Polytopes associated with length restricted directed circuits

Diplomarbeit
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Zusammenfassung

Zur Lösung einer Vielzahl NP-schwerer kombinatorischer Optimierungsprobleme werden erfolgreich Methoden der Linearen Programmierung, wie z.B. Branch-and-Cut-Algorithmen, eingesetzt. Da in Branch-and-Cut-Algorithmen Schnittebenenverfahren zum Einsatz kommen, ist ein gutes Verständnis der Facettialstruktur der konvexen Hülle der Inzidenzvektoren der zulässigen Lösungen eines gegebenen Optimierungsproblems sehr nützlich. Ein bekanntes Beispiel zur Lösung eines kombinatorischen Optimierungsproblems mittels dieses Ansatzes ist das Traveling Salesman Problem. Ferner ist das verwandte Problem in einem Graphen oder Digraphen einen kürzesten Kreis mit Hilfe dieses Ansatzes zu finden, bereits weitgehend untersucht worden.

Die vorliegende Diplomarbeit befaßt sich mit dem längenbeschränkten gerichteten Kreisproblem, d.h. dem Problem, in einem gerichteten Graphen $D = (V, A)$ mit Bogengewichten $c_a \in \mathbb{R}$, $a \in A$, einen gerichteten Kreis C einer zulässigen Länge $|C| \in L$ mit minimalem Gewicht $\sum_{a \in C} c_a$ zu bestimmen. Dabei ist L die Menge der zulässigen Kreislängen. Das zugehörige Polytop ist das Polytop $P_C^L(D)$ der längenbeschränkten gerichteten Kreise C in einem Digraphen D, das ist die konvexe Hülle der Inzidenzvektoren der gerichteten Kreise C, die eine zulässige Länge $|C|$ haben.

In speziellen Fällen, z.B. wenn nur eine bestimmte Kreislänge $k \in \mathbb{N}$ zugelassen wird, kann das Problem in polynomialer Zeit gelöst werden. Aber im allgemeinen ist das Problem NP-schwer, weil wir für $L = \{|V|\}$ das Asymmetrische Traveling Salesman Problem erhalten.

Die Diplomarbeit ist wie folgt aufgebaut:

In Kapitel 1 geben wir eine Einführung in die Problemstellung, diskutieren die Komplexität des längenbeschränkten gerichteten Kreisproblems und präsentieren die Hauptergebnisse der vorliegenden Diplomarbeit.

Kapitel 2 dokumentiert die bisherigen Ergebnisse der polyedrischen Untersuchung des gerichteten Kreispolytops, d.h. dem Polytop der gerichteten Kreise ohne Längenbeschränkung. Insbesondere sind alle facetteninduzierenden Ungleichungen der Form $a^T x \geq a_0$ mit $a_0 \geq 0$ bekannt. Wir werden jedoch ergänzen, dass durch diese Ungleichungen die Dominante des Kreispolytops nicht vollständig charakterisiert ist.

Kapitel 3 umfaßt die Untersuchung der Facettialstruktur des Polytops $P_C^L(D)$. Dabei setzen wir voraus, dass D ein vollständiger Digraph ist. Wir bestimmen die Dimension des Polytops in Abhängigkeit der Längenbeschränkung L und der Ordnung des Digraphen $|V|$, beschäftigen uns mit dem Problem wie man es als ganzzahliges Programm schreiben kann und geben eine vollständige Klassifizierung der Ungleichungen, die Bestandteil der IP-Formulierung sind, d.h. in Abhängigkeit von L und der Ordnung des Digraphen werden wir ermitteln, ob eine Ungleichung der IP-Formulierung für $P_C^L(D)$ facettendefinierend ist

oder nicht.

Ferner werden wir uns mit der Verwandtschaft des Polytops der längenbeschränkten gerichteten Kreise und seines symmetrischen Pendants, d.h. des Polytops der längenbeschränkten Kreise über einen Graphen, auseinandersetzen. Basierend auf den partiellen Beschreibungen für (längenbeschränkte und nicht längenbeschränkte) Kreispolytope über einen Graphen leiten wir weitere Facetten von $P_C^L(D)$ her. Allerdings werden wir die Ungleichungen nicht vollständig klassifizieren, weil die Beweise immer sehr lang und wenig spannend wären.

Außerdem werden wir eine Verbindung zum Polytop der längenbeschränkten gerichteten s-t-Wege ziehen, das ist die konvexe Hülle der Inzidenzvektoren der gerichteten Wege P mit Startknoten s und Endknoten t, die eine zulässige Länge haben. Dieses Polytop kann - einen geeigneten Digraphen vorrausgesetzt - als Facette von $P_C^L(D)$ interpretiert werden. Basierend auf den vorangegangenen Ergebnissen können wir die Dimension dieses Polytops bestimmen und Facetten dieses Polytops zum Polytop $P_C^L(D)$ liften.

Schließlich werden wir uns mit dem Separierungsproblem für das Polytop der längenbeschränkten gerichteten Kreise beschäftigen. Das Separierungsproblem ist das Problem für einen gegebenen Punkt $y \in \mathbb{R}^A$ zu entscheiden, ob er alle (facetteninduzierenden) Ungleichungen einer gegebenen Klasse erfüllt, und falls nicht, mindestens eine Ungleichung aus der Klasse anzugeben, die von y verletzt wird. In Kapitel 4 werden wir das Separierungsproblem für die in den vorangegangenen Kapiteln erörterten Ungleichungen untersuchen.

Schlüsselwörter: Polytop der längenbeschränkten gerichtete Kreise, Dimension, Ganzzahlige Programmierung, Facetten, Liftung von Facetten, Separierung.

Preface

In this thesis I investigate the polyhedral approach to length restricted directed circuit problems as an instance of a combinatorial optimization problem with additional restrictions. The thesis, titled *Polytopes associated with length restricted directed circuits*, is motivated originally from the line planning in public and rail transport. The problems appearing in this field are frequently conjoint with length restricted paths. But since the polytopes of directed circuits as well as of undirected circuits without length restrictions are already well studied, I prefer it to build on these results and to investigate the closely related length restricted directed circuits.

I give no explanation of standard terminology and notations or basic concepts of polyhedral theory and optimization. Instead, I have tried to resort to standard and in particular to the book Grötschel, Lovász, and Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer Verlag, Berlin 1988.

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Contents

Chapter 1

Introduction

One of the most successfull attempts to solve NP-hard combinatorial optimization problems using Linear Programming methods has been the development of a brunch and cut algorithm for the traveling salesman problem which is based on the partial knowledge of the facets of the traveling salesman polytope, that is, the convex hull of the incidence vectors of tours. Further, the related problem to find in a graph or digraph a shortest circuit by means of this approach is already well investigated (See Bauer [7], Balas and Oosten [6]).

This thesis is about the *length restricted directed circuit problem*: Given a digraph $D = (V, A)$ on n nodes, a *set of lengths* $L \subseteq \{2, \dots, n\}$, and costs or weights c_a, $a \in A$, the *L-restricted circuit problem* (LRCP) is to find a circuit C^* with $|C^*| \in L$ such that $\sum_{a \in C^*} c_a$ is minimum.

The LRCP is a generalization of the *(weighted) circuit problem* (CP) which we obtain for $L = \{2, \dots, n\}$. The LRCP is NP-hard, because for $L = \{n\}$ we obtain the *asymmetric traveling salesman problem*, which is the problem of finding a minimum cost tour in D. However, there are polynomially solvable cases of the LRCP, for example, when $L = \{k\}$ for k fixed.

Likewise the CP is NP-hard, as it subsumes the asymmetric traveling salesman problem as a special case, but it can be solved in $\mathcal{O}(n^3)$ by application of shortest-path algorithms if all circuits have nonnegative cost: Find for each arc (i, j) the shortest path from j to i in $D \setminus \{(i, j)\}$ (with respect to the objective function d) and add to it d_{ij}. This yields the shortest circuit C^{ij} containing (i, j), and hence, choosing the shortest among the circuits C^{ij}, $(i, j) \in A$, solves the problem.

As is usually done in polyhedral combinatorics, we define a polytope whose vertices correspond to the feasible solutions of the LRCP. Then it is well-known from Linear Programming that the solution of a particular instance of the LRCP is equivalent to minimizing a linear objective function over this polytope.

Let $D = (V, A)$ be a digraph on n nodes and $L \subseteq \{2, \dots, n\}$ a set of lengths. Then we denote by

$$\mathcal{C}^L(D) := \{C \in \mathcal{C}^L(D) \mid |C| \in L\}$$

the set of all circuits of D with a feasible length with respect to L and by

$$P_C^L(D) := \text{conv}\{\chi^C \mid C \in \mathcal{C}^L(D)\}$$

the *L-restricted circuit polytope*, i.e., $P_C^L(D)$ is the convex hull of the incidence vectors of all circuits $C \in \mathcal{C}^L(D)$. Since the LRCP is NP-hard, we may not expect to have a decent description of the inequalities determining $P_C^L(D)$. That is, the separation problem for $P_C^L(D)$ is NP-hard. So we have to be contend with finding some valid inequalities, as possible facets, of $P_C^L(D_n)$.

Let us fix the notation for some cases of L. Clearly, by definition, $P_C^L(D) = P_C(D)$ and $\mathcal{C}^L(D) = \mathcal{C}(D)$ if $L = \{2, \ldots, n\}$.

$$\text{If } \left\{ \begin{array}{l} L = \{k\} \\ L = \{2, \ldots, k\} \\ L = \{k, \ldots, n\} \end{array} \right\} \text{ we write } \left\{ \begin{array}{l} P_C^k(D) \\ P_C^{\leq k}(D) \\ P_C^{\geq k}(D) \end{array} \right\} \text{ instead of } \left\{ \begin{array}{l} P_C^{\{k\}}(D) \\ P_C^{\{2,\ldots,k\}}(D) \\ P_C^{\{k,\ldots,n\}}(D) \end{array} \right\}.$$

Analogous we define $\mathcal{C}(D)$, $\mathcal{C}^k(D)$, $\mathcal{C}^{\leq k}(D)$, and $\mathcal{C}^{\geq k}(D)$. Further we denote by ${P_C}^{\text{odd}}(D_n)$ and ${P_C}^{\text{even}}(D_n)$ the convex hull of the incidence vectors of odd and even circuits, respectively.

A directed circuit will be considered as a set of arcs and will be denoted by its arcset or by an ordered list of nodes between parenthesis whose origin and terminus are identical. Analogous, a directed will be considered as a set of arcs and will be denoted by its arcset or by an ordered list of nodes between parenthesis.

This thesis deals with some more polytopes. The symmetric counterpart of $P_C^L(D)$ is the *L-restricted circuit polytope*

$$P_C^L(G) := \text{conv}\{\chi^C \in \mathbb{R}^E \mid C \in \mathcal{C}^L(G)\}$$

of a graph $G = (V, E)$. Here, $L \subseteq \{3, \ldots, n\}$, and $\mathcal{C}^L(G)$ denotes the set of all circuits C in G with $|C| \in L$.

