and N, $M \triangle N := (M \cup N) \setminus (M \cap N)$ denotes their symmetric difference.

2. The clique partitioning problem

An instance of the *clique partitioning problem* (CPP, for short) can be described as follows:

(I1) Given a complete graph $K_n = (V_n, E_n)$ with weights $w_e \in \mathbb{R}$ for all $e \in E_n$, find a clique partitioning $A \subseteq E_n$ of minimum weight.

Let us remark that the clique partitioning problem is also meaningful for general (not necessarily complete) graphs. We will in this paper, however, restrict our attention to the problem defined in (I1). The reason is that all applications we came across give rise to clique partitionings of complete graphs. So we developed the theory to be described here for this special case.

Our motivation to study the CPP came from certain clustering problems in economics posed to us by O. Opitz (Augsburg). The standard way to handle such problems is to view them as instances of a problem of *aggregating binary relations into an equivalence relation.* This problem is a well-known model in data analysis

[1, 10] and has a wide range of applications. An instance of it can be formulated as follows:

(I2) Given a family of m binary relations $R_1, R_2, \ldots, R_m$ defined on a set N, find an equivalence relation R^* on N such that $\mathcal{F}(R^*) := \sum_{k=1}^{m} |R^* \triangle R_k|$ is as small as possible.

It is not difficult to prove that (I2) can be reduced to CPP (see [3] or [11]). Assuming that $N = \{1, 2, \ldots, n\}$ and $R_1, R_2, \ldots, R_m$ are given as an instance of (I2), the corresponding instance of CPP is the following: a complete graph with node set N and weights w_{ij} assigned to each edge ij defined as $w_{ij} := \tilde{w}_{ij} + \tilde{w}_{ji}$, where $\tilde{w}_{ij} := m - 2|\{k \in \{1, \ldots, m\}$: i is related to j in relation $R_k\}|$.

In many applications of problem (I2) in marketing, zoology, politics, etc., N is a set of objects (e.g. computers, animals, states, etc.) and each of the m binary relations R_k $(1 \leq k \leq m)$, defined on N, describes whether the object pairs in N are *similar* or not with respect to a certain characteristic k. In this case, the desired equivalence relation R^* can be interpreted as being the one that determines the best partition of N into classes (or clusters) of similar objects.

This approach of clustering objects by considering m similarity relations which are to be aggregated into an equivalence relation R^* that best approximates them has been widely investigated and dates back to (at least) 1965 (see [7]). For the reader interested in applications and algorithms for problem (I2) we refer to [1, 3, 5, 6, 8, 10, 11].

We want to remark here that the clique partitioning problem (on complete graphs) can be equivalently viewed as a certain "multicut problem" as follows: Given a complete graph $K_n = (V_n, E_n)$ with edge weights w_e, find a partition $\Gamma = \{W_1, \ldots, W_k\}$ of V such that the sum of the weights of the edges not contained in $\bigcup_{i=1}^{k} E_n(W_i)$ (these edges form a *multicut*) is as large as possible. Clearly, if A is a minimum weight clique partitioning of K_n, then $E_n \setminus A$ is a maximum weight multicut, and vice versa. The case where only partitions with a fixed number k of node sets are allowed is also of combinatorial and practical interest. In particular, if k is fixed and equal to 2 we obtain the well-known max-cut problem. In our case, however, the number k is not fixed. All possible partitions of V are feasible.

3. The clique partitioning polytope

We will now describe the polyhedral approach to the clique partitioning problem, give an integer linear programming formulation of the CPP, and present some elementary facts about the associated polyhedron.

Let $K_n = (V_n, E_n)$ be the complete graph of order n. We will assume throughout the paper that $n \geq 3$. Let $\mathcal{P}_n$ denote the convex hull of the incidence vectors of the clique partitionings of K_n, i.e.,

$$\mathcal{P}_n = \operatorname{conv}\{\chi^A \in \mathbb{R}^{E_n} \mid A \text{ is a clique partitioning of } K_n\}. \tag{3.1}$$

$\mathcal{P}_n$ is called the *clique partitioning polytope* (of order n). Its number of vertices is equal to the number of different partitionings of a set with n objects into subsets. This number is known to be

$$p_n := \sum_{k=0}^{n} S(n, k),$$

where

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

is the Stirling number of the second kind. The numbers p_n grow quite rapidly. For instance, for $n = 158$ (this is the size of the largest real world instance of CPP we know) we have $p_{158} \approx 5.82 \times 10^{205}$. (Of course, the number of vertices does not say much about the "complexity" of a polyhedron.)

The clique partitioning problem can be viewed as a linear program of the form

$$\begin{aligned} &\text{minimize} && w^{\mathrm{T}} x \\ &\text{subject to} && x \in \mathcal{P}_n, \end{aligned}$$

since every basic solution of this LP is the incidence vector of a clique partitioning and vice versa. However, in order to be able to apply linear programming techniques to solve this problem, we need a description of $\mathcal{P}_n$ by means of a system of linear inequalities. As the CPP is an $\mathcal{NP}$-hard problem (see [11]), it is very unlikely that we can find a good (or "$\mathcal{NP}$"-) description of $\mathcal{P}_n$ (cf. [4]). The aim of this paper is to present a partial characterization of $\mathcal{P}_n$ by exhibiting several classes of facet-defining inequalities for $\mathcal{P}_n$.

Let us begin with formulating CPP as an integer linear programming problem. Since $\mathcal{P}_n$ is contained in the unit hypercube, the *trivial inequalities*

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E_n \tag{3.2}$$

are clearly valid. Moreover, if A is a clique partitioning and if $a = uv$ and $b = vw$ are two edges in A with a common endnode v then the edge $c = uw$ must also be in A. Thus for every triangle $\{a, b, c\}$ of K_n, the *triangle inequality*

$$x_a + x_b - x_c \leq 1 \tag{3.3}$$

is satisfied by every incidence vector of a clique partitioning, and hence it is valid for $\mathcal{P}_n$. Note that every triangle $\{a, b, c\}$ induces in fact three triangle inequalities, namely

$$x_a + x_b - x_c \leq 1, \qquad x_a - x_b + x_c \leq 1, \qquad -x_a + x_b + x_c \leq 1.$$

For ease of notation, we will further on just speak of the triangle inequality $x_a + x_b - x_c \leq 1$ induced by a triangle $\{a, b, c\}$ and assume that it stands for all the three possible triangle inequalities associated with $\{a, b, c\}$.

Consider now the polytope

$$\mathcal{T}_n := \{x \in \mathbb{R}^{E_n} \mid 0 \leq x_e \leq 1 \text{ for all } e \in E_n,$$
$$x_a + x_b - x_c \leq 1 \text{ for all triangles } \{a, b, c\} \text{ of } K_n\}. \tag{3.4}$$

It follows from the remarks above that $\mathcal{P}_n \subseteq \mathcal{T}_n$; and it is easy to see that the integral points in $\mathcal{T}_n$ are exactly the incidence vectors of the clique partitionings of K_n. So $\mathcal{P}_n = \text{conv}\{x \in \mathcal{T}_n \mid x \text{ integral}\}$ and this implies that

$$\begin{aligned} &\text{minimize} && w^{\mathrm{T}}x \\ &\text{subject to} && 0 \leq x_e \leq 1 \ \text{ for all } e \in E_n, \\ &&& x_a + x_b - x_c \leq 1 \ \text{ for all triangles } \{a, b, c\} \text{ of } K_n, \\ &&& x \text{ integral}, \end{aligned} \tag{3.5}$$

is an integer programming formulation of CPP. As our computational experience (see [3]) shows, the linear program obtained from (3.5) by dropping the integrality constraints is a quite reasonable LP-relaxation of CPP. The use of this LP-relaxation is also theoretically justified since all inequalities but the upper bounds define facets of $\mathcal{P}_n$. To prove this, observe first that $\mathcal{P}_n$ contains the zero vector and all unit vectors, so $\mathcal{P}_n$ is full-dimensional, i.e.,

$$\dim \mathcal{P}_n = |E_n| = \tfrac{1}{2}n(n-1). \tag{3.6}$$

This implies that for each facet of $\mathcal{P}_n$ there exists a unique (up to scaling by a positive constant) inequality defining it.

