Composition of Facets of the Clique Partitioning Polytope

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Abstract

In [1] we have introduced the clique partitioning problem and studied the associated polyhedron, the so-called clique partitioning polytope. In this paper we continue these polyhedral investigations; in particular, we present new classes of facets and methods to construct new facet-defining inequalities from given facet-defining inequalities.

0. Introduction and Notation

Let $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ denote the complete graph on n nodes without loops and multiple edges, i. e., every two different nodes of K_n are linked by exactly one edge. An edge set $A \subseteq \mathcal{E}_n$ is called a **clique partitioning** of K_n if there is a **partition** $\{W_1, \dots, W_k\}$ of $\mathcal{V}_n$ (i. e., each W_i is nonempty, $W_i \cap W_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^k W_i = \mathcal{V}_n$) such that A is the union of all those edges in $\mathcal{E}_n$ that have both endnodes in W_i, for some $i \in \{1, \dots, k\}$. The **clique partitioning problem** (for short: CPP) is the task to find, for a given complete graph $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ with edge weights $c_e \in \mathbf{R}$ for all $e \in \mathcal{E}_n$, a clique partitioning $A^* \subseteq \mathcal{E}_n$ such that $c(A^*) := \sum_{e \in A^*} c_e$ is as small as possible.

The clique partitioning problem is a combinatorial optimization version of a clustering problem in data analysis and has many interesting applications, among others, in zoology, economics, and the political sciences — see, for instance, [2], [3], [4], [5], [6]. This problem is $\mathcal{NP}$-hard. To solve instances coming up in practical applications, we have proposed in [2] an LP-based cutting plane procedure that utilizes our polyhedral investigations [1] of the associated "clique partitioning polytope". This approach works quite well; in particular, we could solve all practical applications we could get hold of to optimality. Although — to date — our method is the only one that is able to solve the larger ones of the problems described in [2], we feel that, to attack really large scale problem instances, more information about the clique partitioning polytope is necessary. Thus this paper is a continuation of our polyhedral work [1] on the clique partitioning problem and aims at enlarging our reservoir of facet-defining inequalities that can be used in a cutting plane procedure.

We use standard graph theory terminology. So a graph is denoted by $G = (V, E)$ where V is the node set and E the edge set of G. For our problems loops and multiple edges are irrelevant, so we assume throughout that all graphs considered are simple. If $H = (W, F)$ and $G = (V, E)$ are graphs with $W \subseteq V$ and $F \subseteq E$ then H is called

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a subgraph of G. We will perform many operations with subgraphs of K_n which we distinguish by using subscripts. Therefore we use the symbol $\mathcal{V}_n$ for the node set and the symbol $\mathcal{E}_n$ for the edge set of K_n in order to create no confusion. For $v \in V$, $G - v$ denotes the graph obtained from G by removing v. For $W \subseteq V$, $G[W]$ is the subgraph of G induced by W. It will be convenient to use the following notation, where $S, T, S_1, \ldots, S_k \subseteq V$ and $F \subseteq E$:

$$\begin{aligned} E(S) &:= \{uv \in E \mid u, v \in S\}, \\ E(S_1, \ldots, S_k) &:= \bigcup_{i=1}^{k} E(S_i), \\ [S:T] &:= \{uv \in \mathcal{E}_n \mid u \in S, v \in T\}, \\ V(F) &:= \{v \in V \mid v \text{ is the endnode of some edge in } F\}. \end{aligned}$$

To denote the set of edges in $G = (V, E)$ with one endnode in S and the other in T we write

$$E[S:T] := E \cap [S:T].$$

Using this notation, an edge set $A \subseteq \mathcal{E}_n$ is a clique partitioning of K_n if and only if there is a partition $\{W_1, \ldots, W_k\}$ of $\mathcal{V}_n$ such that $A = \mathcal{E}_n(W_1, \ldots, W_k)$; moreover, for the complete graph $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ and every two disjoint node subsets S, T of $\mathcal{V}_n$, $[S:T] = \mathcal{E}_n[S:T]$ holds.

A **cycle** C of length k is an edge set of the form $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_1v_k\}$, where $v_i \neq v_j$ if $i \neq j$. For $k \geq 4$, the set $\overline{C} := \{v_iv_{i+2} \mid i = 1, \ldots, k-2\} \cup \{v_1v_{k-1}, v_2v_k\}$ is called the set of **2-chords** of C. A **triangle** is a cycle of length three. A **wheel** is the union of a cycle and the set of edges that link some node not on the cycle with all nodes of the cycle. A graph $G = (V, E)$ is **bipartite** if its node set can be partitioned into two nonempty subsets V_1, V_2 such that all edges of G have one endnode in V_1 and the other in V_2. Every partition of V with this property is called a **bipartition** of V.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs then the graph $(V_1 \cup V_2, E_1 \cup E_2)$ is called the **union** of G_1 and G_2 and is denoted by $G_1 \cup G_2$. (We assume that the union operation does not produce multiple edges, so $G_1 \cup G_2$ is a simple graph.)

1. The Clique Partitioning Polytope

To formulate the clique partitioning problem in polyhedral resp. linear programming terms we associate with it a polyhedron in the following way. Let $\mathbf{R}^{\mathcal{E}_n}$ denote the real vector space where every component x_e of a vector $x \in \mathbf{R}^{\mathcal{E}_n}$ is indexed by an edge e of the complete graph $K_n = (\mathcal{V}_n, \mathcal{E}_n)$. To avoid trivialities, we assume throughout the paper that $n \geq 3$. For every edge set $A \subseteq \mathcal{E}_n$, $\chi^A \in \mathbf{R}^{\mathcal{E}_n}$ denotes its incidence vector, i. e., $\chi^A_e = 1$ if $e \in A$ and $\chi^A_e = 0$ if $e \notin A$. The convex hull of all incidence vectors of clique partitionings of K_n is called the **clique partitioning polytope** (of K_n) and is denoted by $\mathcal{P}_n$, i. e.,

$$\mathcal{P}_n = \operatorname{conv}\{\chi^A \in \mathbf{R}^{\mathcal{E}_n} \mid A \text{ is a clique partitioning of } K_n\}.$$

Since the vertices of $\mathcal{P}_n$ are in one-to-one correspondence with the clique partitionings of K_n, it follows immediately that the CPP can be formulated as the problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x \in \mathcal{P}_n. \end{array}$$

This is a linear program in the sense that a linear objective function is to be minimized over a polytope. To apply LP-techniques this formulation is of no use unless $\mathcal{P}_n$ can be represented by a system of linear inequalities. Since the clique partitioning problem is $\mathcal{NP}$-hard, it follows from general results of complexity theory that it is very unlikely that an explicit complete description can ever be obtained; but we were able to determine large classes of valid and facet-defining inequalities for $\mathcal{P}_n$, see [1], and we continue these investigations in this paper.