Analogous to $\mathcal{C}^L(D)$ and $\mathcal{C}^L(G)$ we define

$$\begin{array}{rcl} \mathcal{P}^L(D) & := & \{P \mid P \text{ is a } s-t \text{ path in } D, |P| \in L\}, \\ \mathcal{P}^L(G) & := & \{P \mid P \text{ is a } s-t \text{ path in } G, |P| \in L\}, \end{array}$$

and analogous to $P_C^L(D)$ and $P_C^L(G)$ we define the $s-t$ path polytopes

$$\begin{array}{rcl} P_{s-t \text{ path}}^L(D) & := & \text{conv}\{\chi^P \in \mathbb{R}^A \mid P \in \mathcal{P}^L(D)\}, \\ P_{s-t \text{ path}}^L(G) & := & \text{conv}\{\chi^P \in \mathbb{R}^E \mid P \in \mathcal{P}^L(G)\}. \end{array}$$

Moreover, let us denote by $\mathcal{W}^L(D)$ the set of all closed walks W (not necessary diwalks) in D,with $|W| \in L$ and with the property that the origin and all internal nodes of such a walk are different; in other words, W is an undirected circuit in the digraph D. Then the polytope

$$P_W^L(D) := \text{conv}(\{\chi^W \in \mathbb{R}^A \mid W \in \mathcal{W}^L(D)\})$$

is called the *polytope of undirected L-restricted circuits of D*.

Note that we restrict ourselves mostly to the complete digraph $D_n = (V, A)$ on n nodes.

1.1 Survey of the main results

1.1.1 Dimension and universally valid inequalities for $P_C^L(D_n)$

It is well known that the dimension of the circuit polytope $P_C(D_n)$ is equal to $(n-1)^2$ [6]. The dimension of $P_C^L(D_n)$ in dependence of L is given in Table 1.1.

Table 1.1. Dimension of $P_C^L(D_n)$

Polytope	$P_C^2(D_n)$	$P_C^3(D_4)$	$P_C^k(D_n), n \geq 5$ $3 \leq k < n$	$P_C^n(D_n)$	$P_C^L(D_n)$, $\lvert L\rvert \geq 2$
Dimension	$\frac{1}{2}n(n-1)-1$	6	$\dim P_C(D_n)-1$	$\dim P_C(D_n)-n$	$\dim P_C(D_n)$

In generally we have not found an integer programming formulation in arc variables. So we suggest an extended formulation (see 3.1). However, integer programming formulations for the circuit polytope $P_C^{\{k,\dots,l\}}(D_n)$ can be given in arc variables:

$$\begin{array}{lrll}
\text{(i)} & x(\delta^+(v)) - x(\delta^-(v)) & = 0 & \forall\ v \in V \\
\text{(ii)} & x(\delta^+(v)) & \leq 1 & \forall\ v \in V \\
\text{(iii)} & x(\delta^+(p)) + x(\delta^+(q)) - x(\delta^+(S)) & \leq 1 & \forall\ S \subset V, 2 \leq |S| \leq n-2, \\
 & & & p \in S, q \in V \setminus S \\
\text{(iv)} & x(A) & \geq k & \\
\text{(v)} & x(A) & \leq l. & \\
\text{(vi)} & x & \in & \{0,1\}^A
\end{array}$$

The *flow constraints* (i) and the *degree constraints* (ii) are satisfied by the incidence vectors of all cycles and the zero vector. The *cardinality constraints* (iv) and (v) ensure that all circuits are of length at least k and at most l. In particular, constraint (iv) excludes the zero vector. The *disjoint circuits elimination constraints* (iii) (short dce-constraints) are satisfied by all circuits, but violated by the union of circuits with more than one member.

For $k = 2$ the cardinality constraint (iv) can be substituted by the *linear ordering constraints*

$$\text{(vii)} \qquad \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i),\pi(j)} \geq 1 \quad \forall \text{ permutations } \pi \text{ of } V.$$

Table 1.2 Characteristics of overall valid inequalities for $P_C^L(D_n)$

Inequality class	Facet defining?	Separation Complexity
degree	if $L \neq \{k\}, k = 2, 3, n$	polynomial
linear ordering	only if $2 \in L$, $\lvert L\rvert \geq 2$, $n \approx 2k-2$ where $k := \min\{l \in L \mid l > 2\}$	NP-hard
dce	for certain cases, e.g., $\lvert S\rvert, \lvert T\rvert \geq k$ where $k = \min\{l \in L \mid l > 2\}$	polynomial
nonnegativity	if $P_C^L(D_n) \neq P_C^4(D_4)$	polynomial
$x(A) \geq \min L$	if $\lvert L\rvert \geq 2, \min L > 2$	polynomial
$x(A) \leq \max L$	if $\lvert L\rvert \geq 2, \max L < n$	polynomial

The integer points of $P_C^k(D_n)$, $k \in \{2, \dots, n\}$, are characterized by (i)-(iii),(vi), and

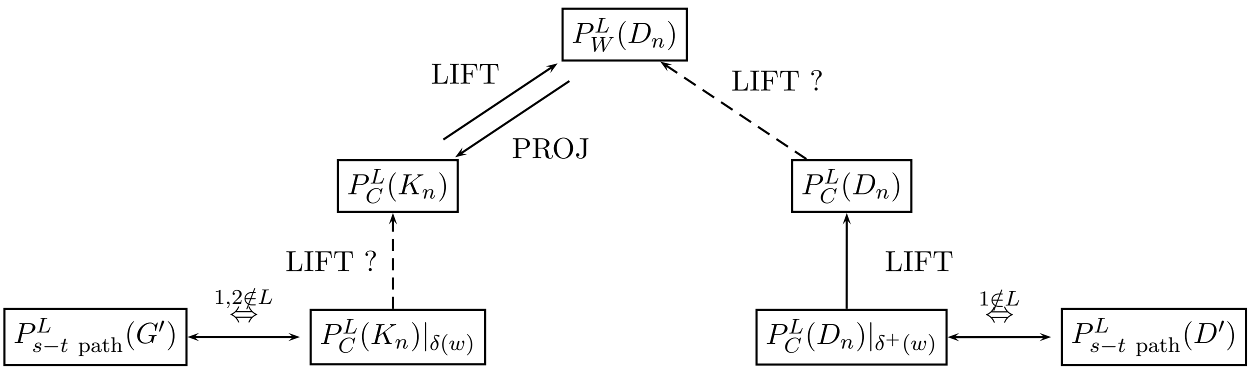
$$x(A) = k.$$

It is well known that the inequalities (ii),(iii),(vii) as well as the *nonnegativity constraints* $x_a \geq 0$, $a \in A$, define facets of the circuit polytope $P_C(D_n)$ (see Balas and Oosten [6]), but they define in generally also facets of $P_C^L(D_n)$ and can be separated in polynomial time, with the exception of the linear ordering constraints (see Table 1.2).

There are some more known classes of facet defining inequalities for $P_C(D_n)$ which can be derived from the asymmetric traveling salesman polytope $P_C^n(D_n)$ by facet lifting. The resulting inequalities are often specific to the polytope $P_C^{\geq k}(D_n)$, e.g., the *primitive SD inequalities* and the *clique tree inequalities* (see Table 1.5).

1.1.2 The relations to other polytopes

To generate facets for $P_C^L(D_n)$ it is very useful to investigate the relations to other polytopes whose facial structure is well studied. The relationships between $P_C^L(D_n)$ and some other polytopes is illustrated in Figure 1.1. An arrow here means that a facet defining inequality for the polytope at the tail of the arrow can be lifted[1] (projected) to (into) a facet defining inequality for the polytope at the head of the arrow, or they are equivalent. The dashed arrow between $P_C^L(D_n)$ and $P_W^L(D_n)$ means that an additional criterion must be satisfied while the arrow between $P_C^L(K_n)$ and $P_C^L(K_n)|_{\delta(w)}$ means that the problem is not sufficient investigated.



$$P_C^L(D_n)|_{\delta^+(v)} := \{x \in P_C^L(D_n) \mid x(\delta^+(v)) = 1\}$$
$$P_C^L(K_n)|_{\delta(v)} := \{x \in P_C^L(K_n) \mid x(\delta(v)) = 2\}$$

Figure 1.1

As already mentioned Balas and Oosten [6] studied the directed circuit polytope $P_C(D_n)$. Bauer [7] and Salazar González [16] gave a partial description of the symmetric

[1] We use standard sequential lifting (see 3.4)

counterpart, i.e., the circuit polytope $P_C(K_n)$ of the complete graph K_n. Further, the undirected cardinality constrained circuit polytope $P_C^{\leq k}(K_n)$ has been studied by Bauer, Linderoth, and Svelsbergh [8].

The connection between length restricted circuit and path polytopes

As already adumbrated, the LRCP can be solved by means of shortest path algorithms by finding for each arc (t, s) the shortest $s-t$ path. A further possibility to solve the LRCP is to find for each node v the shortest circuit C in D, with $|C| \in L$, containing v. This can be interpreted in the following fashion as a shortest path problem: let us denote by D' the digraph obtained by substituting the node v by two new nodes s and t and identifying $\delta^+(v)$ with $\delta^+(s)$ and $\delta^-(v)$ with $\delta^-(t)$. Then each circuit in D containing v is equivalent to a $s-t$ path in D'. Hence, a solution of the LRCP can be obtained by solving n times a length restricted shortest path problem.

This connection admits it to identify the polytopes $P^L_{s-t \text{ path}}(D')$ and

$$P_C^L(D)|_{\delta^+(v)} := \{x \in P_C^L(D) \mid x(\delta^+(v)) = 1\}.$$

Further, in case $D = D_n$ the degree constraint $x(\delta^+(v)) \leq 1$ defines in generally a facet of $P_C^L(D_n)$. Based on this observation we can determine the dimension of $P^L_{s-t \text{ path}}(D')$. Moreover, we can apply sequential lifting to generate facets of $P_C^L(D_n)$ from those for $P^L_{s-t \text{ path}}(D')$.

The connections between $P_C^L(D_n)$, $P_W^L(D_n)$, and $P_C^L(K_n)$

It would be desirable to transform facet defining inequalities for $P_C^L(D_n)$ into facet defining inequalities for $P_C^L(K_n)$, and conversely. For the first direction we can give a partial answer. The link between both polytopes is the polytope $P_W^L(D_n)$. On the one hand we will show namely that facets of $P_C^L(K_n)$ are in 1-1 correspondence with those of $P_W^L(D_n)$ if they are not induced by a nonnegativity constraint. On the other hand $P_C^L(D_n)$ is obviously a subset of $P_W^L(D_n)$. However, $P_C^L(D_n)$ is not a face of $P_W^L(D_n)$; so we have no general working lifting procedure. But we will show that a facet defining inequality $b^Tx \leq b_0$ for $P_C^L(D_n)$ is also facet defining for $P_W^L(D_n)$ if it is symmetric, i.e., $b_{ij} = b_{ji}$ for all $i, j \in V$, $i \neq j$, or if it is equivalent to a symmetric inequality.

1.1.3 Facets derived from the polytopes $P_C(K_n)$ and $P_C^{\leq k}(K_n)$

Many facet defining inequalities for the circuit polytopes $P_C(K_n)$ and $P_C^{\leq k}(K_n)$ can be transformed into facet defining inequalities for the corresponding (directed) circuit polytopes $P_C(D_n)$ and $P_C^{\leq k}(D_n)$, respectively. Moreover, the combinatorial idea of these inequalities is often transferable to other length restricted circuit polytopes. In the following we present the results of these reflections.