Theorem 3.1. *For every clique partitioning polytope $\mathcal{P}_n$, $n \geq 3$, the following holds.*

(a) *Every nonnegativity constraint $x_e \geq 0$ defines a facet of $\mathcal{P}_n$.*

(b) *Every triangle inequality $x_a + x_b - x_c \leq 1$ defines a facet of $\mathcal{P}_n$.*

(c) *No upper bound inequality $x_e \leq 1$ defines a facet of $\mathcal{P}_n$.*

Proof. (a) Let $e \in E_n$. Then $x_e = 0$ is satisfied by the zero vector and all unit vectors $\chi^{\{f\}}$, $f \in E_n$, $f \neq e$. These $|E_n|$ vectors are incidence vectors of clique partitionings and are affinely independent.

(b) Let $\{a, b, c\}$ be a triangle in K_n, say $a = uv$, $b = vw$, $c = uw$. Then the $|E_n|$ incidence vectors of the following clique partitionings

$$\begin{aligned} &\{a\}, \quad \{b\}, \quad \{a, b, c\}, \\ &\{a, e\} \quad \text{for all } e \in E_n,\ e \notin \delta(u) \cup \delta(v), \\ &\{b, e\} \quad \text{for all } e \in E_n,\ e \in \delta(u) \backslash \{a, c\}, \\ &E(\{u, v, w, z\}) \quad \text{for all } z \in V \backslash \{u, v, w\}, \end{aligned}$$

satisfy $x_a + x_b - x_c \leq 1$ with equality and are obviously linearly independent.

(c) Let $e \in E_n$ and let e, f, g be a triangle. Then $2x_e \leq 2$ is the sum of the two facet-defining triangle inequalities $x_e + x_f - x_g \leq 1$ and $x_e - x_f + x_g \leq 1$, and hence $x_e \leq 1$ does not define a facet of $\mathcal{P}_n$. □

The following observation summarizes a few "structural" properties of facet-defining inequalities of $\mathcal{P}_n$.

Corollary 3.2. *Let $a^{\mathrm{T}}x \leq \alpha$ be a nontrivial facet-defining inequality for $\mathcal{P}_n$ and let $E_a := \{e \in E_n \mid a_e \neq 0\}$. Then the following holds.*

(a) $\alpha > 0$.

(b) *a has positive and negative entries.*

(c) *the subgraph $H = (V_n(E_a), E_a)$ of K_n is 2-connected.*

Proof. (a) Clearly $\alpha \geq 0$, since the empty set is a clique partitioning satisfying $a^{\mathrm{T}}\chi^{\phi} = 0$. Assume that $\alpha = 0$. If a had a positive coefficient, say $a_e > 0$, then $a^{\mathrm{T}}\chi^{\{e\}} = a_e > 0$ would contradict the validity of $a^{\mathrm{T}}x \leq \alpha$. Thus all coefficients of a are nonpositive. But then a can be represented as a nonnegative linear combination of the nonnegativity constraints $-x_e \leq 0$, a contradiction.

(b) Assume that the vector a has no negative entries. Since E_n is a clique partitioning it follows that $\sum_{a_e > 0} a_e \leq \alpha$. But then $a^{\mathrm{T}}x \leq \alpha$ can be obtained by a nonnegative linear combination of the constraints $x_e \leq 1$, a contradiction. If a has no positive entries then clearly $a^{\mathrm{T}}x \leq \alpha$ can be obtained by a nonnegative linear combination of the constraints $-x_e \leq 0$, a contradiction.

(c) We prove that $H = (V_n(E_a), E_a)$ is connected. The proof of the 2-connectedness of H follows analogously (assuming that there is cut-vertex). Suppose H is disconnected. Let W_1 be the set of nodes of one of the nontrivial connected components of H, and let $W_2 := V_n \setminus W_1$. For $i = 1, 2$ let $F_i := E_n(W_i)$ be the set of edges in K_n with both endnodes in W_i, and let a^i be a vector defined by $(a^i)_e := a_e$ if $e \in F_i$ and $(a^i)_e := 0$, otherwise. Now let D be a clique partitioning of K_n such that $a^{\mathrm{T}}\chi^D = \alpha$, and let $D_1 := D \cap F_1$ and $D_2 := D \cap F_2$. Clearly, D_1 and D_2 are clique partitionings of K_n. Let $\alpha_1 := a^{\mathrm{T}}\chi^{D_1}$ and $\alpha_2 := a^{\mathrm{T}}\chi^{D_2}$. Thus $\alpha_1 + \alpha_2 = \alpha$, and furthermore $a^{1\mathrm{T}}x \leq \alpha_1$ and $a^{2\mathrm{T}}x \leq \alpha_2$ are valid inequalities for $\mathcal{P}_n$. Since $a^{\mathrm{T}}x \leq \alpha$ is the sum of these two inequalities, we have a contradiction. □

Finally we would like to prove a useful lifting theorem that shows that every inequality that defines a facet of $\mathcal{P}_k$ (and satisfies a certain condition) also defines a facet of $\mathcal{P}_n$, $n > k$.

Theorem 3.3 (Lifting Theorem). *Suppose $\sum_{e \in E_k} a_e x_e \leq \alpha$ defines a nontrivial facet of $\mathcal{P}_k$. Then this inequality also defines a facet of $\mathcal{P}_n$ for all $n > k$, provided the following condition is satisfied:*

(L) *There exist a clique partioning $A = E_k(W_1, \ldots, W_s)$ of K_k and a node $v \in V_k$ such that $\sum_{e \in E_k} a_e \chi_e^A = \alpha$ and $\{v\} = W_i$ for some $i \in \{1, \ldots, s\}$.*

Proof. We will show that the given inequality defines a facet of $\mathcal{P}_{k+1}$. The statement of the theorem then follows by induction, since condition (L) remains satisfied in K_{k+1}.

Set $V_k := \{1, \ldots, k\}$, $V_{k+1} := V_k \cup \{k+1\}$. Let $a := (a_e)_{e \in E_k}$, $\bar{a} := (\bar{a}_e)_{e \in E_{k+1}}$, where $\bar{a}_e := a_e$ for $e \in E_k$ and $\bar{a}_e := 0$ for $e \in E_{k+1} \setminus E_k$. The validity of $\bar{a}^{\mathrm{T}} x \leq \alpha$ for $\mathcal{P}_{k+1}$ is obvious.

Since $a^{\mathrm{T}} x \leq \alpha$ defines a nontrivial facet of $\mathcal{P}_k$ we have $\alpha > 0$ and thus there are $|E_k|$ clique partitionings $A_1, \ldots, A_{|E_k|}$ whose incidence vectors are linearly independent and satisfy $a^{\mathrm{T}} x \leq \alpha$ with equality. Each set A_i is also a clique partitioning of K_{k+1} and satisfies $\bar{a}^{\mathrm{T}} \chi^{A_i} = \alpha$. Let M be the nonsingular $|E_k| \times |E_k|$ matrix whose rows are the vectors χ^{A_i}. We may assume that the rows and columns of M are arranged in such a way that the last $k-1$ columns of M correspond to the edges iv ($i \in V_k \setminus \{v\}$), where v is the special node existing by condition (L), and such that the $(k-1) \times (k-1)$ submatrix N of M in the lower right hand corner is nonsingular.

From the $k-1$ clique partitionings $A_{|E_k|-k+2}, \ldots, A_{|E_k|}$, whose incidence vectors are the last $k-1$ rows of M, we construct $k-1$ new clique partitionings of K_{k+1} as follows. For $i \in \{|E_k|-k+2, \ldots, |E_k|\}$, let $(Y_i, E_k(Y_i))$ be the clique of A_i with $v \in Y_i$; since N is nonsingular, $|Y_i| \geq 2$ holds. Set

$$B_i := A_i \cup \{\{j, k+1\} \mid j \in Y_i\}.$$

Then $\bar{a}^{\mathrm{T}} \chi^{B_i} = \alpha$ holds by construction. Finally, let A be the clique partitioning existing by condition (L); set

$$B_v := A \cup \{\{v, k+1\}\}.$$

Clearly, $\bar{a}^{\mathrm{T}} \chi^{B_v} = \alpha$.