Recall at this point that an inequality $a^T x \leq \alpha$ is called **valid** for $\mathcal{P}_n$ if $\mathcal{P}_n \subseteq \{x \in \mathbf{R}^{\mathcal{E}_n} \mid a^T x \leq \alpha\}$. A valid inequality $a^T x \leq \alpha$ is said to **define a facet** of $\mathcal{P}_n$ if the face $F_a := \{x \in \mathcal{P}_n \mid a^T x = \alpha\}$ of $\mathcal{P}_n$ is a facet, i. e., if F_a is a face of dimension one less than the dimension of $\mathcal{P}_n$ (the dimension of a set S is the cardinality of the largest set of affinely independent points in S minus one). If F is a subset of $\mathcal{E}_n$ then we use the symbol $x(F)$ as a short-hand notation for the sum $\sum_{e \in F} x_e$.

The following theorem is a summary of some of the results presented in [1].

(1.1) Theorem. *Let $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ be a complete graph with $n \geq 3$ nodes, and let $\mathcal{P}_n \subseteq \mathbf{R}^{\mathcal{E}_n}$ be the clique partitioning polytope of K_n.*

(a) *The dimension of $\mathcal{P}_n$ is equal to $|\mathcal{E}_n| = n(n-1)/2$.*

(b) *For every edge $e \in \mathcal{E}_n$, the* **trivial inequalities** *$x_e \geq 0$ and $x_e \leq 1$ are valid for $\mathcal{P}_n$. Every inequality $x_e \geq 0$ defines a facet of $\mathcal{P}_n$, but no inequality $x_e \leq 1$ does.*

(c) *For every three different nodes $i, j, k \in \mathcal{V}_n$, each of the three associated* **triangle inequalities**

$$\begin{aligned} x_{ij} + x_{jk} - x_{ik} &\leq 1 \\ x_{ij} - x_{jk} + x_{ik} &\leq 1 \\ -x_{ij} + x_{jk} + x_{ik} &\leq 1 \end{aligned}$$

defines a facet of $\mathcal{P}_n$.

(d) *For every two disjoint nonempty subsets S, T of $\mathcal{V}_n$, the* **2-partition inequality induced by S and T** *(for short: $[S,T]$-***inequality***)*

$$x([S:T]) - x(\mathcal{E}_n(S)) - x(\mathcal{E}_n(T)) \leq \min\{|S|, |T|\}$$

is valid for $\mathcal{P}_n$. It defines a facet of $\mathcal{P}_n$ if and only if $|S| \neq |T|$.

(e) *For every cycle $C \subseteq \mathcal{E}_n$ of length at least 5 and its set $\overline{C}$ of 2-chords, the* **2-chorded cycle inequality**

$$x(C) - x(\overline{C}) \leq \left\lfloor \frac{|C|}{2} \right\rfloor$$

is valid for $\mathcal{P}_n$. It defines a facet of $\mathcal{P}_n$ if and only if $|C|$ is odd.

(f) *For every even cycle $C \subseteq E_n$ of length at least 8, for every node $z \in \mathcal{V}_n$ not in the node set $\mathcal{V}_n(C)$ of C, and for every bipartition $\{V, \overline{V}\}$ of $\mathcal{V}_n(C)$, the* **2-chorded even wheel inequality**

$$x(C \cup R) - x(\overline{C} \cup \overline{R}) \leq \frac{|C|}{2}$$

defines a facet of P_n, where $\overline{C}$ is the set of 2-chords of C and $R := \{zv \mid v \in V\}$, $\overline{R} := \{zv \mid v \in \overline{V}\}$. □

The aim of this paper is to construct further inequalities defining facets of $\mathcal{P}_n$. We will, in particular, generalize the 2-partition inequalities using some "glueing" and "lifting" techniques.

2. G-Induced $[S, T]$-Inequalities and Construction ∇

Let $G = (V, E)$ be a subgraph of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ and let $\{S, T\}$ be a partition of V, where S or T may possibly be empty. Then the inequality

$$x(E[S:T]) - x(E(S)) - x(E(T)) \leq \min\{|S|, |T|\} \tag{2.1}$$

is called a **general 2-partition inequality induced by G, S, and T**, or for short, a **G-induced $[S, T]$-inequality**. Note that the order of S and T plays no role, so a G-induced $[S, T]$-inequality is also a G-induced $[T, S]$-inequality.

Every $[S, T]$-inequality (introduced in (1.1) (d)) is a $K_{|S \cup T|}$-induced $[S, T]$-inequality where $K_{|S \cup T|}$ is the complete subgraph of K_n induced by the node set $S \cup T$. So every G-induced $[S, T]$-inequality, $S \neq \emptyset \neq T$, can be obtained from the $[S, T]$-inequality by setting some of the positive and negative coefficients to zero. G-induced $[S, T]$-inequalities are not necessarily valid with respect to $\mathcal{P}_n$. They are, however, obviously valid if S or T is empty. This case is trivial and only included for technical reasons.

(2.2) Definition. *Let S, T be two disjoint subsets of $\mathcal{V}_n$ and let $G = (V, E)$ be a subgraph of K_n with $V = S \cup T$. G is called* **$[S, T]$-valid** *(with respect to $\mathcal{P}_n$) if the G-induced $[S, T]$-inequality is valid for $\mathcal{P}_n$. G is called* **strongly $[S, T]$-valid** *(with respect to $\mathcal{P}_n$) if for every node set $W \subseteq V$ the $(G - W)$-induced $[S \setminus W, T \setminus W]$-inequality is valid for $\mathcal{P}_n$.* □

So, for a strongly $[S, T]$-valid graph $G = (V, E)$, $G - W$ is $[S \setminus W, T \setminus W]$-valid for all $W \subseteq V$, in fact, $G - W$ is strongly $[S \setminus W, T \setminus W]$-valid.