(a) The class of *bipartition inequalities* relies on the fact that an odd circuit C has an even number of arcs in $(S : T) \cup (T : S)$ and an odd number of arcs in $A(S) \cup A(T)$ for any bipartition $V = S \cup T$.[2] In the notation specific to $P_C^{\leq k}(D_n)$, k odd, they

[2]For any $X, Y \subset V$ we denote by $(X : Y)$ the set of arcs $(u, v) \in A$ such that $u \in X$ and $v \in Y$.

can be formulated as

$$x((S:T)) \leq \frac{k-1}{2},$$

where $(S:T) := \{(s,t) \in A \mid s \in S, t \in T\}$, while in the notion for $P_C^{\geq k}$ they reads as follows:

$$x(A(S)) + x(A(T)) + x((T:S)) \geq \frac{k+1}{2}.$$

These both representations are equivalent for $P_C^k(D_n)$. The bipartition inequalities are facet defining for $P_C^k(D_n)$, k odd, if $|S|, |T| \geq \frac{k+1}{2}$, and hence they can be lifted into facet defining inequalities for the polytopes $P_C^L(D_n)$, where $L \subseteq \{2, \ldots, k\}$ or $L \subseteq \{k, \ldots, n\}$, since $P_C^k(D_n)$ is a facet of those polytopes $P_C^L(D_n)$.

(b) The *generalized linear ordering constraint* in the notation specific to $P_C^{\geq k}(D_n)$ says that a circuit of length at least k uses at least one arc in $(V_i : V_j)$, $i < j$, where $\bigcup_{i=1}^m V_i$ is a partition of the nodeset V such that $|V_i| \leq k-1$ for all i:

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) \geq 1.$$

For $m = n$ this is exactly a linear ordering constraint. For $P_C^k(D_n)$ the generalized linear ordering constraints are equivalent to the *asymmetric k-partition inequalities* which are specific to the polytope $P_C^{\leq k}(D_n)$ and whose symmetric counterpart (for $P_C^{\leq k}(K_n)$) were introduced by Bauer, Linderoth, and Savelsbergh [8]:

$$\sum_{i=1}^{m} x(A(V_i)) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j) \leq (k-1)$$

The generalized linear ordering constraints are in generally facet defining for $P_C^k(D_n)$, and hence they can be lifted into facet defining inequalities for the polytopes $P_C^L(D_n)$, where $L \subseteq \{2, \ldots, k\}$ or $L \subseteq \{k, \ldots, n\}$.

(c) Bauer, Linderoth, and Savelsbergh [8] introduced the class of *cardinality-path inequalities* for the circuit polytope $P_C^{\leq k}(K_n)$ which says, that a circuit C of length at most k never uses more edges of a path P of length k than inner nodes of P. This is a class of facet defining inequalities for $P_C^{\leq k}(K_n)$, and the corresponding symmetric inequality is also facet inducing for $P_C^{\leq k}(D_n)$. Some experiments with PORTA indicates that these inequalities could be also facet inducing for $P_C^k(D_n)$.

Bauer, Linderoth, and Savelsbergh [8] introduced a further class of facet defining inequalities for the undirected circuit polytope $P_C^{\leq k}(K_n)$ they called *cardinality-tree inequalities*. We have not investigated whether the corresponding symmetric inequality is also facet defining for $P_C^{\leq k}(D_n)$ or not.

1.1.4 Overview: Facets of $P_C^k(D_n)$, $P_C^{\{3,\ldots,k\}}(D_n)$, and $P_C^{\geq k}(D_n)$

For some typical length restricted circuit polytopes we give in the following an overview of known facet defining inequalities.

Table 1.3 Facet inducing inequalities for $P_C^k(D_n)$, $4 \leq k < n$

Inequality class	Facet defining?	Separation Complexity
degree	yes	polynomial
dce	if $\lvert S\rvert, \lvert T\rvert \geq k$	polynomial
nonnegativity	yes	polynomial
bipartition	if k odd and $\lvert S\rvert, \lvert T\rvert \geq \frac{k+1}{2}$	NP-hard
generalized linear ordering	if $\lvert V_i\rvert + \lvert V_j\rvert \geq k$, $1 \leq i < j \leq m$	NP-hard
cardinality-path	?	NP-hard
parity	if $k \leq n - 2$	polynomial
cut	if $\lvert S\rvert \geq k + 1$, $\lvert T\rvert \geq k$	polynomial

Table 1.4 Facet defining inequalities for $P_C^{\{3,\ldots,k\}}(D_n)$, $4 \leq k < n$

Inequality class	Facet defining?	Separation Complexity
degree	yes	polynomial
dce	if $\lvert S\rvert, \lvert T\rvert \geq 3$	polynomial
nonnegativity	yes	polynomial
cardinality	only $x(A) \leq k$	polynomial
bipartition	yes	NP-hard
Asym. maximal set	if $s \geq 2$ and $\lvert V_i\rvert \geq 2$	?
k-partition	if $\lvert V_i\rvert + \lvert V_j\rvert \geq k$, $1 \leq i < j \leq m$	NP-hard
generalized linear ordering	if $\lvert V_i\rvert + \lvert V_j\rvert \geq 3$, $1 \leq i < j \leq m$	NP-hard
cardinality-path	yes	NP-hard
cardinality-tree	?	?

Table 1.5 Facet defining inequalities for $P_C^{\geq k}(D_n)$, $3 \leq k < n$

Inequality class	Facet defining?	Separation Complexity
degree	yes	polynomial
dce	if $\lvert S\rvert, \lvert T\rvert \geq k$	polynomial
nonnegativity	yes	polynomial
$x(A) \geq k$	if $n \geq 5$	polynomial
primitive SD	under certain conditions	?
clique tree	under certain conditions	?
bipartition	if $\lvert S\rvert, \lvert T\rvert \geq \frac{k+1}{2}$	NP-hard
generalized linear ordering	if $\lvert V_i\rvert + \lvert V_j\rvert \geq k$, $1 \leq i < j \leq m$	NP-hard
lifted cardinality-path	?	NP-hard

Chapter 2

The circuit polytope $P_C(D_n)$ and associated polyhedra

Summary. This chapter tries to survey the results so far, with respect to the circuit polytope $P_C(D_n)$ and associated polyhedra. An integer programming formulation as well as some facet lifting procedures for the circuit polytope $P_C(D_n)$ are presented. While the facets of $P_C(D_n)$ which are defined by inequalities of the form $a^T x \geq b_0$ with $b_0 \geq 0$ are all known, we will show that the dominant of $P_C(D_n)$ is not completely described by these inequalities.

2.1 Survey: The circuit polytope $P_C(D_n)$

Let us start with an integer programming formulation for the circuit polytope $P_C(D_n)$ introduced by Balas and Oosten [6].

Theorem 2.1 (IP-formulation, [6]). *The integer points in $P_C(D_n)$ are characterized by the system*

$$x(\delta^+(v)) - x(\delta^-(v)) = 0 \qquad \forall\ v \in V \tag{2.1}$$

$$x(\delta^+(v)) \leq 1 \qquad \forall\ v \in V \tag{2.2}$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} \geq 1 \tag{2.3}$$

$$x(\delta^+(p)) + x(\delta^+(q)) - x((S : V \setminus S)) \leq 1 \qquad \forall\ S \subset V, 2 \leq |S| \leq n-2,\ p \in S, q \in V \setminus S \tag{2.4}$$

$$x_a \in \{0, 1\} \qquad \forall\ a \in A. \tag{2.5}$$

Moreover, all given inequalities as well as the nonnegativity constraints

$$x_a \geq 0 \qquad \forall\, a \in A \tag{2.6}$$

define facets of $P_C(D_n)$ for $n \geq 4$.

□

The *flow constraints* (2.1) and the *degree constraints* (2.2) are satisfied by the incidence vectors of all cycles and the zero vector. Constraint (2.3) excludes the zero vector, and the *disjoint circuits elimination constraints* (2.4) (short dce-constraints) are satisfied by all circuits, but violated by the union of circuits with more than one member. In Chapter 3 we will delve into the inequalities of the IP-formulation with respect to the length restricted circuit polytope $P_C^L(D_n)$.

The flow constraints give already some information about the dimension of $P_C(D_n)$. As is easily seen, the matrix associated to the flow constraints is of rank $n-1$, since the rows corresponding to the first $n-1$ degree equations

$$x(\delta^+(v)) - x(\delta^-(v)) = 0, \qquad v = 1, \ldots, n-1,$$

are linearly independent and the last equation of the flow constraints is the sum of the former. Thus the rank of the equality set of $P_C(D_n)$ is at least $n-1$, i.e., dim $(P_C(D_n)) \leq |A| - (n-1)$. Indeed, Balas and Oosten showed [6] that dim $P_C(D_n) = |A| - (n-1) = (n-1)^2$. A proof is also given in 2.1.

Constraint (2.3) can be extended to a large class of facet defining inequalities, since for any ordering of the nodes an analogous statement holds.

Theorem 2.2 ([6]). *For any permutation π of the nodeset V, the inequality*

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i),\pi(j)} \geq 1 \tag{2.7}$$

defines a facet of $P_C(D_n)$, $n \geq 3$.

□

Let us call the inequalities (2.7) *linear ordering constraints.* Notice that the class (2.7) contains $n!$ facets of $P_C(D_n)$, one for each linear ordering of V. We have already mentioned that Balas and Oosten investigated successfully the "lower" side of the circuit polytope $P_C(D_n)$. Indeed, they showed that the nonnegativity constraints (2.6) and the linear ordering constraints (2.7) are the only facet inducing inequalities of the form $b^T x \geq b_0$, with $b_0 \geq 0$. This strong result we hold on in a theorem.

Theorem 2.3 ([6]). *Let $b^T x \geq b_0$ with $b_0 \geq 0$ be a facet defining inequality for $P_C(D_n)$. Then it is equivalent to a nonnegativity constraint* (2.6) *or to some linear ordering constraint* (2.7).

□

Note that no such characterization is known for the circuit polytope of an undirected graph. Several classes of such facets for the latter polytope are identified (see Bauer [7]), but it seems at this point in time to be out of reach to give an exhaustive characterization.

Adding to (2.3) the analogous inequality

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ji} \geq 1$$

yields exactly the inequality $x(A) \geq 2$, i.e., $x(A) \geq 2$ is not facet defining for $P_C(D_n)$. Likewise $x_a \leq 1$ defines not a facet of $P_C(D_n)$. If we take any arc $a = (u, v) \in A$ and sum up the inequalities $x(\delta^+(u)) \leq 1$ and $-x_{uw} \leq 0$ for all $w \in V \setminus \{u, v\}$ we get the inequality $x_a \leq 1$.

There is a direct relationship between the circuit polytope and the *asymmetric traveling salesman polytope* $P_C^n(D_n)$. The latter is namely a face of the former, since it is the restriction of $P_C(D_n)$ to the hyperplane defined by

$$x(A) = n.$$

The circuit polytope $P_C(D_n)$ is also closely related to the *circuit-and-loops polytope* $P_0(D_n^0)$ which is defined to be the convex hull (of incidence vectors) of spanning disjoint unions of a circuit and loops on a complete digraph on n nodes with loops, $P_0(D_n^0) := (V, A \cup \mathcal{L})$, where $\mathcal{L} := \{(1,1), (2,2), \ldots, (n,n)\}$ is the set of loops. It can be interpreted as a restriction of the *assignment polytope* which is the convex hull of spanning unions of disjoint circuits. An IP-formulation of $P_0(D_n^0)$ is given in the next theorem.