Let $\bar{M}$ be the $|E_{k+1}| \times |E_{k+1}|$-matrix whose rows are the incidence vectors (in $\mathbb{R}^{E_{k+1}}$) of the clique partitionings $A_1, \ldots, A_{|E_k|}$, $B_{|E_k|-k+2}, \ldots, B_{|E_k|}$, B_v. Then $\bar{M}$ can be put into the form shown in Figure 3.1 where M and N are nonsingular. Obviously, $\bar{M}$ is nonsingular, and thus there are $|E_{k+1}|$ clique partitionings in K_{k+1} whose incidence vectors satisfy $\bar{a}^{\mathrm{T}} x \leq \alpha$ with equality and are linearly independent. This implies that $\bar{a}^{\mathrm{T}} x \leq \alpha$ defines a facet of $\mathcal{P}_{k+1}$. $\square$

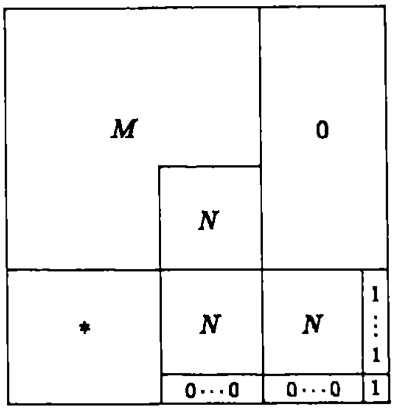

Fig. 3.1. Matrix $\bar{M}$.

Condition (L) in Theorem 3.3 is a sufficient condition for "trivial lifting". We believe it is not necessary, but could not prove it. However, all classes of facet-defining inequalities we have found satisfy it. Maybe all nontrivial facet-defining inequalities satisfy (L).

4. 2-Partition inequalities

In this section we introduce a class of facet-defining inequalities that generalizes the class of triangle inequalities. This class of inequalities turns out to be of particular importance from a computational point of view.

Let $K_n = (V_n, E_n)$ be a complete graph. For every two disjoint nonempty subsets S and T of V_n, the inequality

$$\sum_{s\in S}\sum_{t\in T} x_{st} - \sum_{\substack{s,t\in S\\ s\neq t}} x_{st} - \sum_{\substack{s,t\in T\\ s\neq t}} x_{st} \leq \min\{|S|, |T|\} \tag{4.1}$$

is called a *2-partition inequality*. If we want to stress that the inequality is the one corresponding to S and T we say that $x([S:T]) - x(E_n(S)) - x(E_n(T)) \leq \min\{|S|, |T|\}$ is the *2-partition inequality induced by S and T* (or short: $[S, T]$-*inequality*). The graph of the support of a 2-partition inequality with $S = \{u, v\}$ and $T = \{t, y, z\}$ is shown in Figure 4.1. Note that, if $|S| = 1$ and $|T| = 2$ then the corresponding $[S, T]$-inequality is nothing but a triangle inequality.

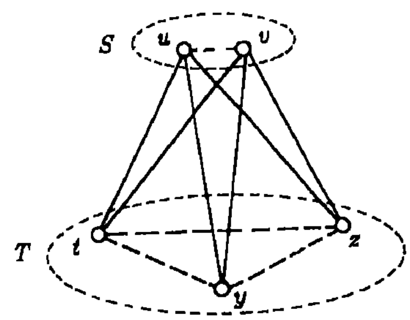

Fig. 4.1. Graph of the support of a 2-partition inequality with $S = \{u, v\}$ and $T = \{t, y, z\}$. The corresponding 2-partition inequality is $x(\{ut, uy, uz, vt, vy, vz\}) - x_{uv} - x(\{ty, tz, yz\}) \leq 2$.

Theorem 4.1. *For every $n \geq 3$ and every two nonempty disjoint subsets S, T of V_n, the corresponding 2-partition inequality*

$$x([S:T]) - x(E_n(S)) - x(E_n(T)) \leq \min\{|S|, |T|\}$$

is valid for $\mathcal{P}_n$. It defines a facet if and only if $|S| \neq |T|$.

Proof. Assume w.l.o.g. that $|S| \leqslant |T|$. We prove the validity of (4.1) by induction on $|S|+|T|$. Let $|S|=1$ and $|T| \geqslant 1$. For $|T| \leqslant 2$ the result is immediate. So assume that $|T|=t \geqslant 3$. By induction hypothesis, for every $v \in T$ the $[S, T\backslash\{v\}]$-inequality

$$x([S:T\backslash\{v\}])-x(E_n(T\backslash\{v\}))\leqslant 1$$

is valid for $\mathscr{P}_n$. Adding these inequalities up for all $v \in T$ we obtain

$$(t-1)(x([S:T]))-(t-2)(x(E_n(T)))\leqslant t.$$

Since $-x(E_n(T)) \leqslant 0$ is also valid for $\mathscr{P}_n$, adding this inequality to the above one, we get

$$(t-1)(x([S:T])-x(E_n(T)))\leqslant t,$$

and hence

$$x([S:T])-x(E_n(T))\leqslant \frac{t}{t-1},$$

which implies that

$$x([S:T])-x(E_n(T))\leqslant \left\lfloor \frac{t}{t-1} \right\rfloor = 1$$

is valid for $\mathscr{P}_n$.

Now let $|S|=s\geqslant 2$, $|T|=t\geqslant 2$, $|S|+|T|=k$, and suppose that (4.1) is valid for $|S|+|T|\leqslant k-1$.

For every $v \in S$ consider the $[S\backslash\{v\}:T]$-inequality,

$$x([S\backslash\{v\}:T])-x(E_n(S\backslash\{v\}))-x(E_n(T))\leqslant s-1, \tag{4.2}$$

and for every $v\in T$ consider the $[S:T\backslash\{v\}]$-inequality

$$x([S:T\backslash\{v\}])-x(E_n(S))-x(E_n(T\backslash\{v\}))\leqslant \min\{s, t-1\}. \tag{4.3}$$

By induction hypothesis, all these inequalities are valid for $\mathscr{P}_n$. Adding up the inequalites (4.2) for every $v\in S$ and (4.3) for every $v \in T$ we obtain

$$\begin{aligned}&(s+t-2)(x([S:T])-x(E_n(S))-x(E_n(T)))\\&\quad\leqslant s(s-1)+t(\min\{s, t-1\}).\end{aligned} \tag{4.4}$$

If $|S|<|T|$, then (4.4) yields

$$x([S:T])-x(E_n(S))-x(E_n(T))\leqslant \left\lfloor \frac{s(s+t-1)}{s+t-2} \right\rfloor = |S|.$$

If $|S|=|T|$, i.e., $s=t$, then (4.4) can be written as

$$(2s-2)(x([S:T])-x(E_n(S))-x(E_n(T)))\leqslant s(2s-2),$$

which implies that

$$x([S:T])-x(E_n(S))-x(E_n(T))\leqslant |S|.$$

This completes the proof that the inequality (4.1) is valid for P_n. When $|S|=|T|$ the proof given above shows that the inequality (4.1) can be obtained by a nonnegative linear combination of other valid inequalities, and therefore it does not define a facet of $\mathcal{P}_n$.

Now assume that $|S|<|T|$. We prove first that (4.1) defines a facet of $\mathcal{P}_k$, where $k=|S|+|T|$.