(2.3) Remark. (a) If $G = (V, E)$ is a subgraph of K_n with $E = \emptyset$ then G is strongly $[S, T]$-valid for every partition $\{S, T\}$ of V.

(b) It follows immediately from (1.1) (d) that every complete subgraph $G = (V, E)$ of K_n is strongly $[S, T]$-valid for every partition $\{S, T\}$ of V (S or T possibly empty). □

We will now introduce a construction that can be used to combine strongly $[S_i, T_i]$-valid graphs G_i ($i = 1, 2$) into new strongly $[S, T]$-valid graphs. Let us first describe it in terms of an operation on two graphs.

Suppose we have two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ and, for each $i \in \{1,2\}$, we are given two subsets S'_i, T'_i of V_i such that S'_i and T'_i are disjoint, $|T'_1| = |T'_2|$, and the induced subgraph $G[T'_i]$ of G_i is complete. Assume furthermore that a bijection $\varphi : T'_1 \to T'_2$ is given. Let $G = (V, E)$ be the graph obtained from G_i, S'_i, T'_i $(i = 1,2)$ by identifying each node v of $G_1[T'_1]$ with the corresponding node $\varphi(v)$ of $G_2[T'_2]$ and adding all edges with one endnode in S'_1 and the other in S'_2. We call this operation **Construction** ∇ — see Figure 2.1 for a pictorial description.

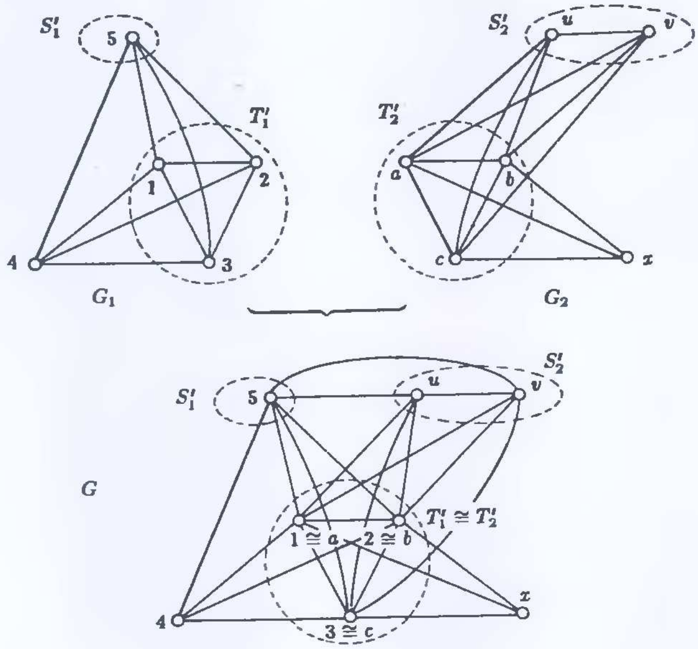

Figure 2.1 Example of Construction ∇

In order to avoid the necessity of specifying the bijection φ it is more convenient for us to work on subgraphs G_1 and G_2 of the complete graph K_n whose intersection is a complete subgraph. So let us redefine Construction ∇ for that case.

(2.4) Definition. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ and let $S'_1 \subseteq V_1$, $S'_2 \subseteq V_2$, and $T' \subseteq \mathcal{V}_n$ be node sets such that*

(A.1) $V_1 \cap V_2 = T'$;

(A.2) $S'_1 \subseteq V_1 \setminus T'$, $S'_2 \subseteq V_2 \setminus T'$;

(A.3) $G_i[T']$ *is complete for* $i = 1,2$.

Let $G = (V, E)$ be the subgraph of K_n obtained from the union $G_1 \cup G_2$ of G_1 and G_2 by adding all edges with one endnode in S'_1 and the other in S'_2. We will say that G is obtained from G_1 and G_2 by Construction $\nabla(S'_1, S'_2; T')$ and write $G = G_1 \nabla G_2$. □

Note that in (2.4) we replace the identification process (which depends on φ) by assuming that the two subgraphs overlap in T'. This way φ is given implicitly. Also observe that the cases $S_1 = \emptyset$, $S_2 = \emptyset$, or $T' = \emptyset$ are allowed in Construction $\nabla(S'_1, S'_2; T')$. It is immediately clear from the above definition that the following holds.

(2.5) Remark. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be subgraphs of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$ satisfying the assumptions of (2.4), and let $G = G_1 \nabla G_2$ be the subgraph of K_n obtained by Construction $\nabla(S'_1, S'_2; T')$. Then, for all $W \subseteq \mathcal{V}_n$, the graphs $G_1 - W$ and $G_2 - W$ and the node sets $S'_1 \setminus W$, $S'_2 \setminus W$, and $T' \setminus W$ satisfy the assumptions of (2.4). So Construction $\nabla(S'_1 \setminus W, S'_2 \setminus W; T' \setminus W)$ is well-defined, and $G - W = (G_1 - W)\nabla(G_2 - W)$ holds. □

(2.6) Theorem. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$. For $i = 1, 2$, let $\{S_i, T_i\}$ be a partition of V_i and let $S'_i \subseteq V_i$, $T' \subseteq \mathcal{V}_n$ be node sets such that*

(B.1) $V_1 \cap V_2 = T_1 \cap T_2 = T'$;

(B.2) $S'_i \subseteq S_i$;

(B.3) *$G_i[T']$ is complete;*

(B.4) *G_i is strongly $[S_i, T_i]$-valid with respect to $\mathcal{P}_n$;*

(B.5) *no node in $S_i \setminus S'_i$ is adjacent to a node in T'.*

Let $G = (V, E)$ be the subgraph of K_n obtained from G_1 and G_2 by Construction $\nabla(S'_1, S'_2; T')$ and set $T := T_1 \cup T_2$, $S := S_1 \cup S_2$. Then G is strongly $[S, T]$-valid with respect to $\mathcal{P}_n$.

Proof. Note that assumptions (B.1), (B.2), (B.3) imply assumptions (A.1), (A.2), (A.3). So Construction $\nabla(S'_1, S'_2; T')$ can be performed.