Theorem 2.4 ([6]). *The circuit-and-loops polytope $P_0(D_n^0)$ of D_n^0 is the convex hull of points $\begin{pmatrix} x \\ y \end{pmatrix}$, $x \in \{0,1\}^A$, $y \in \{0,1\}^{\mathcal{L}}$, satisfying*

$$x(\delta^+(v)) + y_v = 1 \qquad \forall\ v \in V \tag{2.8}$$

$$x(\delta^-(v)) + y_v = 1 \qquad \forall\ v \in V \tag{2.9}$$

$$x(A) \geq 2 \tag{2.10}$$

$$x(A(S)) + \sum_{v \in S \setminus \{p\}} y_v - y_q \leq |S| - 1 \qquad \forall\ S \subset V, 2 \leq |S| \leq n-2, \quad p \in S, q \in V \setminus S. \tag{2.11}$$

□

In [6] the authors showed that $P_C(D_n)$ is the projection into the x-space of $P_0(D_n^0)$. Moreover, the facets of $P_C(D_n)$ are in 1-1 correspondence with those of $P_0(D_n^0)$.

Theorem 2.5 ([6]). *Let $a^T x + b^T y \leq a_0$ be a facet defining inequality for the circuit-and-loops polytope $P_0(D_n^0)$. Then, the inequality*

$$\sum_{u \in V} \sum_{v \in V \setminus \{u\}} (a_{uv} - b_u) x_{uv} \leq a_0 - \sum_{v \in V} b_v, \tag{2.12}$$

which is the projection of $a^T x + b^T y \leq a_0$ onto the subspace of x, defines a facet of $P_C(D_n)$.

□

Note that this in general is not the case. When a polyhedron Π is projected onto a subspace, facets of Π do not necessarily project into facets of the subspace-polyhedron. There is a further interesting point of view. As is easily seen, the facets induced by the inequalities (2.11) projects into facets defined by the dce-constraints (2.4), but the inequalities (2.11) corresponds obviously to the well known *subtour elimination constraints*

$$x(A(S)) \leq |S| - 1$$

for the asymmetric traveling salesman polytope $P_C^n(D_n)$.

Lifting Procedures

Balas and Fischetti [5] presented a procedure, which is called *cloning*, to lift a facet inducing inequality $a^T x \leq a_0$ for $P_C^n(D_n)$ into a facet defining inequality for $P_C^{n+1}(D_{n+1})$ and was extended in [4] to a lifting procedure for the circuit-and-loops polytope. Here, we will only sketch the procedure.

Two nodes $p, q \in V$, $p \neq q$, are called *clones* with respect to a valid inequality $b^T x \leq b_0$ for the asymmetric traveling salesman polytope $P_C^n(D_n)$ if

(a) $b_{vp} = b_{vq}$ and $b_{pv} = b_{qv}$ for all $v \in V \setminus \{p, q\}$,

(b) $b_{pq} = b_{qp} = \max\{b_{uq} = b_{qv} - b_{uv} \mid u, v \in V \setminus \{p, q\}\}$,

(c) the restriction $\tilde{b}^T x \leq \tilde{b}_0$ of $b^T x \leq b_0$ to $\tilde{D} := D_n \setminus \{p\}$, with $\tilde{b}_0 := b_0 - b_{pq}$, is valid for $P_C^{n-1}(\tilde{D})$.

We say a valid inequality $b^T x \leq b_0$ for $P_C^n(D_n)$ is *primitive* if D_n has no clones with respect to $b^T x \leq b_0$. Notice, Balas and Fischetti [5] showed that if all the primitive members of a family $\mathcal{F}$ of inequalities define regular facets of the asymmetric traveling salesman polytope, then so do all the members of $\mathcal{F}$. (Here, a facet is regular unless it is trivial or defined by $x_{ij} + x_{ji} \leq 1$.)

In the cloning procedure one obtains from a facet defining inequality $a^T x \leq a_0$ for $P_C^n(D_n)$, $D_n = (V, A)$, a facet defining inequality

$$b^T x + \sum_{i=1}^{n} b_{i,n+1} x_{i,n+1} + \sum_{i=1}^{n} b_{n+1,i} x_{n+1,i}$$

for $P_C^{n+1}(D_{n+1})$ by setting

$$b_{uv} = a_{uv} \qquad \forall\, (u, v) \in A,$$

and making the new node $n+1$ a clone of some node $p \in V$. Applying cloning repeatedly leads to cliques of clones.

A modified version of the cloning procedure can be applied also to $P_0(D_n^0)$. Given a facet defining inequality

$$\sum_{i,j=1}^{n} b_{ij} x_{ij} \leq b_0$$

for $P_0(D_n^0)$, a node $p \in V$, and a new node $n+1$, the lifting coefficients $b_{v,n+1}, b_{n+1,v}$, $v = 1, \ldots, n$, are determined in conformity with conditions (a), (b), and (c), but, in addition,

(d) $b_{n+1,n+1} := \max\{b_{u,n+1} + b_{n+1,v} - b_{uv} \mid u, v \in V \setminus \{p, n+1\}\}$.

Note that the coefficient $b_{n+1,n+1}$ is not necessarily the same as the coefficient b_{pp}. Hence, node $n+1$ is called a *quasi-clone* of p.

Using the fact that the facets of $P_0(D_n^0)$ and $P_C(D_n)$ are in the above 1-1 correspondence, the cloning procedure can be embedded in a lifting procedure by starting with a facet of $P_C^n(D_n)$ and ending with a facet of $P_C(D_n)$.

Procedure 2.6 ([6]).

INPUT: *A facet defining inequality* $a^Tx \leq a_0$ *for* $P_C^n(D_n)$.

OUTPUT: *A facet defining inequality for* $P_C(D_n)$.

1. *Find the primitive inequality* $b^Tx' \leq b_0$ *for* $P_C^k(D)$ *defined on some induced subgraph* D *of* D_n $(k := |D|)$*, from which* $a^Tx \leq a_0$ *can be derived by cloning.*
2. *Derive from* $b^Tx' \leq b_0$ *the primitive facet inducing inequality* $b^Tx' + c^Ty' \leq b_0$ *for* $P_C^k(D^0)$ *defined on the digraph* D^0 *with loops.*
3. *Apply the cloning procedure to* $b^Tx' + c^Ty' \leq b_0$ *to obtain a facet inducing inequality* $a^Tx + \tilde{c}^Ty \leq a_0$ *for* $P_0(D_n^0)$.
4. *Substitute for* y *to obtain the corresponding facet inducing inequality* $d^Tx \leq d_0$ *for* $P_C^n(D_n)$.

Unfortunately the procedure has been not carried out for a complete family of inequalities for the asymmetric traveling salesman polytope but only for the primitive members of some classes, since step 3 seems to be a hard task (not for a member of a family but for the whole family).

First we lift the primitive members of the *clique tree inequalities* to the circuit polytope. A *clique tree* CLIQ of D_n is a strongly connected subgraph of D_n composed of two collections of cliques H_i, $i = 1, \ldots, h$ and T_j, $j = 1, \ldots, t$, called *handles* and *teeth*, respectively, satisfying the following conditions:

(i) The teeth are pairwise disjoint.

(ii) The handles are pairwise disjoint.

(iii) For each handle, the number of teeth intersecting it is odd and at least three.

(iv) Every tooth has at least one node not contained in any handle.

(v) If a tooth T and a handle H have a nonempty intersection, then CLIQ$-(H \cap T)$ is disconnected.

The associated *clique tree inequality*

$$\sum_{i=1}^{h} x(A(H_i)) + \sum_{j=1}^{t} x(A(T_j)) \leq \sum_{i=1}^{h} |H_i| + \sum_{j=1}^{t} (|T_j| - t_j) - (t+1)/2$$

where t_j denotes the number of handles intersecting a tooth T_j is known to be facet defining for the asymmetric traveling salesman polytope $P_C^n(D_n)$, $n \geq 7$ [13].

A clique tree CLIQ is said to be *primitive* if it has the additional properties:

(vi) Every tooth has exactly one node not contained in any handle.

(vii) Every nonempty intersection of a handle and a tooth contains exactly one node.

(viii) Every handle has at most one node not contained in any tooth.

(ix) D_n has at most one node not contained in CLIQ.

Balas lifted the primitive clique tree inequalities to the circuit-and-loops polytope $P_0(D_n^0)$ by adding to the left side the sum $\sum_{i=1}^{h} y(H_i)$ of loop variables, i.e.,

$$\sum_{i=1}^{h} x(A(H_i)) + \sum_{j=1}^{t} x(A(T_j)) + \sum_{i=1}^{h} y(H_i) \leq \sum_{i=1}^{h} |H_i| + \sum_{j=1}^{t} (|T_j| - t_j) - (t+1)/2$$

and showed that they are facet defining for $n \geq 7$ [4].

To this result we add that for a primitive clique tree $|T_j| - t_j = 1$, $j = 1, \ldots, t$. Moreover, projecting out the loop variables results in:

Theorem 2.7. *Let $n \geq 7$. Then the primitive clique tree inequality*

$$\sum_{i=1}^{h} x(A(H_i)) + \sum_{j=1}^{t} x(A(T_j)) - \sum_{i=1}^{h} \sum_{v \in H_i} x(\delta^+(v)) \leq (t-1)/2$$

is facet defining for $P_C(D_n)$. □

Next we consider the *SD inequalities* (SD for source-destination) for $P_C^n(D_n)$ introduced by Balas and Fischetti [5], which generalizes the well-known *comb* (see Grötschel [17]) and *odd CAT inequalities* (see Balas [2]). The primitive members of the family are defined on a partition $V = S \cup D \cup W \cup I \cup E \cup Q$ of the nodeset V and induce in generally facets of $P_C^n(D_n)$. The partition has the following properties:

- S is a (possibly empty) set of sources
- D is a (possibly empty) set of destinations
- $H := W \cup I$ is the (nonempty) handle, with $0 \leq |W| \leq 1$ and $|I| = s \geq 1$
- $I := \{u_1, \ldots, u_s\}$
- $E := \{v_1, \ldots, v_s\}$
- $T_j := \{u_j, v_j\}$, $j = 1, \ldots, s$ are the teeth
- $0 \leq |Q| \leq 1$
- $|S| + |D| + s$ is odd.