For notational convenience we may assume that $S:=\{1,2,\ldots,s\}$. Let $a^{\mathrm{T}}x\leq a_0$ denote the inequality (4.1), i.e., $a^{\mathrm{T}}x:=x([S:T])-x(E_n(S))-x(E_n(T))\leq s=:a_0$, and let $b^{\mathrm{T}}x\leq b_0$ be a facet-defining inequality for $\mathcal{P}_k$ such that $F_a:=\{x\in\mathcal{P}_k \mid a^{\mathrm{T}}x=a_0\}\subseteq F_b:=\{x\in\mathcal{P}_k \mid b^{\mathrm{T}}x=b_0\}$. Clearly, $F_a\neq\mathcal{P}_k$, thus if we can prove that $b=\alpha a$ for some $\alpha\in\mathbb{R}$ then since $F_a\neq\emptyset$ we can conclude that (4.1) defines a facet of $\mathcal{P}_k$. We start by establishing the following:

Claim 1. There exists $\alpha\in\mathbb{R}$ such that $b_e=\alpha$ for all $e\in[S:T]$.

Proof. To prove Claim 1 consider a subset $T'\subset T$ with $|T'|=s$, and let i be a node in S. For every node $w\in T'$, let $M_i(w)$ be an s-matching containing iw with $M_i(w)\subseteq[S:T']$; and for every pair (w,v) with $w\in T'$ and $v\in T\setminus T'$, let

$$M_i(w,v):=(M_i(w)\setminus\{iw\})\cup\{iv\}.$$

It is clear that $\chi^{M_i(w)}$, $\chi^{M_i(w,v)}\in F_a\subseteq F_b$, and therefore $0=b_0-b_0=b^{\mathrm{T}}\chi^{M_i(w)}-b^{\mathrm{T}}\chi^{M_i(w,v)}=b_{iw}-b_{iv}$. Thus, for a (fixed) node $i\in S$ and for every $w\in T'$ we have that $b_{iw}=b_{iv}$ for every $v\in T\setminus T'$. This implies that for every $i\in S$ there exists $\alpha_i\in\mathbb{R}$ such that $b_{it}=\alpha_i$ for all $t\in T$.

To complete the proof of Claim 1 we shall prove that for $s\geq 2$, $\alpha_1=\cdots=\alpha_s$.

For $i,j\in S$, $i\neq j$, let M be an $(s-2)$-matching contained in $[S\setminus\{i,j\}:T]$. If $s=2$ take $M=\emptyset$. Let u, v, w be distinct nodes in T, not covered by the matching M. Let

$$A:=M\cup\{iu,iv,uv\}\cup\{jw\}\quad\text{and}\quad B:=M\cup\{ju,jv,uv\}\cup\{iw\}.$$

Since χ^A, $\chi^B\in F_a\subseteq F_b$, it follows that $b_{iu}+b_{iv}+b_{jw}=b_{ju}+b_{jv}+b_{iw}$, which implies that $\alpha_i=\alpha_j$. This completes the proof of Claim 1.

Our next step is to prove the following:

Claim 2. $b_e=-\alpha$ for all $e\in E_n(S)\cup E_n(T)$.

Proof. Let $e:=uv\in E_n(T)$ and let M be an s-matching covering v but not u, $M\subset[S:T]$. Let $i\in S$ be such that $iv\in M$ and let

$$A_e:=M\cup\{iu,uv\}.$$

Clearly χ^M, $\chi^{A_e}\in F_a\subseteq F_b$, and therefore $b_{iu}+b_{uv}=0$, i.e., $b_e=b_{uv}=-b_{iu}=-\alpha$.

Now let $e:=ij\in E_n(S)$ and let M be an s-matching contained in $[S:T]$. Let u, v be nodes of T such that $iu, jv\in M$ and let

$$B_e:=M\cup\{iv,ij,ju,uv\}.$$

Since $\chi^M,\chi^{B_e}\in F_a\subseteq F_b$, it follows that $b_{iv}+b_{ij}+b_{ju}+b_{uv}=0$. By the previous results, $b_{iv}=b_{ju}=\alpha=-b_{uv}$. Thus, $b_e=b_{ij}=-\alpha$, and this completes the proof of Claim 2.

The two claims imply that $b_e=\alpha a_e$ for all $e\in E_k$, and this proves that (4.1) defines a facet of $\mathcal{P}_k$.

For any node $v \in T$ there exists an s-matching $M \subseteq [S, T]$ not covering v. M is a clique partitioning and v a node as required in condition (L). Thus by Theorem 3.3, the $[S, T]$-inequality defines a facet of $\mathcal{P}_n$ for all $n \geq k$. $\square$

Remark 4.2. It is easy to see that the polytope $\mathcal{T}_n$ (see (3.4)) is equal to $\mathcal{P}_n$ for $n = 2, 3$. Since the 2-partition inequalities for $|S| \neq |T|$ define facets of $\mathcal{P}_n$, we can conclude that $\mathcal{T}_n \neq \mathcal{P}_n$ for $n \geq 4$. In fact, for every two nonempty disjoint subsets S, T of V_n with $|S| \neq |T|$, $|S| + |T| \neq 3$, the point $x^* \in \mathbb{R}^{E_n}$ defined by $x^*_e := \frac{1}{2}$ for all $e \in [S : T]$, $x^*_e = 0$ else, is a vertex of $\mathcal{T}_n$ (that is not contained in $\mathcal{P}_n$).

5. Facets from 2-chorded cycles, paths and even wheels

We will now introduce three further classes of inequalities valid for $\mathcal{P}_n$ and we will show which of these inequalities define facets of $\mathcal{P}_n$.

Let C be a cycle in K_n, say $C = \{e_1, \ldots, e_k\}$ and $e_i = v_i v_{i+1}$ $(i = 1, \ldots, k-1)$, $e_k = v_1 v_k$, then the set

$$\bar{C} := \{v_i v_{i+2} \in E_n \mid i = 1, \ldots, k-2\} \cup \{v_1 v_{k-1}, v_2 v_k\}$$

is called the set of *2-chords of* C. For every cycle $C \subseteq E_n$ of length at least 5 and its associated set $\bar{C}$ of 2-chords,

$$x(C) - x(\bar{C}) \leq \lfloor \tfrac{1}{2}|C| \rfloor \tag{5.1}$$

is called the *2-chorded cycle inequality* (induced by C). Figure 5.1 shows a 7-cycle and its set of 2-chords. The associated 2-chorded cycle inequality is given by

$$C = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,1\}\},$$
$$\bar{C} = \{\{1,3\}, \{3,5\}, \{5,7\}, \{7,2\}, \{2,4\}, \{4,6\}, \{6,1\}\},$$
$$x(C) - x(\bar{C}) \leq 3.$$

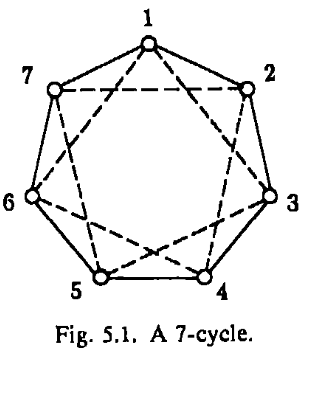

Fig. 5.1. A 7-cycle.

Theorem 5.1. *Let $C \subseteq E_n$ be a cycle of length at least 5 and let $\bar{C}$ be the set of 2-chords of C. Then the 2-chorded cycle inequality induced by C,*

$$x(C) - x(\bar{C}) \leq \lfloor \tfrac{1}{2}|C| \rfloor$$

is valid for $\mathcal{P}_n$. It defines a facet of $\mathcal{P}_n$ if and only if $|C|$ is odd.

Proof. To establish the validity of (5.1) we proceed as follows. For each edge $e \in \bar{C}$, let e_1 and e_2 be the two edges in C such that $\{e_1, e_2, e\}$ is a triangle in K_n. Consider the associated triangle inequality $x_{e_1} + x_{e_2} - x_e \leq 1$. Adding up, for all $e \in \bar{C}$, these triangle inequalities we obtain $\sum_{e \in \bar{C}} (x_{e_1} + x_{e_2} - x_e) = 2x(C) - x(\bar{C}) \leq |\bar{C}| = |C|$. Since $-x(\bar{C}) \leq 0$ is valid for $\mathcal{P}_n$, adding this inequality to the latter one and then dividing by 2, we obtain that $x(C) - x(\bar{C}) \leq \frac{1}{2}|C|$ is valid for $\mathcal{P}_n$. Since for every vertex of $\mathcal{P}_n$ the left-hand side of this inequality is an integer, we can round the right-hand side down to the next integer and obtain the validity of the 2-chorded cycle inequality induced by C.