We prove the theorem by induction on $\nu := |V_1| + |V_2| - |T'| = |V|$. The result is obvious for $\nu \leq 3$. (Actually, the only interesting case is $\nu = 3$ and $|S_i| = |S'_i| = |T'| = 1$ where we obtain a triangle inequality (1.1) (c) from two trivial inequalities of type $x_e \leq 1$. Observe also that, for $|S_i| = |T'| = 1$ and $S'_i = \emptyset$, we have $E = \emptyset$ by (B.5).)

Assume now that the theorem holds for $\nu \geq 3$ and let G_1, G_2, T' be such that $|V_1| + |V_2| - |T| = \nu + 1$. We have to prove that $G - W$ is $[S \setminus W, T \setminus W]$-valid with respect to $\mathcal{P}_n$ for all $W \subseteq V$.

Case 1. $W \neq \emptyset$. By Remark (2.5), $G - W = (G_1 - W)\nabla(G_2 - W)$; by assumption (B.4), $G_i - W$ is strongly $[S_i \setminus W, T_i \setminus W]$-valid; and thus (since $|V_1 \setminus W| + |V_2 \setminus W| - |T' \setminus W| < \nu + 1$) $G - W$ is strongly $[S \setminus W, T \setminus W]$-valid by induction hypothesis.

Case 2. $W = \emptyset$. To prove that G is $[S, T]$-valid we use the fact — proved in Case 1 — that $G - v$ is $[S \setminus \{v\}, T \setminus \{v\}]$-valid for all $v \in V$. By adding the sum of the left-hand sides of the $|S|$ valid inequalities

$$x(E[S \setminus \{v\} : T]) - x(E(S \setminus \{v\})) - x(E(T)) \leq \min\{|S| - 1, |T|\}, \ v \in S$$

to the sum of the left-hand sides of the $|T|$ valid inequalities

$$x(E[S : T \setminus \{v\}]) - x(E(S)) - x(E(T \setminus \{v\})) \leq \min\{|S|, |T| - 1\}, \ v \in T$$

and estimating the sum of the $|S|+|T|$ right-hand sides from above we obtain the valid inequality

$$(\nu-1)(x(E[S:T])-x(E(S))-x(E(T)))\le(\nu+1)\min\{|S|,|T|\}-\min\{|S|,|T|\}.$$

Dividing by $\nu-1$ we get

$$x(E[S:T])-x(E(S))-x(E(T))\le\min\{|S|,|T|\}+\tfrac{1}{\nu-1}\min\{|S|,|T|\}.$$

It follows from $\nu\ge 3$ and $\min\{|S|,|T|\}\le\lfloor\nu/2\rfloor$ that $\min\{|S|,|T|\}/(\nu-1)<1$ which implies that $x(E[S:T])-x(E(S))-x(E(T))\le\min\{|S|,|T|\}$ is valid for $\mathcal{P}_n$. □

We will now prove the main (technical) result of our paper that will be used later to derive interesting classes of facet-defining inequalities for $\mathcal{P}_n$. Recall that a **matching** is a subset M of the edges of a graph such that no two edges in M have a common endnode; an s-**matching** is a matching with s elements. A node that is in some edge of a matching M is said to be **covered** by M.

(2.7) Theorem. *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two subgraphs of $K_n=(V_n,\mathcal{E}_n)$. For $i=1,2$, let $\{S_i,T_i\}$ be a partition of V_i and let $S_i'\subseteq V_i$, $T'\subseteq V_n$ be node sets such that*

(C.1) $V_1\cap V_2=T_1\cap T_2=T'$, $|T'|\ge 2$;

(C.2) $\emptyset\ne S_i'\subseteq S_i$, $|S_i|\le|T_i\setminus T'|$;

(C.3) *$G_i[S_i'\cup T']$ is complete;*

(C.4) *G_i is strongly $[S_i,T_i]$-valid with respect to $\mathcal{P}_n$ and the associated $[S_i,T_i]$-inequality defines a facet of $\mathcal{P}_n$;*

(C.5) *no node in $S_i\setminus S_i'$ is adjacent to a node in T';*

(C.6) *for every pair of nodes w,z with $w\in T_i\setminus T'$ and $z\in T'$, G_i has an $|S_i|$-matching $M_i(w,z)$ contained in $E_i[S_i:(T_i\setminus T')\cup\{z\}]$ that does not cover w;*

(C.7) *G_i has an $|S_i|$-matching N_i contained in $E_i[S_i:T_i\setminus T']$.*

Let $G=(V,E)$ be the subgraph of K_n obtained from G_1 and G_2 by Construction $\nabla(S_1',S_2';T')$ and let $S:=S_1\cup S_2$, $T:=T_1\cup T_2$. Then G is strongly $[S,T]$-valid and the G-induced $[S,T]$-inequality defines a facet of $\mathcal{P}_n$.

Proof. The assumptions (C.1),...,(C.5) obviously imply the assumptions (B.1),..., (B.5) of Theorem (2.6). So, Construction $\nabla(S_1',S_2';T')$ is well-defined and the graph $G=G_1\nabla G_2$ is strongly $[S,T]$-valid with respect to $\mathcal{P}_n$.

Let $a^Tx\le\alpha$ be the G-induced $[S,T]$-inequality, i. e., $a^Tx=x(E[S:T])-x(E(S))-x(E(T))\le\min\{|S|,|T|\}=\alpha$, and let $F_a:=\{x\in\mathcal{P}_n\mid a^Tx=\alpha\}$. Assume that $b^Tx=\beta$ defines a hyperplane such that $F_b:=\{x\in\mathcal{P}_n\mid b^Tx=\beta\}$ is a facet of $\mathcal{P}_n$ with $F_a\subseteq F_b$. We will show that there exists a real number $\pi\ne 0$ such that $b=\pi a$. This will prove the theorem.

Let us, for $i=1,2$, denote the G_i-induced $[S_i,T_i]$-inequality by $(a^i)^Tx\le\alpha_i$. So $a^Tx=(a^1)^Tx+(a^2)^Tx+x(E(T'))-x([S_1':S_2'])$. It follows from (C.2) that $\alpha_i=|S_i|$ $(i=1,2)$ and $\alpha=|S|=|S_1|+|S_2|$.

We prove that $b_e = \pi a_e$ for all $e \in \mathcal{E}_n$ and some $\pi \in \mathbf{R}$.

Case 1. $e \in \mathcal{E}_n(V_1) \cup \mathcal{E}_n(V_2)$.