Then, lifting the primitive SD inequality

$$x((S \cup H : H \cup D)) + \sum_{j=1}^{s} x(A(T_j)) \leq (|S| + |D| + 2|H| + s - 1)/2$$

to the circuit-and-loops polytope $P_0(D_n^0)$ (that can be done by adding to the left side the sum $\sum_{i \in H} y_i$ of loop variables) and projecting out the loop variables yields:

Theorem 2.8 ([6])**.** *Let $n \geq 7$, $|S|+|D|+s \geq 3$, and $||S|-|D|| \leq \max\{0, s-3\}$. Then the primitive SD inequality*

$$x((S : H \cup D)) + \sum_{j=1}^{s} x(A(T_j)) - x((H : V \setminus (D \cup H))) \leq (|S|+|D|+s-1)/2 \quad (2.13)$$

defines a facet of $P_C(D_n)$. □

Finally, we turn to the *lifted cycle inequalities.* Two known members of this family are the D_k^+ and D_k^- inequalities (see Grötschel and Padberg [20]), where k is any integer satisfying $3 \leq k \leq n-2$:

$$\begin{aligned} \sum_{j=1}^{k-1} x_{v_j,v_{j+1}} + x_{v_k,v_1} + 2x((v_1 : \{v_3, \ldots, v_k\})) & \\ + \sum_{j=4}^{k} x((v_j : \{v_3, \ldots, v_{j-1}\})) & \leq k-1 \end{aligned}$$

and

$$\begin{aligned} \sum_{j=1}^{k-1} x_{v_j,v_{j+1}} + x_{v_k,v_1} + 2x((\{v_2, \ldots, v_{k-1}\} : v_1)) & \\ + \sum_{j=3}^{k-1} x((v_j : \{v_2, \ldots, v_{j-1}\})) & \leq k-1 \end{aligned}$$

These inequalities are strictly speaking not primitive, but their only clones are isolated nodes. Applying the lifting procedure to D_k^+ inequalities yields:

Theorem 2.9 ([6])**.** *For any $k \in \{3, \ldots, n-2\}$ and any $l \in \{k+1, \ldots, n\}$, the D_k^+ inequality*

$$\begin{aligned} & x_{v_2v_3} + x_{v_3v_4} + x_{v_kv_1} \\ + \; & x((v_1 : \{v_3, \ldots, v_k\})) + x(\delta^+(v_l)) - x((v_1 : V \setminus \{v_2, \ldots, v_k\})) \\ - \; & \sum_{j=4}^{k} x((v_j : V \setminus \{v_3, \ldots, v_{j-1}, v_{j+1}\})) \\ - \; & x((v_1 : V \setminus \{v_2, \ldots, v_{k-1}\})) - x(\delta^+(v_3)) \leq 1 \end{aligned} \quad (2.14)$$

defines a facet of $P_C(D_n)$. □

An analogous result can be derived for the D_k^- inequalities [6].

Balas and Oosten [6] gave also a procedure whereby facet defining inequalities with respect to $P_C(D_n)$ defined on the complete digraph $D_n = (V, A)$ can be lifted into facet defining inequalities for $P_C(D_{n+1})$ defined on the complete digraph D_{n+1} on $n+1$ nodes. It based on the fact that $P_C(D_n)$ is a face of $P_C(D_{n+1})$:

$$P_C(D_n) = \{x \in P_C(D_{n+1}) \mid x_{v,n+1} = x_{n+1,v} = 0 \; \forall \; v \in V\}.$$

Let $d^T x \leq d_0$ be a valid inequality that defines a nontrivial facet F_d of $P_C(D_n)$. For our purposes here, a lifting of $d^T x \leq d_0$ is an inequality

$$d^T x + \sum_{i=1}^{n} b_i x_{i,n+1} + \sum_{j=1}^{n} c_j x_{n+1,j} \leq d_0, \tag{2.15}$$

which is valid. The polyhedron

$$\text{LIFT}_d := \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{R}^{2n} \mid (2.15) \text{ is valid} \right\}$$

is called the *lifting set* of $d^T x \leq d_0$. LIFT_d is fulldimensional, and it can be shown that

$$\text{LIFT}_d = \left\{ \begin{pmatrix} b \\ c \end{pmatrix} \in \mathbb{R}^{2n} \;\middle|\; \begin{array}{ccccccc} b_i & + & c_j & \leq & a_{ij} & \forall & (i,j) \in A \\ b_i & + & c_i & \leq & a_0 & \forall & i \in \mathbb{N} \end{array} \right\} .$$

Balas and Oosten proved that a lifting (2.15) defines a facet of $P_C(D_{n+1})$ if and only if (b^T, c^T) lies on a one-dimensional face of LIFT_d. This characterization leads to a sequential lifting procedure for generating all the liftings (2.15) of $d^T x \leq d_0$ in $\mathcal{O}(n^2)$ time for each inequality.

Procedure 2.10 ([6]).

INPUT: *$p, q \in \{1, 2, \ldots, n\}$, a valid and facet defining inequality for $P_C(D_n)$ $d^T x \leq d_0$, a permutation π of $(b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n)$, with $\pi_1 = c_p$ and $\pi_2 = b_q$.*

OUTPUT: *A facet defining lifting (2.15) for $P_C(D_{n+1})$.*

1. *Set $c_p := 0$.*
2. *FOR $k = 2, 3, \ldots, 2n$ DO:*
   (a) *IF $\pi_k = b_i$, THEN*

   $$b_i = \min_j \{d_{ij} - c_j \mid c_j = \pi_l \text{ for some } l < k\};$$

   (b) *IF $\pi_k = c_j$, THEN*

   $$c_j = \min_i \{d_{ij} - b_i \mid b_i = \pi_l \text{ for some } l < k\},$$

   where $d_{ii} = d_{jj} = a_0$.

□

We have $2n$ variables $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$, but the rank of the system defining LIFT_d is $2n - 1$. So we can fix the value of one variable, say c_p, to zero. The sequence produces a solution satisfying $2n - 1$ linearly independent inequalities at equality. Thus (b^T, c^T) lies on a one-dimensional face of LIFT_d, and hence, (2.15) defines a facet of $P_C(D_{n+1})$. Moreover, all liftings of the form (2.15) are obtainable this way by choice of an appropriate permutation.

2.2 Related circuit polyhedra

We will study now some extensions of the circuit polytope $P_C(D)$, namely the circuit cone, the upper circuit polyhedron and the dominant. They are of interest, since some instances of the CP correspond to the problem to minimize an objective function over one of these polyhedra.

2.2.1 The dominant of the circuit polytope

Let $D = (V, A)$ be a digraph. The *dominant* $\mathrm{dmt}(P_C(D))$ of the circuit polytope $P_C(D)$ is defined by $\mathrm{dmt}(P_C(D)) := P_C(D) + \mathbb{R}^A_+$. Clearly, the dominant of the circuit polytope $P_C(D) \subseteq \mathbb{R}^A_+$ is a fulldimensional polyhedron.

If we have a linear objective function $c : A \to \mathbb{R}$ with nonnegative coefficients $c_a \geq 0$, $a \in A$, then the CP is equivalent to minimizing $c^T x$ over its dominant $\mathrm{dmt}(P_C(D))$. Since the minimizing problem

$$\min_{C \in \mathcal{C}(D)} \sum_{a \in C} c_a$$

is solvable in polynomial time, we have some hope to find a complete and tractable linear description of $\mathrm{dmt}(P_C(D))$ and this all the more as the "lower" side of the circuit polytope $P_C(D)$ is determined by the nonnegativity constraints (2.6) and the linear ordering constraints (2.7). However, up to now, no such description is known. Clearly, it is not hard to see that the nonnegativity constraints and the linear ordering constraints define facets of $\mathrm{dmt}(P_C(D_n))$, but we will later show that these both classes of inequalities do not determine the $\mathrm{dmt}(P_C(D_n))$. This negative result is at first view astonishing. However, if (2.6) and (2.7) would determine $\mathrm{dmt}(P_C(D_n))$, one could conclude P=NP.

To see this we consider the linear ordering problem (LOP) and apply some results of the theory of blocking polyhedra. We suggest Grötschel, Jünger, and Reinelt [18] as a reference for the LOP and Borndörfer [9] as well as Schrijver [25] as a reference for the blocking theory.

A polyhedron P in the Euclidean space $\mathbb{R}^m$ is of *blocking type* if $P \subseteq \mathbb{R}^m_+$ and if $y \geq x \in P$ implies $y \in P$. It follows directly that a polyhedron P in $\mathbb{R}^m$ is of blocking type if and only if there are vectors $c_1, \ldots, c_p$ in $\mathbb{R}^m$ such that $P = \mathrm{conv}\{c_1, \ldots, c_p\} + \mathbb{R}^m_+$. For any polyhedron P in $\mathbb{R}^m$, the blocking polyhedron $\mathrm{bl}(P)$ is defined by

$$\mathrm{bl}(P) := \{x \in \mathbb{R}^m_+ \mid y^T x \geq 1 \; \forall \, y \in P\},$$

and blocking theory says that

$$\mathrm{bl}(P) := \{x \in \mathbb{R}^m_+ \mid x^T c_i \geq 1 \text{ for } i = 1, \ldots, p\}$$

and $\mathrm{bl}(\mathrm{bl}(P)) = P$ if P is of blocking type, i.e., $P = \mathrm{conv}\{c_1, \ldots, c_p\} + \mathbb{R}^m_+$ (see Schrijver [25]).

We bend now the bow to the linear ordering problem. Let $D_n = (V, A)$ be the complete digraph on n nodes. A tournament is a subset of A containing for each pair of nodes $i, j \in V$ either the arc (i, j) or the arc (j, i), but no circuit. The *linear ordering problem* (LOP) is to find a tournament of maximum weight. The LOP is known to be NP-hard. The associated *linear ordering polytope* $P_{LO}(D_n)$ is defined as the convex hull of the incidence vectors of tournaments in D_n. By definition, each tournament corresponds to exactly one linear ordering constraint (2.7), since the arcset of the support graph of a linear ordering constraint is a tournament. Since the cardinality of each tournament is equal to $|A|/2$, one can assume w.l.o.g. that the objective function is nonnegative, that is, we can assume that we optimize over the dominant $\text{dmt}(P_{LO}(D_n))$ of the linear ordering polytope.

The blocking polyhedron $\text{bl}(\text{dmt}(P_{LO}(D_n)))$ is determined by the nonnegativity constraints (2.6) and the linear ordering constraints (2.7), i.e.,

$$\text{bl}(\text{dmt}(P_{LO}(D_n))) = \{x \in \mathbb{R}^A \mid x \text{ satisfies (2.6) and (2.7)}\}.$$

Assuming that $\text{dmt}(P_C(D_n))$ is determined by (2.6) and (2.7) would imply that

$$\text{bl}(\text{dmt}(P_{LO}(D_n))) = \text{dmt}(P_C(D_n)).$$

Since $\text{dmt}(P_{LO}(D_n))$ is of blocking type, it follows that

$$\text{dmt}(P_{LO}(D_n)) = \text{bl}(\text{bl}(\text{dmt}(P_{LO}(D_n)))) = \text{bl}(\text{dmt}(P_C(D_n))).$$

Also $\text{bl}(\text{dmt}(P_C(D_n)))$ is of blocking type, and hence

$$\text{dmt}(P_{LO}(D_n)) = \{z \in \mathbb{R}^A_+ \mid z^T v \geq 1 \text{ for all vertices } v \text{ of } P_C(D_n)\},$$

i.e.,

$$\text{dmt}(P_{LO}(D_n)) = \{z \in \mathbb{R}^A_+ \mid z^T \chi^C \geq 1 \, \forall \, C \in \mathcal{C}(D_n)\}.$$

The optimization problem over $\text{dmt}(P_{LO}(D_n))$ is NP-hard, the circuit inequalities $z^T\chi^C \geq 1$ can be separated in polynomial time, but optimization and separation over a polyhedron are polynomially equivalent (see Grötschel, Lovász, and Schrijver[19]), i.e., $\text{dmt}(P_C(D_n)) = \{x \mid x \text{ satisfies } (2.6), (2.7)\}$ would imply P=NP.