If $|C|$ is even, the above proof shows that the inequality (5.1) can be obtained by a nonnegative linear combination of other facet-defining inequalities, and therefore it does not define a facet.

Now assume that $|C| = k$ is odd (and at least 5). We first show that (5.1) defines a facet of $\mathcal{P}_k$. Let $k := 2p+1$, $V := \{1, \ldots, 2p+1\}$, $C := \{\{i, i+1\} \mid i = 1, 2, \ldots, 2p+1\}$, and consider all the additions of node numbers taken modulo $2p+1$. Denote by $a^{\mathrm{T}}x \leq a_0$ the inequality $x(C) - x(\bar{C}) \geq p$ and let $F_a := \{x \in \mathcal{P}_k \mid a^{\mathrm{T}}x = a_0\}$ be the face defined by it. Note that $F_a \neq \mathcal{P}_k$. Assume further that $b^{\mathrm{T}}x \leq b_0$ is a facet-defining inequality for $\mathcal{P}_k$ such that $F_a \subseteq F_b := \{x \in \mathcal{P}_k \mid b^{\mathrm{T}}x = b_0\}$. We want to show that $b = \alpha a$ for some $\alpha \in \mathbb{R}$.

For $i \in V$, let M_i be the unique perfect p-matching contained in $C \backslash \{\{i, i+1\}, \{i, i-1\}\}$, that is,

$$M_i = \{\{i+1, i+2\}, \{i+3, i+4\}, \ldots, \{i+2p-1, i+2p\}\}.$$

Clearly, $\chi^{M_i} \in F_a \subseteq F_b$ for all $i \in V$, and therefore $b^{\mathrm{T}}\chi^{M_1} = b^{\mathrm{T}}\chi^{M_2} = \cdots = b^{\mathrm{T}}\chi^{M_{2p+1}} = b_0$. Note that for every $i \in V$, $M_i \triangle M_{i+2} = \{\{i, i+1\}, \{i+1, i+2\}\}$ holds. This fact together with $b^{\mathrm{T}}\chi^{M_i} = b^{\mathrm{T}}\chi^{M_{i+2}}$ imply that $b_{i,i+1} = b_{i+1,i+2}$. Thus we can conclude that there exists $\alpha \in \mathbb{R}$ such that

$$b_e = \alpha \quad \text{for all } e \in C. \tag{5.2}$$

Now for every $i \in V$, consider the clique partitioning

$$A_i := M_i \cup \{\{i, i+1\}, \{i, i+2\}\}.$$

Since χ^{A_i}, $\chi^{M_i} \in F_a$ and $F_a \subseteq F_b$, it follows that $b^{\mathrm{T}}\chi^{A_i} = b^{\mathrm{T}}\chi^{M_i}$, and therefore $b_{i,i+1} + b_{i,i+2} = 0$. Then using (5.2) we get that $b_{i,i+2} = -\alpha$, and since this holds for every $i \in V$, we can conclude that

$$b_e = -\alpha \quad \text{for all } e \in \bar{C}. \tag{5.3}$$

Our next step is to prove that $b_e = 0$ for all $e \in E_k \backslash (C \cup \bar{C})$. Since $E_5 = C \cup \bar{C}$ we assume from now on that $k \geq 7$. Set

$$J := \{3, 5, \ldots, 2p-3\}.$$

For every $i \in V$ and $j \in J$ consider the clique partitioning

$$A_{ij} := M_i \cup \{\{i, i+j\}, \{i, i+j+1\}\}.$$

Since $\chi^{A_{ij}}, \chi^{M_i} \in F_a \subseteq F_b$, we have $b^T\chi^{A_{ij}} = b^T\chi^{M_i}$, and therefore

$$b_{i,i+j} = -b_{i,i+j+1} \quad \text{for all } i \in V, j \in J. \tag{5.4}$$

Let $i \in V, j \in J$, and $r = 2p-j$. Then $r \in J$, $i+j+1 \in V$ and $i+j+1+r = i+2p+1 = i \bmod (2p+1)$. Thus (5.4) yields

$$b_{i+j+1,i} = -b_{i+j+1,i+1} \quad \text{for all } i \in V, j \in J, \tag{5.5}$$

and hence by (5.4) and (5.5),

$$b_{i,i+j} = -b_{i,i+j+1} = b_{i+1,i+j+1} \quad \text{for all } i \in V, j \in J. \tag{5.6}$$

From (5.6) we conclude that

$$\begin{aligned} &\text{for every } j \in J \text{ there exists } \beta_j \in \mathbb{R} \text{ such that} \\ &\beta_j = b_{i,i+j} = -b_{i,i+j+1} \quad \text{for all } i \in V. \end{aligned} \tag{5.7}$$

Note that the edges $e \in E_k \backslash (C \cup \bar{C})$ are exactly those of the form $\{i, i+j\}$ or $\{i, i+j+1\}$ for $i \in V$ and $j \in J$. Thus if we can prove that $\beta_j = 0$ for all $j \in J$ we have the desired result.

Let $j \in J$. By (5.7), $\beta_{2p-j} = -b_{i,i+2p-j+1}$ for all $i \in V$. Hence, in particular, for $i = j+1$, it follows that $\beta_{2p-j} = -b_{j+1,2p+2} = -b_{j+1,1} = -b_{1,1+j} = -\beta_j$, i.e.,

$$\beta_j = -\beta_{2p-j} \quad \text{for all } j \in J. \tag{5.8}$$

In case $k=7$, we have $J = \{3\}$ and so $j = 2p-j = 3$. Thus (5.8) implies that $\beta_3 = 0$, and the proof is finished for $k=7$. Hence from now on we assume $k \geq 9$. Equation (5.8) shows that we only have to prove that $\beta_j = 0$ for those $j \in J$ with $3 \leq j \leq q$, where $q = p$ if p is odd and $q = p+1$ if p is even. For $s \in \{5, 7, \ldots, q\}$, consider the clique partitioning

$$B_s := M_{2p+1} \cup \{\{1, s\}, \{1, s+1\}, \{2, s\}, \{2, s+1\}\}.$$

Since $\chi^{B_s}, \chi^{M_{2p+1}} \in F_a \subseteq F_b$, we conclude that

$$b_{1,s} + b_{1,s+1} + b_{2,s} + b_{2,s+1} = 0. \tag{5.9}$$

Taking $j = s-2$ and $i = 1$ (resp. $i = 2$) in (5.7) we obtain $\beta_{s-2} = -b_{1,s}$ (resp. $\beta_{s-2} = b_{2,s} = -b_{2,s+1}$). Analogously, taking $j = s$ and $i = 1$ in (5.7) we get $\beta_s = b_{1,1+s}$. These equations and (5.9) imply that

$$\beta_s = \beta_{s-2} \quad \text{for all } s \in \{5, 7, \ldots, q\}.$$

Thus for p odd (since $q = p$) we have that $\beta_3 = \beta_5 = \cdots = \beta_p$, and since by (5.8), $\beta_p = -\beta_p$, it follows that $\beta_p = 0$, and therefore $\beta_j = 0$ for all $j \in J$. If p is even (since $q = p+1$) then $\beta_3 = \beta_5 = \cdots = \beta_{p-1} = \beta_{p+1}$, and since by (5.8) $\beta_{p-1} = -\beta_{p+1}$, it follows that $\beta_j = 0$ for all $j \in J$. Thus we have proved that $b_e = 0$ for all $e \in E_k \backslash C \cup \bar{C}$.

Altogether we have shown that there exists $\alpha \in \mathbb{R}$ such that

$$b_e = \begin{cases} \alpha & \text{if } e \in C, \\ -\alpha & \text{if } e \in \bar{C}, \\ 0 & \text{if } e \in E_k \backslash (C \cup \bar{C}), \end{cases}$$

i.e., $b_e = \alpha a_e$ for all $e \in E_k$. This completes the proof that $x(C) - x(\bar{C}) \leq p$ defines a facet of $\mathcal{P}_k$.