Since $(a^1)^T x \leq \alpha_1$ defines a facet of $\mathcal{P}_n$ there are $m := |\mathcal{E}_n|$ clique partitionings $A_1, \ldots, A_m$ whose incidence vectors are linearly independent and satisfy $(a^1)^T x \leq \alpha_1$ with equality. By (C.7) there exists an $|S_2|$-matching $N_2 \subseteq [S_2 : T_2 \setminus T']$. Hence the edge sets $B_j := (A_j \cap \mathcal{E}_n(V_1)) \cup N_2$ are clique partitionings of K_n such that $a^T \chi^{B_j} = \alpha$ for $j = 1, \ldots, m$; and therefore

$$b^T(\chi^{B_j} - \chi^{B_m}) = 0 \quad \text{for } j = 1, \ldots, m-1$$

holds. Let X be the $(m-1) \times |\mathcal{E}_n|$ matrix whose rows are the vectors $\chi^{B_j} - \chi^{B_m}$, $j = 1, \ldots, m-1$. All columns of X corresponding to edges $e \in \mathcal{E}_n \setminus \mathcal{E}_n(V_1)$ are zero, and it follows from the fact that the vectors $\chi^{A_1}, \ldots, \chi^{A_m}$ are linearly independent that the $(m-1) \times |\mathcal{E}_n(V_1)|$ submatrix Y of X corresponding to the edges $e \in \mathcal{E}_n(V_1)$ has rank $|\mathcal{E}_n(V_1)| - 1$. So the kernel $\{y \in \mathbf{R}^{\mathcal{E}_n(V_1)} \mid Yy = 0\}$ of Y has dimension 1. Since the vector $\hat{a}^1 \in \mathbf{R}^{\mathcal{E}_n(V_1)}$ obtained from a^1 by deleting all components corresponding to edges in $\mathcal{E}_n \setminus \mathcal{E}_n(V_1)$ and the vector $\hat{b} \in \mathbf{R}^{\mathcal{E}_n(V_1)}$ obtained from b in the same way satisfy $Y\hat{a}^1 = Y\hat{b} = 0$ and since $\hat{a}^1 \neq 0$ we know that there exists a real number π such that $\hat{b} = \pi \hat{a}^1$. This implies $b_e = \pi a_e^1$ for all $e \in \mathcal{E}_n(V_1)$.

By symmetry we obtain that there exists a real number π' such that $b_e = \pi' a_e^2$ for all $e \in \mathcal{E}_n(V_2)$. By (C.1), $T' = V_1 \cap V_2$ and $|T'| \geq 2$, and by (C.3) $G[T']$ is complete. So there is an edge $f \in E(T') = E_1(T') = E_2(T')$. Since $a_f = a_f^1 = a_f^2 = -1$ we can conclude that $\pi = \pi'$, and thus there exists a real number π such that

(1) $$b_e = \pi a_e \;\text{ for all } e \in \mathcal{E}_n(V_1) \cup \mathcal{E}_n(V_2).$$

Case 2. $e = uv$ with $u \in S_1'$ and $v \in S_2'$.

Let z_1 and z_2 be two different nodes in T'. Let N_1, N_2 be the two matchings existing by (C.7) and let $u' \in T_1 \setminus T'$, $v' \in T_2 \setminus T'$ be the nodes such that $uu' \in N_1$ and $vv' \in N_2$. Set

$$A := N_1 \cup N_2 \text{ and}$$
$$B := (A \setminus \{uu', vv'\}) \cup \{uv, uz_1, uz_2, vz_1, vz_2, z_1z_2\}.$$

Then A and B are clique partitionings; χ^A obviously satisfies $\chi^A \in F_a \subseteq F_b$, while (C.3) yields that $\chi^B \in F_a \subseteq F_b$. Thus (1) implies $0 = b^T\chi^A - b^T\chi^B + b_{uu'} + b_{vv'} - b_{uv} - b_{uz_1} - b_{uz_2} - b_{vz_1} - b_{vz_2} - b_{z_1z_2} = -b_{uv} - \pi$. From this we obtain

(2) $$b_e = -\pi \;\text{ for all } e \in [S_1' : S_2'].$$

Case 3. $e = uv$ with $u \in T_1 \setminus T'$ and $v \in T_2 \setminus T'$.

Let z_1, z_2 be any two nodes in T' and let $M_1(u, z_1) \subseteq E_1[S_1 : (T_1 \setminus T') \cup \{z_1\}]$, $M_2(v, z_2) \subseteq E_2[S_2 : (T_2 \setminus T') \cup \{z_2\}]$ be $|S_i|$-matchings ($i = 1, 2$) not covering u and v, respectively. Such matchings exist by (C.6). Set

$$A := M_1(u, z_1) \cup M_2(v, z_2), \; B := A \cup \{e\}.$$

Then A and B are clique partitionings with $\chi^A, \chi^B \in F_a \subseteq F_b$, and we can conclude from $0 = \beta - \beta = b^T\chi^B - b^T\chi^A = b_e$ that

$$b_e = 0 \text{ for all } e \in [T_1 \setminus T' : T_2 \setminus T']. \tag{3}$$

Case 4. $e = uv$ with $u \in T_i \setminus T'$ and $v \in S_j$ for $i, j \in \{1,2\}$, $i \neq j$.

Let $N_j \subseteq E_j[S_j : T_j \setminus T']$ be the $|S_j|$-matching existing by (C.7) and let $v' \in T_j \setminus T'$ be the node with $vv' \in N_j$. Let z be any node in T' and $M_i(u,z)$ be the $|S_i|$-matching existing by (C.6). Set

$$A := M_i(u,z) \cup N_j \text{ and } B := A \cup \{uv, uv'\}.$$

Then A and B are clique partitionings with $\chi^A, \chi^B \in F_a \subseteq F_b$. So $0 = b^T\chi^B - b^T\chi^A = b_{uv} + b_{uv'}$, and (3) implies

$$b_e = 0 \text{ for all } e \in [T_i \setminus T' : S_j] \text{ with } i, j \in \{1,2\}, i \neq j. \tag{4}$$

Case 5. $e = uv$ with $u \in S_i \setminus S'_i$ and $v \in S_j$ for $i, j \in \{1,2\}$, $i \neq j$.