To prove that the dominant $\text{dmt}(P_C(D_n))$ is not determined by the nonnegativity constraints and the linear ordering constraints consider the case $n = 6$. We examine the point

$$x^M := \frac{1}{2}[e_{21} + e_{14} + e_{43} + e_{32} + e_{36} + e_{65} + e_{54} + e_{52} + e_{16}],$$

whose support graph G is illustrated in Figure 2.1. The support graph of $2x^M$ is known as *Möbius ladder*. Clearly, x^M satisfies the nonnegativity constraints, but it is not hard to see that it also satisfies the linear ordering constraints: Suppose that there is any order π of the nodeset $V = \{1, \ldots, 6\}$ such that the associated linear ordering constraint is violated. Then follows immediately that $\pi(3) < \pi(2)$ or $\pi(5) < \pi(2)$, since $x^M_{23} + x^M_{25} = 1$.

First suppose that $\pi(3) < \pi(2) < \pi(5)$. Then follows further $\pi(4)|\pi(3)$ and $\pi(1) < \pi(4)$, but then $\pi(1) < \pi(2) < \pi(5)$, and x^M would satisfy the linear ordering constraint associated to π. Analogous is the case $\pi(5) < \pi(2) < \pi(3)$, because then follows immediately $\pi(1) < \pi(6) < \pi(5) < \pi(2) < \pi(3)$. Hence, $\pi(3) < \pi(2)$ and $\pi(5) < \pi(2)$ and analogous $\pi(1) < \pi(4)$, $\pi(5) < \pi(4)$, and $\pi(1) < \pi(6)$, $\pi(3) < \pi(6)$. But then for any order of the nodes 2, 4, and 6, the associated linear ordering constraint is satisfied by x^M. Let for example $\pi(2) < \pi(4) < \pi(6)$. Then $x^M_{\pi(5),\pi(6)} + x^M_{\pi(1),\pi(2)} = 1$. The other five possibilities yield similar results by reasons of symmetry.

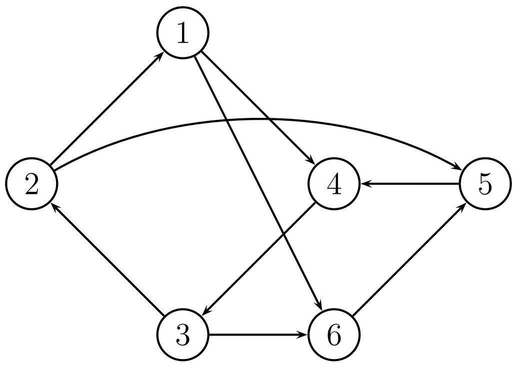
$\frac{1}{2}\cdot$ 

Figure 2.1 Möbius ladder

Now we show that x^M is not in the dominant $\mathrm{dmt}(P_C(D_n))$. Suppose, for the sake of contradiction, that $x^M \in \mathrm{dmt}(P_C(D_n))$. Then there are $x \in P_C(D_n)$ and $z \geq \mathbf{0}$ with $x^M = x + z$, where x is a convex combination of circuits in G, that is, $x = \sum_{i=1}^m \lambda_i \chi^{C_i}$, $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i \geq 0$, $i = 1, \ldots, m$.

The point x satisfies the flow constraints (2.1) at each point. Thus,

$$\begin{array}{ccccccccccccc}
z_{14} & + & z_{16} & & & & & & & & & \geq & 1/2 \\
z_{14} & & & + & z_{54} & & & & & & & \geq & 1/2 \\
& & & & & & z_{32} & + & z_{36} & & & \geq & 1/2 \\
& & & & & & z_{32} & & & + & z_{52} & \geq & 1/2 \\
& & z_16 & & & & & + & z_{36} & & & \geq & 1/2 \\
& & & & z_{54} & & & & & + & z_{52} & \geq & 1/2,
\end{array}$$

$$\begin{array}{rrcl}
\Rightarrow & 2 \cdot \mathbf{1}^T z & \geq & 3 \\
\Rightarrow & \mathbf{1}^T z & \geq & \frac{3}{2} \\
\Rightarrow & \mathbf{1}^T x & = & \mathbf{1}^T x^M - \mathbf{1}^T z \\
& & = & \frac{9}{2} - \mathbf{1}^T z \\
& & \leq & \frac{9}{2} - \frac{3}{2} = 3.
\end{array}$$

But each circuit in G is of length at least four, and thus $\mathbf{1}^T x \geq 4$. Contradiction.

Theorem 2.11. *Let $D_n = (V, A)$, $n \geq 2$, be the complete digraph on n nodes. Then the nonnegativity constraints* (2.6) *and the linear ordering constraints* (2.7) *define facets of* $\mathrm{dmt}(P_C(D_n))$.

Proof. The inequalities(2.6) and (2.7) are valid, since they are valid for $P_C(D_n)$ and since all coefficients of them are nonnegative.

In order to show that $x_a \geq 0$, $a \in A$, induces a facet F of $\mathrm{dmt}(P_C(D_n))$, consider the incidence vector x^* of a 2-circuit not containing the arc a and the points $y^i := x^* + e_i$, $i = 1, \ldots, |A|$, $i \neq a$, where e_i is the i-th unit vector in $\mathbb{R}^A$. All given points are in F, and the vectors y^i, $i = 1, \ldots, |A|$, $i \neq a$, are linearly independent. Next we show that x^* is affinely independent of the vectors y^i, $i = 1, \ldots, |A|$, $i \neq a$. Suppose, for the sake of contradiction, that

$$x^* = \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i y^i, \tag{2.16}$$

$$\text{with } \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i = 1 \tag{2.17}$$

Then, it follows from (2.16) that

$$\begin{aligned} x^* &= \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i x^* + \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i e_i \\ &= x^* + \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i e_i \\ \Rightarrow &\quad \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i e_i = 0 \\ \Rightarrow &\quad \lambda_i = 0, i = 1, \ldots, |A|, i \neq a \\ \Rightarrow &\quad \sum_{\substack{i=1\\i\neq a}}^{|A|} \lambda_i = 0. \text{ Contradiction to (2.17)!} \end{aligned}$$

Hence, the nonnegativity constraint $x_a \geq 0$ defines a facet of $\text{dmt}(P_C(D_n))$.

To prove the facet defining property of (2.3) is easy. We assume w.l.o.g. that $(\pi(1), \ldots, \pi(n)) = (1, \ldots, n)$, that is, (2.7) is (2.3). The $\frac{|A|}{2}$ incidence vectors x^{ij} of the 2-circuits $\{(i,j),(j,i)\}$, $1 \leq i < j \leq n$, are linearly independent and satisfy (2.3) at equality. The $\frac{|A|}{2}$ points $y^{ij} := x^{ij} + e_{ji}$, $1 \leq i < j \leq n$, where e_{ji} is the ji-th unit vector, are also linearly independent and satisfy (2.3) at equality. Since they are also linearly independent of the former points, we constructed $|A|$ linearly independent points satisfying (2.3) at equality. □

It is possible to make some statements about the coefficients of a facet defining inequality $d^T x \geq d_0$ with respect to $\text{dmt}(P_C(D_n))$.

Theorem 2.12. *Let*

$$d^T x \geq d_0 \tag{2.18}$$

be a facet defining inequality for $\text{dmt}(P_C(D_n))$. *Then* $d_a \geq 0$ *for all* $a \in A$ *and* $d_0 \geq 0$. *Moreover, the following statements are valid:*

(a) If $d_0 = 0$, *then* (2.18) *is equivalent to a nonnegativity constraint.*

(b) If $d_0 > 0$, *then* $d_{ij} + d_{ji} = d_0$ *for all* $i, j \in V$, $i \neq j$. *Moreover, there are* $n-1$ *pairs of nodes* i, j *with* $d_{ij} = 0$ *and* $d_{ji} = d_0$.

Proof. First we show that $d_a \geq 0$ for all $a \in A$ and $d_0 \geq 0$. Since the incidence vectors of all 2-circuits are in $\text{dmt}(P_C(D_n))$, it follows that $d_{ij} + d_{ji} \geq d_0$. Suppose, for the sake of contradiction, that there is $(i,j) \in A$ with $d_{ij} < 0$. Then the point $x^* := \chi^{(i,j)(j,i)} + \lambda e_{ij} \in \text{dmt}(P_C(D_n))$, where e_{ij} is the ij-th unit vector and $\lambda = 1 + \frac{d_0 - d_{ij} - d_{ji}}{d_{ij}}$, violates $d^T x \geq d_0$, since $d_{ij} + d_{ji} \geq d_0$. Further, $d^T x \geq d_0$ defines a facet, and thus $d_0 \geq 0$.

(a): Clearly, there is at least one coefficient d_a, $a \in A$, with $d_a > 0$, and, as is easily seen, this implies $\{x \in \text{dmt}(P_C(D_n)) \mid d^T x = 0\} \subseteq \{x \in \text{dmt}(P_C(D_n)) \mid x_a = 0\}$. Since (2.18) defines a facet, we get even $\{x \in \text{dmt}(P_C(D_n)) \mid d^T x = d_0\} = \{x \in \text{dmt}(P_C(D_n)) \mid x_a = 0\}$, i.e., (2.18) is equivalent to the nonnegativity constraint $x_a \geq 0$, $a \in A$.

(b): Recall that

(i) $d_{ij} + d_{ji} \geq d_0$ for all $i, j \in V$, $i \neq j$.

Further, the inequality $d^T x \geq d_0$ is obviously not equivalent to a nonnegativity constraint, as $d_0 > 0$ and $\text{dmt}(P_C(D_n))$ is full-dimensional, and hence

(ii) $d_{ij} \leq d_0$ for all $(i,j) \in A$.

Moreover, for each arc (i,j) there exists $x' \in \text{dmt}(P_C(D_n))$, with $x'_{ij} > 0$, satisfying $d^T x' = d_0$. Let now $i, j \in V$, $i \neq j$.

Case 1: $d_{ij} = 0$ or $d_{ji} = 0$

Let w.l.o.g. $d_{ij} = 0$. Then (i) and (ii) imply directly $d_{ji} = d_0$, and thus we obtain $d_{ij} + d_{ji} = d_0$.

Case 2: $d_{ij} > 0$ and $d_{ji} > 0$

First we show that there are circuits $C^{ij}, C^{ji} \in \mathcal{C}(D_n)$, with $(i,j) \in C^{ij}$ and $(j,i) \in C^{ji}$, whose incidence vectors $y^{ij} := \chi^{C^{ij}}$ and $y^{ij} := \chi^{C^{ij}}$ satisfy (2.18) at equality, i.e.,

(iii) $d^T y^{ij} = d_0$ and $d^T y^{ji} = d_0$.

To see this, suppose, for the sake of contradiction, that $d^T \chi^C = d_0$ implies $(i,j) \notin C$ for all $C \in \mathcal{C}(D_n)$. Now let $y' + z' \in \text{dmt}(P_C(D_n))$, with $(y' + z')_{ij > 0}$, satisfying $d^T(y' + z') = d_0$. This implies, in particular, $d^T y' = d_0$, and thus, by the assumption, $y'_{ij} = 0$, since $y' \in P_C(D_n)$. So we conclude $z'_{ij} > 0$, which contradicts $d^T(y' + z') = d_0$, since $d_{ij} > 0$. Thus there is such a circuit C^{ij}. Analogous follows that there is a circuit C^{ji} satisfying the above conditions.