To prove that it also defines a facet of $\mathcal{P}_n$ for all $n > k$, observe that every node i together with the clique partitioning M_i satisfies the condition (L) of Theorem 3.3. Thus, by Theorem 3.3 the result follows. □

Given an odd cycle C of length at least 5, then the point $x^* \in \mathbb{R}^{E_n}$ with $x^*_e = \frac{1}{2}$ if $e \in C$ and $x^*_e = 0$ if $e \in E_n \setminus C$ is contained in $\mathcal{T}_n$ and satisfies all 2-partition inequalities. But x^* violates the 2-chorded cycle inequality $x(C) - x(\bar{C}) \leq \frac{1}{2}(|C| - 1)$.

A further class of inequalities can be derived from paths and a universal node as follows. Let $P = \{e_1, \ldots, e_{k-2}\}$ be a path in K_n, $n \geq k$, of length at least two and assume that $e_i = v_i v_{i+1}$, $i = 1, \ldots, k-2$, then the set

$$\bar{P} := \{v_i v_{i+2} \mid i = 1, \ldots, k-3\}$$

is called the set of *2-chords of P*. Let $z \in V_n$ be a node different from $v_1, \ldots, v_{k-1}$ and set

$$R := \{v_i z \mid i \in \{1, \ldots, k-1\}, i \text{ even}\},$$

$$\bar{R} := \{v_i z \mid i \in \{1, \ldots, k-1\}, i \text{ odd}\}.$$

We call $P \cup \bar{P} \cup R \cup \bar{R}$ a *2-chorded path with a universal node* (see Figure 5.2) and the inequality

$$x(P \cup R) - x(\bar{P} \cup \bar{R}) \leq \lfloor \tfrac{1}{2}(|P| + 1) \rfloor \tag{5.10}$$

the *2-chorded path inequality* induced by $P \cup \bar{P} \cup R \cup \bar{R}$.

Note that, when P has length two, the corresponding 2-chorded path inequality is an $[S, T]$-inequality with $S = \{v_2\}$ and $T = \{v_1, v_3, z\}$.

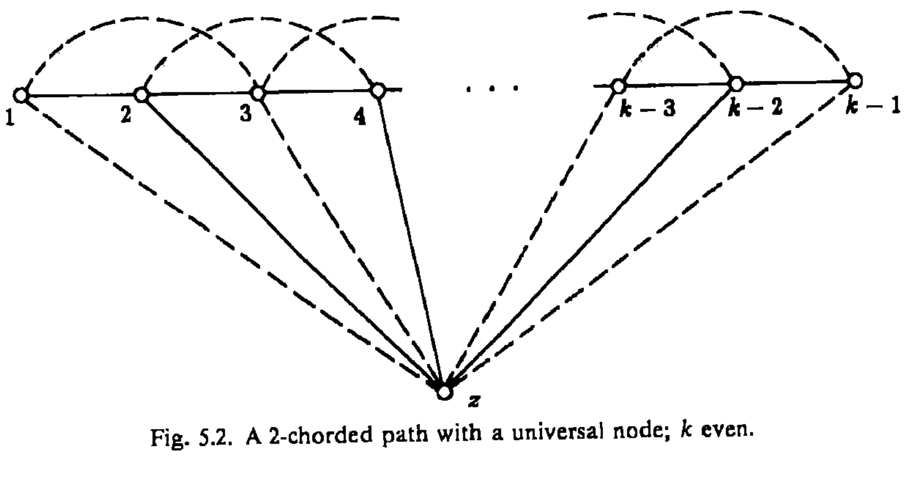

Fig. 5.2. A 2-chorded path with a universal node; k even.

Theorem 5.2. *Let $P \cup \bar{P} \cup R \cup \bar{R} \subseteq E_n$ be a 2-chorded path of length at least two with a universal node. Then the associated 2-chorded path inequality*

$$x(P \cup R) - x(\bar{P} \cup \bar{R}) \leq \lfloor \tfrac{1}{2}(|P| + 1) \rfloor$$

is valid for $\mathcal{P}_n$. It defines a facet of $\mathcal{P}_n$ if and only if $|P|$ is even.

Proof. For notational convenience let us assume that $V := V(P \cup \bar{P} \cup R \cup \bar{R}) = \{1, 2, \ldots, k-1\} \cup \{z\}$, where z is the universal node; so $|V| = k \leq n$.

(a) *Validity.* We prove validity of (5.10) by induction on k.

(a1) $k=4$. The corresponding 2-chorded path inequality is—as remarked before—an $[S, T]$-inequality. Thus the result follows from Theorem 4.1.

(a2) $k=5$. In this case, consider the partition of V into the sets $S=\{2,4\}$ and $T=\{1,3,z\}$. Then, by Theorem 4.1 the 2-partition inequality induced by S and T,

$$x_{21}+x_{23}+x_{2z}+x_{41}+x_{43}+x_{4z}-x_{24}-x_{13}-x_{1z}-x_{3z} \leq 2$$

is valid for $\mathcal{P}_n$. Adding the inequality $-x_{41} \leq 0$ and the above inequality we obtain the inequality (5.10).

(a3) Assume that $k \geq 6$. Let

$$P_1 := P\backslash\{\{k-2, k-1\}\}, \qquad P_2 := P_1\backslash\{\{k-3, k-2\}\},$$

$$R_1 := R\backslash\{z, k-1\}, \qquad R_2 := R_1\backslash\{z, k-2\},$$

$$\bar{R}_1 := \bar{R}\backslash\{z, k-1\}, \qquad \bar{R}_2 := \bar{R}_1\backslash\{z, k-2\}.$$

By induction hypothesis, the 2-chorded path inequality induced by $P_1 \cup \bar{P}_1 \cup R_1 \cup \bar{R}_1$,

$$x(P_1 \cup R_1) - x(\bar{P}_1 \cup \bar{R}_1) \leq \lfloor \tfrac{1}{2}|P| \rfloor,$$

and the 2-chorded path inequality induced by $P_2 \cup \bar{P}_2 \cup R_2 \cup \bar{R}_2$,

$$x(P_2 \cup R_2) - x(\bar{P}_2 \cup \bar{R}_2) \leq \lfloor \tfrac{1}{2}(|P|-1) \rfloor,$$

are valid for $\mathcal{P}_n$.

Let $d^{\mathrm{T}}x \leq d_0$ be the inequality obtained by adding these two inequalities and the following three inequalities (which define facets of $\mathcal{P}_n$ by Theorem 3.1):

$$x_{k-3,k-2}+x_{k-2,k-1}-x_{k-3,k-1} \leq 1,$$

$$-x_{k-3,k-1} \leq 0,$$

$$-x_{k-4,k-2} \leq 0.$$

If k is odd, we add to $d^{\mathrm{T}}x \leq d_0$ the inequalities

$$x_{k-2,k-1}+x_{k-1,z}-x_{k-2,z} \leq 1 \quad \text{and} \quad x_{k-1,z} \leq 1,$$

and obtain

$$2(x(P \cup R) - x(\bar{P} \cup \bar{R})) \leq \lfloor \tfrac{1}{2}|P| \rfloor + \lfloor \tfrac{1}{2}(|P|-1) \rfloor + 3 = |P| + 2.$$

This yields that the inequality (5.10) is valid for $\mathcal{P}_n$.

If k is even, then we add the inequalities

$$x_{k-2,k-1}+x_{k-2,z}-x_{k-1,z} \leq 1 \quad \text{and} \quad -x_{k-1,z} \leq 0,$$

to $d^{\mathrm{T}}x \leq d_0$ and obtain

$$2(x(P \cup R) - x(\bar{P} \cup \bar{R})) \leq \lfloor \tfrac{1}{2}|P| \rfloor + \lfloor \tfrac{1}{2}(|P|-1) \rfloor + 2 = |P| + 1.$$

Hence we can conclude that (5.10) is valid for $\mathcal{P}_n$.