Let N_i and N_j be the matchings existing by (C.7) and let $u' \in T_i \setminus T'$, $v' \in T_j \setminus T'$ be the nodes with $uu' \in N_i$ and $vv' \in N_j$. Set

$$A := N_i \cup N_j, \; B := A \cup \{uv, u'v', uv', u'v\}.$$

Then A and B are clique partitionings with $\chi^A, \chi^B \in F_a \subseteq F_b$. Therefore, $0 = b_{uv} + b_{u'v'} + b_{uv'} + b_{u'v}$ and (3) and (4) imply

$$b_e = 0 \text{ for all } e \in [S_i \setminus S' : S_j] \text{ with } i, j \in \{1,2\}, i \neq j. \tag{5}$$

Case 6. $e = uv$ with $u \in \mathcal{V}_n \setminus V$.

This case is trivial and we obtain

$$b_e = 0 \text{ for all } e \notin \mathcal{E}_n(V).$$

Altogether we have now shown that $b = \pi a$, and clearly $\pi \neq 0$. Thus $a^T x \leq \alpha$ defines a facet of $\mathcal{P}_n$. □

We would like to remark that the statement of Theorem (2.7) holds under slightly more general conditions. These are, however, rather complicated and technical. We have decided to present here the systems (C.1), ... ,(C.7). These assumptions are relatively easy to understand and are sufficient for the derivation of our main classes of facet-defining inequalities. An immediate consequence of Theorem (2.7) is the following.

(2.8) Theorem. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two complete subgraphs of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$. For $i = 1, 2$, let $\{S_i, T_i\}$ be a partition of V_i and let $T' \subseteq \mathcal{V}_n$ be node sets such that*

(D.1) $V_1 \cap V_2 = T_1 \cap T_2 = T'$, $|T'| \geq 2$;

(D.2) $1 \leq |S_i| \leq |T_i \setminus T'|$.

Let $G = (V, E)$ be the subgraph of K_n obtained from G_1 and G_2 by Construction $\nabla(S_1, S_2; T')$ and let $S := S_1 \cup S_2$, $T := T_1 \cup T_2$. Then G is strongly $[S, T]$-valid and the G-induced $[S, T]$-inequality defines a facet of $\mathcal{P}_n$.

Proof. For $i = 1, 2$, set $S'_i = S_i$ then the assumptions (C.1), (C.2), (C.3), (C.5), (C.6), (C.7) are obviously satisfied; (C.4) is satisfied by Remark (2.3) (b). Thus (2.8) follows from (2.7). □

Figure 2.2 shows two graphs that are obtained by Construction ∇ from two complete subgraphs of K_n. The associated general $[S, T]$-inequalities define facets of $\mathcal{P}_n$ for $n \geq 6$ and $n \geq 9$, respectively, by Theorem (2.8).

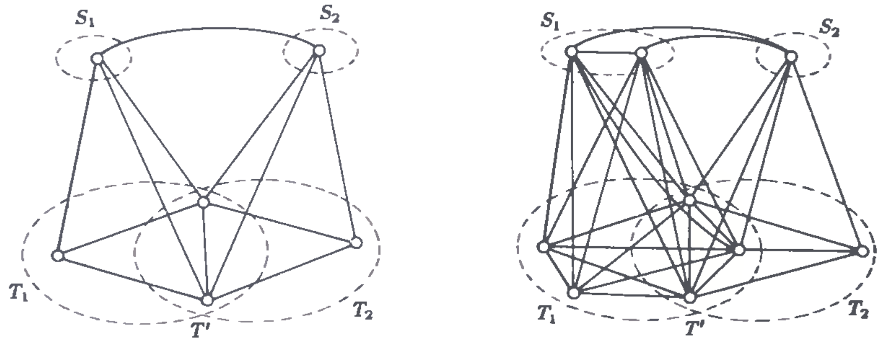

Figure 2.2 Graphs inducing general 2-partition inequalities.

Theorem (2.8) has been cast in a way that Theorem (2.7) is directly applicable. The following version of it is probably easier to remember.

(2.9) Corollary. *Let S_1, S_2, T_1, T_2, T be five mutually disjoint subsets of the node set $\mathcal{V}_n$ of K_n such that $|T| \geq 2$ and $1 \leq |S_i| \leq |T_i|$ for $i = 1, 2$. Then the (general 2-partition) inequality*

$$x([S_1 : T_1 \cup T]) + x([S_2 : T_2 \cup T]) - x([S_1 : S_2]) - x(\mathcal{E}_n(S_1)) - x(\mathcal{E}_n(S_2)) -$$
$$x([T_1 : T]) - x([T_2 : T]) - x(\mathcal{E}_n(T_1)) - x(\mathcal{E}_n(T_2)) - x(\mathcal{E}_n(T)) \leq |S_1| + |S_2|$$

defines a facet of $\mathcal{P}_n$. □

3. Two Further Compositions

We will now describe two ways of applying Construction ∇ iteratively that can be used to produce new facet-defining inequalities for $\mathcal{P}_n$.

(3.1) Definition. *Let $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$, $p \geq 2$, be complete subgraphs of K_n, let $\{S_i, T_i\}$ be a partition of V_i, $i = 1, \ldots, p$, and let T_i', $i = 1, \ldots, p-1$, be disjoint subsets of $\mathcal{V}_n$ such that*

(E.1) $V_i \cap V_{i+1} = T_i \cap T_{i+1} = T_i'$ *for* $i = 1, \ldots, p-1$;

(E.2) $V_i \cap V_{i+k} = \emptyset$ *for* $i = 1, \ldots, p-2$ *and* $k = 2, \ldots, p-i$.

Set $\tilde{G}_1 := G_1$, and for $i = 2, \ldots, p$, let $\tilde{G}_i$ be the graph obtained from $\tilde{G}_{i-1}$ and G_i by Construction $\nabla(S_{i-1}, S_i; T_{i-1}')$. Let us denote the graph $\tilde{G}_p$ constructed this way by $G = (V, E)$. We say that G is the graph obtained from complete graphs $G_1, \ldots, G_p$ by **Construction** $\nabla(S_1, \ldots, S_p; T_1', \ldots, T_{p-1}')$, *or (not specifying details) by a* **repeated nonoverlapping ∇-construction**. □

Figure 3.1 shows the scheme of a repeated nonoverlapping ∇-construction.