It remains to be shown that $d_{ij} + d_{ji} = d_0$. If $C^{ij} = C^{ji}$, then follows immediately that $C^{ij} = C^{ji} = \{(i,j), (j,i)\}$, and hence,

$$d_{ij} + d_{ji} = d^T y^{ij} \overset{(iii)}{=} d_0.$$

Finally, let $C^{ij} \neq C^{ji}$. Then it follows immediately that $C^{ij} \neq \{(i,j), (j,i)\} \neq C^{ji}$. Now we can continue as Balas and Oosten in [6]. Let us denote by P_1 the path in C^{ij} running from j to i, and by P_2 the path in C^{ji} running from i to j. The union of P_1 and

P_2 can be interpreted as the union of (possibly overlapping) circuits, say $K_1, \dots, K_s$, and thus

$$\begin{array}{rcl} 2d_0 & = & d^T(y^{ij} + y^{ji}) \\ & = & (d_{ij} + d_{ji}) + [d^T y^{ij} + d^T y^{ji} - (d_{ij} + d_{ji})] \\ & = & (d_{ij} + d_{ji}) + \sum\limits_{(u,v)\in P_1} d_{uv} + \sum\limits_{(u,v)\in P_2} d_{uv} \\ & = & (d_{ij} + d_{ji}) + \sum\limits_{r=1}^{s} \underbrace{\sum\limits_{(u,v)\in K_r} d_{uv}}_{\geq d_0} \\ & \geq & (d_{ij} + d_{ji}) + d_0 \\ & \overset{(i)}{\geq} & 2d_0, \end{array}$$

that is, all terms are equal. In particular,

$$\begin{array}{rrcl} & 2d_0 & = & (d_{ij} + d_{ji}) + d_0 \\ \Leftrightarrow & d_0 & = & d_{ij} + d_{ji}. \end{array}$$

Now $d^T x \geq d_0$, with $d_0 > 0$, is facet defining for $\mathrm{dmt}(P_C(D_n))$, but there are at most $n^2 - 2n + 2$ affinely independent circuits satisfying (2.18) at equality. In fact, it is not hard to see that there are at most $n^2 - 2n + 1$ affinely independent circuits satisfying (2.18) at equality. Otherwise there would be a equation with right side greater than zero which is valid for all $x \in P_C(D)$. Thus there are at least $n - 1$ arcs $(i.j)$ with $d_{ij} = 0$, and this implies that there are at least $n - 1$ pairs of nodes i, j with $d_{ij} = 0$ and $d_{ji} = d_0$. ☐

2.2.2 The circuit cone

For a digraph D, the *circuit cone* $C_C(D)$ is the cone generated by the incidence vectors of all (directed) circuits of D. Clearly, the dimension of the circuit cone $C_C(D)$ is equal to that of $P_C(D)$. A complete linear description of the circuit cone $C_C(G)$ of an undirected graph $G = (V, E)$ was given by Seymour [27] by so called cut inequalities (see 3.6) and the nonnegativity constraints $x_e \geq 0$, $e \in E$. We will show that the circuit cone of a directed graph is determined by the degree equalities and the nonnegativity constraints.

Theorem 2.13. *Let $D = (V, A)$ be a digraph. A complete linear description of the circuit cone $C_C(D)$ is given by*

$$\begin{array}{llcll} (i) & x(\delta^+(v)) - x(\delta^-(v)) & = & 0 & \forall\, v \in V, \\ (ii) & x_a & \geq & 0 & \forall\, a \in A. \end{array}$$

Proof. We have to show $C_C(D) = \{x \in \mathbb{R}^A \mid x \text{ satisfies } (i) \text{ and } (ii)\}$.

"$\subseteq$": Trivial.

"$\supseteq$": Let $x \in \mathbb{R}^A$ satisfying (i) and (ii). If $x = \mathbf{0}$ then $x \in C_C(D)$. Otherwise set $x^0 := x$ and generate a sequence of points x^i by

$$x^{i+1} := x^i - \lambda_i \chi_i^C,$$

while $\text{supp}(x^i) \neq \emptyset$, whereat $\lambda_i := \min\{x^i_a \mid a \in \text{supp}(x^i)\}$ and C_i a circuit on the support of x^i, i.e., $C \subseteq \text{supp}(x^i)$. First, observe that for each point x^i, with $\text{supp}(x^i) \neq \emptyset$, such a circuit exist, since with $x^{i-1}(\delta^+(v)) - x^{i-1}(\delta^-(v)) = 0$ follows $x^i(\delta^+(v)) - x^i(\delta^-(v)) = 0$. Secondly, note that each λ_i is greater than zero. Finally, the sequence is finite, because $|\text{supp}(x^{i+1})| < |\text{supp}(x^i)|$, and thus the last point, say x^k, is the zero vector. Hence,

$$\sum_{i=0}^{k-1} \lambda_i \chi_i^C = \sum_{i=0}^{k-1} x^i - x^{i+1} = x^0 - x^k = x,$$

i.e., $x \in C_C(D)$.

□

Given the optimization problem

$$\min d^T x, x \in C_C(D), \tag{2.19}$$

the minimum is equal to zero if $d^T \chi^C \geq 0$ for all $C \in \mathcal{C}(D)$ and unbounded below if there is a circuit with negative cost, i.e., the optimization problem (2.19) corresponds to the decision problem whether there exists a circuit with negative cost or not.

2.2.3 The upper circuit polyhedron

The *upper circuit polyhedron* $U_C(D)$ of a digraph D is the sum of the circuit polytope $P_C(D)$ and the circuit cone $C_C(D)$, i.e., $U_C(D) := P_C(D) + C_C(D)$. Given a linear objective function d with the property $d^T \chi^C \geq 0$ for all $C \in \mathcal{C}(D)$, the CP is equivalent to minimizing $d^T x$ over the vertices of $U_C(D)$.

Between the dominant $\text{dmt}(P_C(D))$ and the upper circuit polyhedron $U_C(D)$ we have the following connection:

Theorem 2.14. *Let $D = (V, A)$ be a digraph on n nodes. Then*

$$U_C(D) = \{x \in \text{dmt}(P_C(D)) \mid x(\delta^+(v)) - x(\delta^-(v)) = 0 \,\forall\, v \in V\}.$$

Proof.
" $\subseteq$ " : Let $x \in U_C(D)$. Then there are $y \in P_C(D)$ and $z \in C_C(D)$ with $x = y + z$. The flow constraints (2.1) are satisfied by y and z, and thus also by x. Moreover, $z \geq \mathbf{0}$. Hence, $x \in \text{dmt}(P_C(D))$, and x satisfies the flow constraints.

" $\supseteq$ " : Let $x \in \text{dmt}(P_C(D))$ satisfying the flow constraints (2.1), and $x = y + z$, $y \in P_C(D)$, $z \geq \mathbf{0}$. Since x and y satisfy the flow constraints (2.1), also z satisfies the flow constraints (2.1). Thus $x \in U_C(D)$. □

That is, in particular, if we have a linear description of $\text{dmt}(P_C(D))$ then we have also a linear characterization of $U_C(D)$.

Chapter 3

Length restricted circuit polytopes $P_C^L(D_n)$

Summary. This is the main part of the thesis. As is usually done by study of facets of polytopes defined on graphs, we restrict ourselves to the complete digraph $D_n = (V, A)$. We determine the dimension of $P_C^L(D_n)$ subject to L and n, give adequate integer programming formulations, so far as possible, for $P_C^L(D_n)$, and classify the inequalities given in the IP-formulation 2.1 in view of validity and study in which cases they are facet defining. Further, we derive facets from the circuit polytopes $P_C(K_n)$ and $P_C^{\leq k}(K_n)$, from the asymmetric traveling salesman polytope $P_C^n(D_n)$, and from specific path polytopes. Moreover, we investigate the relations between $P_C^L(D_n)$ and these polytopes.

3.1 Integer programming formulations for $P_C^L(D_n)$

In 2.1 we presented an integer programming formulation for the circuit polytope $P_C(D)$ of a digraph $D = (V, A)$. It can be extended to an IP-formulation for the polytopes $P_C^{\{k,\dots,l\}}(D)$ trouble-free.

Theorem 3.1 (IP-formulation).
The integer points in $P_C^{\{k,\dots,l\}}(D_n)$, $2 \leq k \leq l \leq n$, are determined by the system

$$
\begin{aligned}
x(\delta^+(v)) - x(\delta^-(v)) &= 0 && \forall\ v \in V && (2.1)\\
x(\delta^+(v)) &\leq 1 && \forall\ v \in V && (2.2)\\
x(\delta^+(p)) + x(\delta^+(q)) - x((S : V \setminus S)) &\leq 1 && \forall\ S \subset V, 2 \leq |S| \leq n-2, && (2.4)\\
& && p \in S, q \in V \setminus S &&\\
x_a &\in \{0,1\} && \forall\ a \in A && (2.5)\\
x(A) &\geq k && && (3.1)\\
x(A) &\leq l. && && (3.2)
\end{aligned}
$$

In particular, the integer points of $P_C^k(D_n)$, $k \in \{2,\dots,n\}$, are characterized by (2.1), (2.2), (2.4), (2.5), *and*

$$x(A) = k. \tag{3.3}$$

□

For $P_C^3(D_n)$ constraint (2.4) is redundant, and for $P_C^2(D_n)$ we give a complete linear description.

Theorem 3.2. *$P_C^2(D_n)$ is determined by the system*

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_{ij} = 1 \tag{3.4}$$

$$x_{ij} - x_{ji} = 0 \qquad \forall\ (i,j) \in A \tag{3.5}$$

$$x_{ij} \geq 0 \qquad \forall\ (i,j) \in A. \tag{3.6}$$

Proof. Clearly, each vector in $P_C^2(D_n)$ satisfies (3.4) - (3.6) and each vector x^* satisfying (3.4) - (3.6) is a convex combination of the incidence vectors of 2-circuits, because

$$\begin{aligned} x^* &\overset{(3.5)}{=} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x^*_{ij}\chi^{\{(i,j),(j,i)\}}, \\ 1 &\overset{(3.4)}{=} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x^*_{ij}, \\ x^*_{ij} &\overset{(3.6)}{\geq} 0. \end{aligned}$$

□

To model general length restricted circuit polytopes we suggest an extended formulation. The background yields the theory of *disjunctive programming* which is optimization over unions of polyhedra. A short review article about disjunctive programming is given in [3]. For our purposes here, we give only the extended formulation.

Theorem 3.3 (Extended formulation)**.**
The integer points in $P_C^L(D_n)$ are those vectors $x \in \{0,1\}^A$ for which there exist vectors $\begin{pmatrix} y^k \\ \lambda_k \end{pmatrix} \in \{0,1\}^{A+1}, k \in L$, satisfying

$$\begin{aligned} x - \sum_{k\in L} \lambda_k y^k &= 0 \\ \sum_{k \in L} \lambda_k &= 1 \\ y^k \text{ satisfies } (2.1), (2.2), (2.4), \text{ and } y^k(A) &= k \quad \forall\, k \in L \end{aligned}$$

□

Of course, if L is given by an union of integer intervals, say $L = \bigcup_{j=1}^m I_j$, where $I_j := \{k_j, \ldots, l_j\}$, then one can give also an extended formulation similar to the above.