For k odd, it is easy to see that each incidence vector of a clique partitioning satisfying the 2-chorded path inequality with equality also satisfies the triangle inequality $x_{k-2,k-1} + x_{z,k-1} - x_{z,k-2} \leq 1$ with equality. So in this case the 2-chorded path inequality does not define a facet of $\mathcal{P}_n$.

(b) *Facet.* We will now prove that the inequality (5.10) defines a facet of $\mathcal{P}_n$ when $|P|$ is even. Assume that $|P| = k-2$ is even and $k \geq 6$. We first show that the 2-chorded path inequality induced by $P \cup \bar{P} \cup R \cup \bar{R}$, let us denote it by $a^{\mathrm{T}}x \leq a_0$, defines a facet of $\mathcal{P}_k$. Let F_a denote the face of $\mathcal{P}_k$ defined by $a^{\mathrm{T}}x \leq a_0$. Clearly $F_a \neq \mathcal{P}_k$. Consider a facet-defining inequality $b^{\mathrm{T}}x \leq b_0$ and assume that $F_a \subseteq F_b := \{x \in \mathcal{P}_k \mid b^{\mathrm{T}}x = b_0\}$.

Let $V_k = \{1, 2, \ldots, k\}$, so the universal node z is equal to node k. Let $I_1 := \{1, 3, \ldots, k-1\}$ and $I_2 := \{2, 4, \ldots, k-2\}$. For $i \in I_2 \cup \{k\}$ consider the matching

$$M_i := \{\{j, j-1\}: j \in I_2, j < i\} \cup \{\{j, j+1\}: j \in I_2, j \geq i\}. \tag{5.11}$$

Then clearly,

$$\chi^{M_i} \in F_a \subseteq F_b \quad \text{for } i \in I_2 \cup \{k\}. \tag{5.12}$$

Since $M_i \triangle M_{i+2} = \{\{i, i-1\}, \{i, i+1\}\}$, using (5.12) we obtain

$$b_{i,i-1} = b_{i,i+1} \quad \text{for } i \in I_2. \tag{5.13}$$

For $i \in I_2$, consider the clique partitionings

$$A_i := (M_i \setminus \{i, i+1\}) \cup \{i, z\},$$

$$B_i := M_i \cup \{\{i, i-1\}, \{i-1, i+1\}\},$$

$$C_i := M_i \cup \{\{i, z\}, \{i+1, z\}\},$$

$$D_i := M_{i+2} \cup \{\{i, z\}, \{i-1, z\}\}.$$

Since χ^{A_i}, χ^{B_i}, χ^{C_i}, $\chi^{D_i} \in F_a \subseteq F_b$, using (5.12) and (5.13) we conclude that for every $i \in I_2$ there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i = b_{i,i-1} = b_{i,i+1} = b_{i,z} = -b_{i-1,i+1} = -b_{i-1,z} = -b_{i+1,z}$. This implies that there exists $\alpha \in \mathbb{R}$ such that

$$b_e = \begin{cases} \alpha & \text{for all } e \in P \cup R, \\ -\alpha & \text{for all } e \in \bar{R} \cup \{e \in \bar{P} \mid e = \{i, i+2\}, i \in I_1\}. \end{cases} \tag{5.14}$$

Now we want to prove that $b_e = -\alpha$ for all $e \in \bar{P}$, $e = \{i, i+2\}$, $i \in I_2$. Let $e = \{i, i+2\}$ be such an edge. The following clique partitioning derived from the matching M_k defined in (5.11),

$$A := (M_k \setminus \{i, i-1\}) \cup \{\{i, i+2\}, \{i, i+1\}, \{i, z\}, \{i+1, z\}, \{i+2, z\}\}$$

is such that its incidence vector is in $F_a \subseteq F_b$, and therefore by (5.12) and (5.14), it follows that $b_e = -\alpha$.

Our next step is to prove that $b_e = 0$ for all $e \in \bar{E} := E_k \setminus (P \cup \bar{P} \cup R \cup \bar{R})$.

Case 1. $e \in \bar{E}$ is incident to two nodes in I_2. We may assume that $e := \{i, j\}$, where $i, j \in I_2$ and $j \geq i+4$. For

$$B := (M_k \setminus \{\{i, i-1\}, \{j, j-1\}\}) \cup \{\{i, j\}, \{i, z\}, \{j, z\}\}$$

we have $\chi^B \in F_a \subseteq F_b$. Using (5.12) and (5.14) we can conclude that $b_e = 0$.

Case 2. $e \in \bar{E}$ is incident to two nodes in I_1. We may assume that $e := \{i, j\}$, where $i, j \in I_1$ and $j \geq i+4$. In this case,

$$C := (M_{i+1} \setminus \{j-1, j\}) \cup \{\{j-1, z\}, \{i, j\}\}$$

is such that $\chi^C \in F_a \subseteq F_b$, and thus by (5.12) and (5.14) it follows that $b_e = 0$.

Case 3. $e \in \bar{E}$ is incident to a node in I_2 and a node in I_1. Assume first that $e := \{i, j\}$, where $i \in I_2$, $j \in I_1$ and $j \geq i+3$. Let

$$D := M_{j+1} \cup \{\{i, j\}, \{i-1, j\}\}.$$

Then $\chi^D \in F_a \subseteq F_b$. Since $\chi^{M_{j+1}} \in F_a \subseteq F_b$ and $b_{i-1,j} = 0$, we get $b_e = 0$.

Now assume that $e := \{i, j\}$, where $i \in I_2$, $j \in I_1$ and $j \leq i-3$. Set

$$D' := M_{j+1} \cup \{\{i, j\}, \{j, i+1\}\}.$$

Since $\chi^{D'}$, $\chi^{M_{j+1}} \in F_a \subseteq F_b$ and $b_{j,i+1} = 0$, then $b_e = 0$.

We have shown that $b = \alpha a$ for an $\alpha \in \mathbb{R}$, and thus we can conclude that $a^{\mathrm{T}} x \leq a_0$ defines a facet of $\mathcal{P}_k$. To prove that it also defines a facet of $\mathcal{P}_n$ for $n > k$, observe that, for every $i \in I_2$, the node $i-1$ and the clique partitioning M_i satisfy condition (L) of Theorem 3.3. This finishes the proof. □

A *wheel* consists of the edges of a cycle plus the edges that link a further node to all the nodes of the cycle. We can derive another class of facet-defining inequalities from even wheels and the 2-chords of their cycles as follows.

Let $C \subseteq E_n$ be a cycle of even length $2p$ with $p \geq 4$ and let $z \in V_n \setminus V_n(C)$ be a further node (the *center*). Let $\bar{C}$ be the set of 2-chords of C and let $\{V, \bar{V}\}$ be a bipartition of the node set of C (i.e., $|V| = |\bar{V}|$ and every edge of C has one endnode in V and the other in $\bar{V}$). Set

$$R := \{zv \mid v \in V\}, \qquad \bar{R} := \{zv \mid v \in \bar{V}\},$$

then $C \cup \bar{C} \cup R \cup \bar{R}$ is called a *2-chorded even wheel* and the inequality

$$x(C \cup R) - x(\bar{C} \cup \bar{R}) \leq \tfrac{1}{2}|C| \tag{5.15}$$

is called the *2-chorded even wheel inequality* (induced by $C \cup \bar{C} \cup R \cup \bar{R}$). Figure 5.3 shows two 2-chorded even wheels.