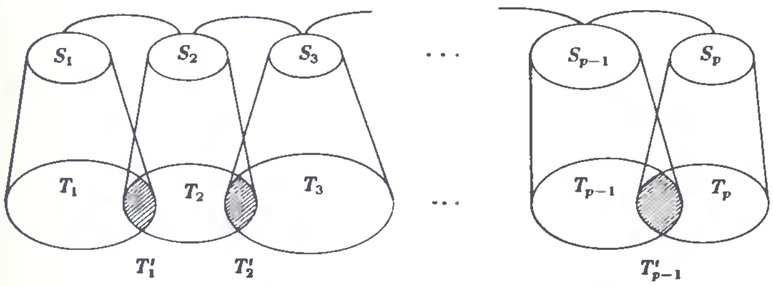

Figure 3.1

(3.2) Theorem. *Let $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$, $p \geq 2$, be complete subgraphs of $K_n = (\mathcal{V}_n, \mathcal{E}_n)$, let $\{S_i, T_i\}$ be a partition of V_i for $i = 1, \ldots, p$; and let T_i', $i = 1, \ldots, p-1$, be disjoint subsets of $\mathcal{V}_n$ such that*

(F.1) $V_i \cap V_{i+1} = T_i \cap T_{i+1} = T_i'$ *and* $|T_i'| \geq 2$ *for* $i = 1, \ldots, p-1$;

(F.2) $V_i \cap V_{i+k} = \emptyset$ *for* $i = 1, \ldots, p-2$ *and* $k = 2, \ldots, p-i$;

(F.3) $1 \leq |S_i| \leq |T_i| - \max\{|T_{i-1}'|, |T_i'|\}$ *for* $i = 2, \ldots, p-1$,
$1 \leq |S_1| \leq |T_1| - |T_1'|$, $1 \leq |S_p| \leq |T_p| - |T_{p-1}'|$.

Let $G = (V, E)$ be the subgraph of K_n obtained from $G_1, \ldots, G_p$ by Construction $\nabla(S_1, \ldots, S_p; T_1', \ldots, T_{p-1}')$ and let $S := \bigcup_{i=1}^p S_i$, $T := \bigcup_{i=1}^p T_i$. Then G is strongly $[S,T]$-valid and the G-induced $[S,T]$-inequality defines a facet of $\mathcal{P}_n$.

Proof. The assumptions imply that the repeated nonoverlapping ∇-construction (3.1) can be performed. Let us denote the graphs constructed in this process by $\tilde{G}_i = (\tilde{V}_i, \tilde{E}_i)$ and set $\tilde{S}_i := \bigcup_{j=1}^i S_i$, $\tilde{T}_i := \bigcup_{j=1}^i T_i$, $\tilde{S}_i' := S_i$, $i = 1, \ldots, p$.

By Theorem (1.1) (d) all G_i-induced $[S_i, T_i]$-inequalities define facets of $\mathcal{P}_n$, and applying Theorem (2.8) to $\tilde{G}_1 = G_1$ and G_2 we get that, for the graph $\tilde{G}_2 = (\tilde{V}_2, \tilde{E}_2)$ obtained by Construction $\nabla(\tilde{S}_1', S_2; T_1')$ from $\tilde{G}_1$ and G_2, the $\tilde{G}_2$-induced $[\tilde{S}_2, \tilde{T}_2]$-inequality is strongly valid for $\mathcal{P}_n$ and defines a facet of $\mathcal{P}_n$. It is easy to see that $\tilde{G}_2 = (\tilde{V}_2, \tilde{E}_2)$ and $G_3 = (V_3, E_3)$ with the partitions $\{\tilde{S}_2, \tilde{T}_2\}$ of $\tilde{V}_2$ and $\{S_3, T_3\}$ of V_3 and additional sets

$S_1' := \tilde{S}_2'$, $S_2' := S_3$, $T' := T_2'$ satisfy all assumptions of Theorem (2.7). So, for the graph $\tilde{G}_3 = (\tilde{V}_3, \tilde{E}_3)$ obtained from $\tilde{G}_2$ and G_3 by Construction $\nabla(S_1', S_2'; T') = \nabla(\tilde{S}_2', S_3; T_2')$, the $\tilde{G}_3$-induced $[\tilde{S}_3, \tilde{T}_3]$-inequality is strongly valid for $\mathcal{P}_n$ and defines a facet of $\mathcal{P}_n$.

Repeating this process iteratively we can conclude that for $G = \tilde{G}_p = \tilde{G}_{p-1} \nabla G_p$ the G-induced $[S, T]$-inequality is strongly valid and defines a facet of $\mathcal{P}_n$. □

An easier to read version of the above theorem — which includes Corollary (2.9) as a special case — is the following.

(3.3) Corollary. *Let $S_1, \ldots, S_p, T_1, \ldots, T_p, T_1', \ldots, T_{p-1}'$, $p \geq 2$, be mutually disjoint subsets of the node set $\mathcal{V}_n$ of K_n. Set (for notational convenience) $T_0' := T_p' := \emptyset$ and assume that*

$$|T_i'| \geq 2 \quad \text{for } i = 1, \ldots, p-1;$$
$$1 \leq |S_i| \leq |T_i| + \min\{|T_i'|, |T_{i-1}'|\} \quad \text{for } i = 1, \ldots, p$$

is satisfied. Then the (general 2-partition) inequality

$$\sum_{i=1}^{p} x([S_i : T_i \cup T_i' \cup T_{i-1}']) - \sum_{i=1}^{p-1} x([S_i : S_{i+1}]) - \sum_{i=1}^{p} x([T_i : T_i' \cup T_{i-1}'])$$
$$- \sum_{i=1}^{p} (x(\mathcal{E}_n(S_i)) + x(\mathcal{E}_n(T_i)) + x(\mathcal{E}_n(T_i'))) \leq \sum_{i=1}^{p} |S_i|$$

defines a facet of $\mathcal{P}_n$.

□

Figure 3.2 shows a graph obtained by a repeated nonoverlapping ∇-construction of complete graphs.

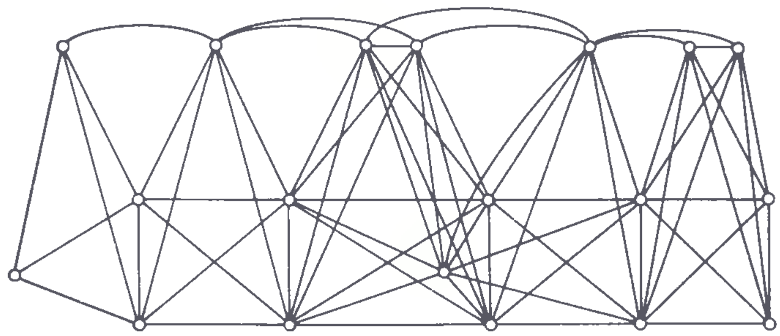

Figure 3.2 Nonoverlapping ∇-composition of complete graphs.

Another way of making iterative use of Construction ∇ is the following.

(3.4) Definition. *Let $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$, $p \geq 2$, be complete subgraphs of K_n, let $\{S_i, T_i\}$ be a partition of V_i for $i = 1, \ldots, p$ and let T' be a subset of $\mathcal{V}_n$ such that*

(G.1) $\quad V_i \cap V_j = T_i \cap T_j = T'$ *for* $1 \leq i < j \leq p$.

Set $\overline{G}_1 := G_1$, and for $i = 2, \ldots, p$, let $\overline{G}_i$ be the subgraph of K_n obtained from $\overline{G}_{i-1}$ and G_i by Construction $\nabla(S_1 \cup \ldots \cup S_{i-1}, S_i; T')$. Let us denote the graph $\overline{G}_p$ obtained *this way by $G = (V, E)$. We say that G is the graph obtained from complete graphs $G_1, \ldots, G_p$ by* **Construction** $\nabla(S_1, \ldots, S_p; T')$, *or (without specifying details) by a* **repeated totally overlapping ∇-construction**. □

The basic scheme of a repeated totally overlapping ∇-construction is displayed in Figure 3.3.

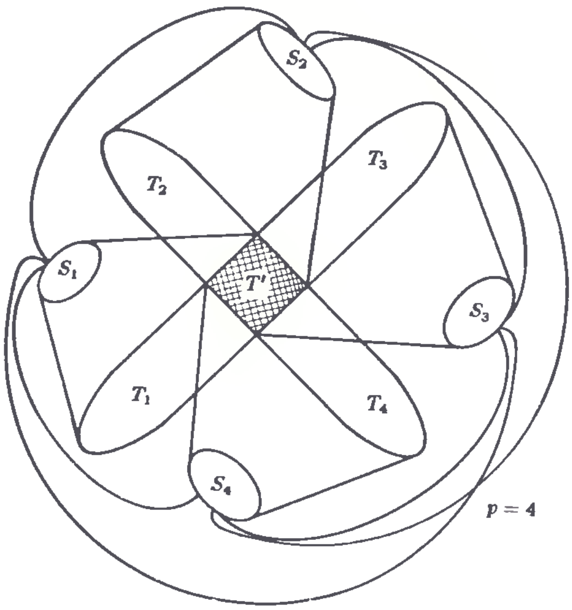

Figure 3.3

Facet-defining inequalities can be obtained with the repeated totally overlapping ∇-construction as follows.

(3.5) Theorem. *Let $G_1 = (V_1, E_1), \ldots, G_p = (V_p, E_p)$, $p \geq 2$, be complete subgraphs of K_n, let $\{S_i, T_i\}$ be a partition of V_i for $i = 1, \ldots, p$ and let $T' \subseteq \mathcal{V}_n$ such that*

(H.1) $V_i \cap V_j = T_i \cap T_j = T'$ *for* $1 \leq i < j \leq p$, $|T'| \geq 2$;

(H.2) $1 \leq |S_i| \leq |T_i| - |T'|$ *for* $i = 1, \ldots, p$.

Let $G = (V, E)$ be the subgraph of K_n obtained from $G_1, \ldots, G_p$ by Construction $\nabla(S_1, \ldots, S_p; T')$ and let $S := \bigcup_{i=1}^{p} S_i$, $T := \bigcup_{i=1}^{p} T_i$. Then G is strongly $[S, T]$-valid and the G-induced $[S, T]$-inequality defines a facet of $\mathcal{P}_n$.

Proof. The result follows — as in the proof of (3.2) — by induction from Theorem (2.7). The proof is straightforward and is left to the reader. □

A more digestible form of Theorem (3.5) is the following.

(3.6) Corollary. *Let $S_1, \ldots, S_p, T_1, \ldots, T_p, T'$, $p \geq 2$, be mutually disjoint subsets of the node set $\mathcal{V}_n$ of K_n. Set $S := \bigcup_{i=1}^{p} S_i$ and assume that*

$$|T'| \geq 2;$$
$$1 \leq |S_i| \leq |T_i| - |T'| \quad \text{for } i = 1, \ldots, p$$

holds. Then the (general 2-partition) inequality

$$\sum_{i=1}^{p} x([S_i : T_i \cup T']) - x(\mathcal{E}_n(S)) - x(\mathcal{E}_n(T')) - \sum_{i=1}^{p}(x(\mathcal{E}_n(T_i)) + x([T_i : T'])) \leq |S|$$

defines a facet of $\mathcal{P}_n$. □

A graph obtained by a repeated totally overlapping ∇-construction from 3 complete graphs is shown in Figure 3.4. Here $|T'| = 2$ and the edge forming $\mathcal{E}_n(T')$ is drawn by a thick line. The sets S_1, S_2, and S_3 have cardinality 1, 1, and 2 respectively.

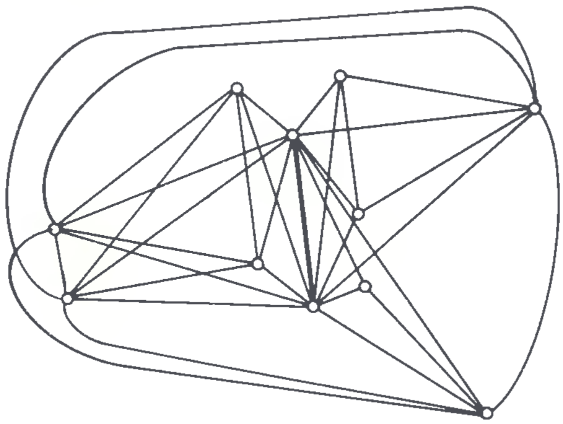

Figure 3.4 Totally overlapping ∇-composition of complete graphs.

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