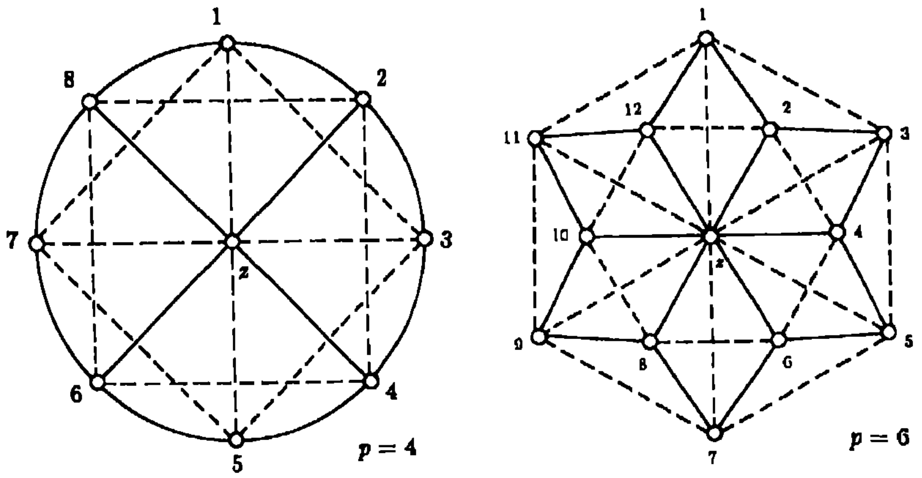

Fig. 5.3. Two 2-chorded even wheels. The edges in $C \cup R$ are drawn solid, those in $\bar{C} \cup \bar{R}$ dashed.

Theorem 5.3. *Let $C \cup \bar{C} \cup R \cup \bar{R} \subseteq E_n$ be a 2-chorded even wheel, where C is an even cycle of length at least 8. Then the 2-chorded even wheel inequality*

$$x(C \cup R) - x(\bar{C} \cup \bar{R}) \leq \tfrac{1}{2}|C|$$

induced by $C \cup \bar{C} \cup R \cup \bar{R}$ defines a facet of $\mathcal{P}_n$.

Proof. For ease of notation, let us assume that $V_n(C) := \{1, 2, \ldots, 2p\}$, $V := \{2, 4, \ldots, 2p\}$, $\bar{V} := \{1, 3, \ldots, 2p-1\}$, $z \in V_n \setminus V(C)$, $C := \{\{i, i+1\} \mid i = 1, \ldots, 2p\}$, $\bar{C} := \{\{i, i+2\} \mid i = 1, \ldots, 2p\}$, where (here and in the following) summation of integers representing nodes is considered modulo $2p$.

To establish the validity of (5.15) for $\mathcal{P}_n$, consider, for every $i \in V$, the path

$$P_i := C \setminus \{\{i, i-1\}, \{i, i+1\}\}$$

from $i+1$ to $i-1$ of length $2(p-1)$, its set $\bar{P}_i$ of 2-chords, and $\bar{R}_i := \{zv \mid v \in \bar{V}\}$, $R_i := \{zv \mid v \in V \setminus \{i\}\}$. By Theorem 5.2, for every $i \in V$, the 2-chorded path inequality induced by $P_i \cup \bar{P}_i \cup R_i \cup \bar{R}_i$,

$$x(P_i \cup R_i) - x(\bar{P}_i \cup \bar{R}_i) \leq p - 1$$

is valid for $\mathcal{P}_n$. The sum of these p inequalities (over all $i \in V$) is

$$(p-1)(x(C \cup R) - x(\bar{C} \cup \bar{R})) + (x_{24} + x_{46} + x_{68} + \cdots + x_{2p,2}) - x(\bar{R})$$
$$\leq p(p-1).$$

Adding to this inequality the following inequalities:

(a) the p triangle inequalities:

$$x_{i,i+1} + x_{i+1,i+2} - x_{i,i+2} \leq 1 \quad \text{for all } i \in V;$$

(b) the $\lfloor \frac{1}{2}p \rfloor$ triangle inequalities:

$$x_{i,z} + x_{i+2,z} - x_{i,i+2} \leq 1 \quad \text{for } i = 2 + 4j,\ j = 0, \ldots, \lfloor \tfrac{1}{2}p \rfloor - 1;$$

(c) the trivial inequalities:

$$-x_{i,i+2} \leq 0 \quad \text{for all } i \in \bar{V} \text{ and for } i = 4+4j,\ j = 0, \ldots, \lfloor \tfrac{1}{2} p \rfloor - 1;$$

and if p is odd, also the inequalities $x_{2,2p} \leq 1$ and $-x_{2p,2} \leq 0$; we obtain

$$p(x(C \cup R) - x(\bar{C} \cup \bar{R})) \leq p(p-1) + p + \lfloor \tfrac{1}{2}(p+1) \rfloor,$$

which implies that the 2-chorded even wheel inequality (5.15) is valid for $\mathcal{P}_n$.

The proof that $x(C \cup R) - x(\bar{C} \cup \bar{R}) \leq p$ defines a facet of $\mathcal{P}_n$ runs along the same lines as the two previous proofs of such facts in this section. It is technically not more complicated and therefore omitted. □

6. Final remarks

There are a number of further questions that could be asked about the clique partitioning polytope $\mathcal{P}_n$ studied in this paper. We briefly mention a few interesting aspects.

All facet-defining inequalities described in this paper have coefficients that are either 0 or ± 1. Moreover, the subgraphs of K_n they arise from are quite "symmetric" resp. "regular". Do all the facet-defining inequalities of $\mathcal{P}_n$ have this property? They do not. We have found some "nonsymmetric" facet-defining inequalities with coefficients in $\{0, \pm 1, \pm 2\}$. One of these can be described as follows.

Theorem 6.1. *Let V be a subset of seven nodes of V_n for $n \geq 7$, say $V = \{v_1, v_2, \ldots, v_7\}$. Set*

$$E_1 := \{v_1v_2,\ v_1v_4,\ v_2v_3,\ v_2v_5,\ v_3v_4,\ v_4v_5\},$$
$$E_2 := \{v_1v_3,\ v_1v_5,\ v_2v_4,\ v_3v_5\},$$
$$E_3 := \{v_1v_6,\ v_5v_6,\ v_6v_7\},$$
$$E_4 := \{v_1v_7,\ v_2v_6,\ v_4v_6,\ v_5v_7\}.$$

Then the inequality

$$x(E_1) - x(E_2) + 2x(E_3) - 2x(E_4) \leq 4 \tag{6.1}$$

defines a facet of $\mathcal{P}_n$. □

Figure 6.1 shows a picture of the graph $(V, E_1 \cup E_2 \cup E_3 \cup E_4)$, where $v = \{1, \ldots, 7\}$, that induces the facet-defining inequality (6.1). A proof of Theorem 6.1 as well as some indications of how further (quite complicated) facets of this type can be constructed can be found in [11].

Another issue are "lifting" or "glueing" results. A simple (though useful) lifting theorem has been stated in Theorem 3.3. There are further procedures with which edge sets that induce facets can be "glued" together so that the resulting edge set also induces a facet. These procedures are rather complex and will be investigated in a forthcoming paper.

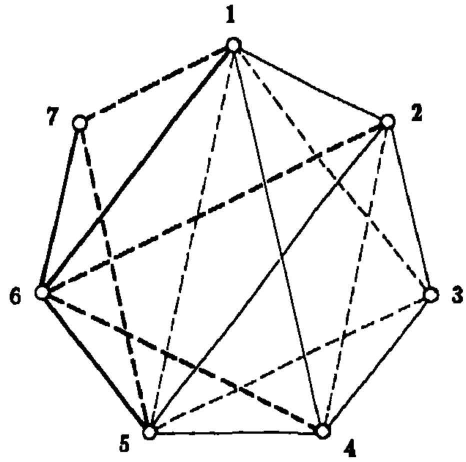

Fig. 6.1. The graph $(V, E_1 \cup E_2 \cup E_3 \cup E_4)$, where $V = \{1, \ldots, 7\}$.

Finally, there is the "applied" aspect of the results presented here. Can one use the inequalities in the framework of a cutting plane procedure for the clique partitioning problem? One can, and the algorithm we have implemented works surprisingly well, although we were not able so far to design fast exact separation procedures for the classes of inequalities presented in Sections 4 and 5. But efficient heuristics worked very well. Our computational experiments show that the triangle inequalities (3.3) and the 2-partition inequalities (4.1) are very often sufficient to produce optimal clique partitionings. These computational results can be found in [3] and [11].

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