There is yet another possibility of an IP-formulation if L can be interpreted as a coset with respect to a number $k \in \{2, \ldots, n\}$. Let us denote by

$$[m] := \{z \in \{2, \ldots, n\} \mid \exists p \ \in \mathbb{N} : z = pk + m\}$$

the (finite) cosets, $m = 0, \ldots, k-1$. For $P_C^{[0]}(D)$ the model is:

Theorem 3.4. *Let $D = (V, A)$ be a digraph with nodeset $V = \{v_1, \ldots, v_n\}$, $k \in \{2, \ldots, n\}$, and L the coset $[0]$ with respect to k. Let $V_0, \ldots, V_{k-1}$ be copies of V and s, t two additional nodes. Define the digraph $D' := (V', A')$ by*

$$\begin{aligned} V' &:= \{s,t\} \cup \left(\bigcup_{j=0}^{k-1} V_j\right), \\ A' &:= (s, V_0) \cup (V_0, t) \cup \left(\bigcup_{j=0}^{k-1} (V_j, V_{j+1})\right) \end{aligned}$$

where $V_k := V_0$ and

$$\begin{aligned} (s, V_0) &:= \{(s, v_{i_0}) \mid i = 1, \ldots, n\}, \\ (V_0, t) &:= \{(v_{i_m}, t) \mid i = 1, \ldots, n\}, \ \text{and} \\ (V_j, V_{j+1}) &:= \{(v_{p_j}, v_{q_{j+1}}) \mid (v_p, v_q) \in A\}, \quad j = 0, \ldots, k-1. \end{aligned}$$

Then y is an integer point of $P_C^{[0]}(D)$ if and only if there is $x \in \{0,1\}^{A'}$ satisfying

$$\sum_{i=1}^{n} x_{s,v_{i_0}} = 1 \tag{3.7}$$

$$x_{v_{i_0},t} - x_{s,v_{i_0}} = 0 \qquad i = 1, \ldots, n \tag{3.8}$$

$$x(\delta^+(v)) - x(\delta^-(v)) = 0 \qquad \forall\, v \in V' \setminus \{s,t\} \tag{3.9}$$

$$\sum_{j=0}^{k-1} x(\delta^+(v_{i_j})) - x_{s,v_{i_0}} \le 1 \qquad i = 1, \ldots, n \tag{3.10}$$

$$\begin{aligned} &\sum_{j=0}^{k-1} x(\delta^+(v_{p_j})) + \sum_{j=0}^{k-1} x(\delta^+(v_{q_j})) \\ &-x_{v_{p_0},t} - x_{v_{q_0},t} - x((S : V' \setminus (S \cup \{s,t\})) \le 1 \qquad \forall\, S \subset V' \setminus \{s,t\}, \\ &\qquad v_{p_j} \in S, v_{q_j} \in V' \setminus (S \cup \{s,t\}), \\ &\qquad v_{i_0} \in S \Leftrightarrow v_{i_j} \in S, \\ &\qquad j = 0, \ldots, k-1 \end{aligned} \tag{3.11}$$

$$x(A') \ge k + 2. \tag{3.12}$$

$$\sum_{l=0}^{k-1} x_{v_{i_l}, v_{j_{l+1}}} - y_{v_i, v_j} = 0 \qquad \forall\, (v_i, v_j) \in A. \tag{3.13}$$

Proof. Necessity. Let y^* be an integer point of $P_C^L(D)$. Then there is a circuit $C = (v^{(1)}, \ldots, v^{(tk)}) \in \mathcal{C}^{[0]}(D)$, $t \in \{1, \ldots, \lfloor \frac{n}{k} \rfloor\}$, with $y^* = \chi^C$. Set $v^{(tk+1)} := v^{(1)}$, and define the coefficients of a vector $x^* \in \mathbb{R}^{A'}$ by

$$x^*_{v_{i-1}^{(sk+i)}, v_i^{(sk+i+1)}} \;=\; 1 \quad s = 0, \ldots, t-1, i = 1, \ldots, k,$$

and otherwise zero. Then x^* is an integer point, which satisfies (3.7)-(3.10), (3.12), and (3.13). Constraint (3.11) is a subclass of the dce constraints and thus satisfied.

Suffiency. Consider first the following system on the digraph $D'' = (V'', A'') := D' \setminus \{s, t\}$:

$$\begin{aligned}
x(\delta^+(v)) - x(\delta^-(v)) &= 0 \quad \forall\, v \in V'' && (3.14)\\
\sum_{j=0}^{k-1} x(\delta^+(v_{i_j})) &\leq 1 \quad i = 1, \ldots, n && (3.15)\\
\sum_{j=0}^{k-1} x(\delta^+(v_{p_j})) + \sum_{j=0}^{k-1} x(\delta^+(v_{q_j})) & \\
-x((S : V'' \setminus S) &\leq 1 \quad \forall\, S \subset V'', && (3.16)\\
& \qquad v_{p_j} \in S, v_{q_j} \in V'' \setminus S, \\
& \qquad v_{i_0} \in S \Leftrightarrow v_{i_j} \in S, \\
& \qquad j = 0, \ldots, k-1 \\
x(A'') &\geq k. && (3.17)\\
\sum_{l=0}^{k-1} x_{v_{i_l}, v_{j_{l+1}}} - y_{v_i, v_j} &= 0 \quad \forall\, (v_i, v_j) \in A. && (3.18)
\end{aligned}$$

Constraints (3.14) and (3.15) are satisfied by the zero vector and the incidence vectors of unions of circuits of length $l \in [0]$ such that at most one copy of each node v_j, $j = 0, \ldots, k-1$, is covered. Constraint (3.17) excludes the zero vector. Constraints (3.16) are satisfied by all circuits of length $l \in [0]$ covering at most one copy of v_p and v_q, respectively, but violated by unions of those circuits with more than one member. Hence, x is the incidence vector of a circuit (in D") of length $l \in [0]$ which covers at most one copy of each node v_j, $j = 0, \ldots, k-1$. By definition of y, it follows that y is the incidence vector of a circuit (in D) of length $l \in [0]$.

The modified system (3.7)-(3.13) ensures that the circuit corresponding to x starts end ends with a node in V_0. □

For the polytopes $P_C^{[m]}(D)$, $m = 1, \ldots, k-1$, we give a (s,t)-path model:

Theorem 3.5. *Let $D = (V, A)$ be a digraph with nodeset $V = \{v_1, \ldots, v_n\}$, $k \in \{2, \ldots, n\}$, and $L := [m]$, $m \in \{1, \ldots, k-1\}$, a coset with respect to k. Let $V_0, \ldots, V_{k-1}$ be copies of*

V, s,t two additional nodes, and define the digraph $D' := (V', A')$ by

$$\begin{aligned} V' &:= \{s,t\} \cup \left(\bigcup_{j=0}^{k-1} V_j \right), \\ A' &:= (s, V_0) \cup (V_m, t) \cup \left(\bigcup_{j=0}^{k-1} (V_j, V_{j+1}) \right) \end{aligned}$$

where $V_k := V_0$ and

$$\begin{aligned} (s, V_0) &:= \{(s, v_{i_0}) \mid i = 1, \dots, n\}, \\ (V_m, t) &:= \{(v_{i_m}, t) \mid i = 1, \dots, n\}, \text{ and} \\ (V_j, V_{j+1}) &:= \{(v_{p_j}, v_{q_{j+1}}) \mid (v_p, v_q) \in A\}, \quad j = 0, \dots, k-1. \end{aligned}$$

Then y is an integer point of $P_C^L(D)$ if and only if there is $x \in \{0,1\}^{A'}$ satisfying

$$\sum_{i=1}^{n} x_{s,v_{i_0}} = 1 \tag{3.19}$$

$$x_{v_{i_m},t} - x_{s,v_{i_0}} = 0 \qquad i = 1, \dots, n \tag{3.20}$$

$$x(\delta^+(v)) - x(\delta^-(v)) = 0 \qquad \forall\, v \in V' \setminus \{s,t\} \tag{3.21}$$

$$\sum_{j=0}^{k-1} x(\delta^+(v_{i_j})) - x_{s,v_{i_0}} \leq 1 \qquad i = 1, \dots, n \tag{3.22}$$

$$x(A'(S)) \leq |S| - 1 \qquad \forall S \subseteq V' \setminus \{s,t\}\ \exists\ p \in \{1, \dots, \lfloor \frac{n}{k} \rfloor\} \text{ with } |S \cap V_j| = p, j = 0, \dots, k-1, \text{ and } i,j = 0, \dots, k-1, i \neq j \tag{3.23}$$

$$\sum_{l=0}^{k-1} x_{v_{i_l}, v_{j_{l+1}}} - y_{v_i, v_j} = 0 \qquad \forall\, (v_i, v_j) \in A. \tag{3.24}$$

Proof.

Necessity. Let y^* be an integer point of $P_C^L(D)$. Then there is a circuit $C = (v^{(1)}, \dots, v^{(qk+m)}) \in \mathcal{C}^{[m]}(D)$, $q \in \{0, \dots, \lfloor \frac{n}{k} \rfloor\}$, with $y^* = \chi^C$. Set $v^{(qk+m+1)} := v^{(1)}$, and define the coefficients of a vector $x^* \in \mathbb{R}^{A'}$ by

$$\begin{aligned} x^*_{s,v_0^{(1)}} &= x^*_{v_m^{(1)},t} = 1, & \\ x^*_{s,v_{i_0}} &= x^*_{v_{i_m},t} = 0 & \forall\, v_i \in V, v_i \neq v^{(1)}, \\ x^*_{v_{i-1}^{(pk+i)}, v_i^{(pk+i+1)}} &= 1 & p = 0, \dots, q-1, i = 1, \dots, k; \\ x^*_{v_{i-1}^{(qk+i)}, v_i^{(qk+i+1)}} &= 1 & i = 1, \dots, m-1; \end{aligned}$$

all other coefficients set to zero. The point x^* is integer and satisfies obviously (3.19)-(3.22) and (3.24). Constraints (3.23) are a choice of the subtour elimination constraints, and hence x^* satisfies also (3.23).

Suffiency. Let $x \in \{0,1\}^{A'}$ satisfying the constraints (3.19)-(3.24). Constraints (3.19),(3.20), and (3.22) ensure that x is the sum of incidence vectors of a (s,t)-path P and circuits $C_1, \dots, C_f$. Clearly, there is exactly one $r \in \{1, \dots, n\}$ with $(s, v_{r_0}), (v_{r_m}, t) \in P$. Constraint (3.21) ensures that for each $i \in \{1, \dots, n\}$ at most one copy of a node v_i is covered by $P \cup_{j=1}^{f} C_j$ excepting v_r; there are exactly two copies which are covered, both by the path P. Constraint (3.23) excludes the circuits $C_1, \dots, C_f$, and hence, y is an incidence vector of a circuit of length $l \in L$ by definition (3.24). □

3.2 Equivalence of inequalities

The polytope $P_C^L(D_n)$ is obviously not full dimensional, since all $x \in P_C^L(D_n)$ satisfy the flow constraints (2.1). Thus, two valid inequalities can define the same face of $P_C^L(D_n)$, although they may be quite different in appearance. So it is very helpful to define a canonical form of valid inequalities for $P_C^L(D_n)$.

A system $Cx = d$ of linear equations whose solution set is the affine hull of a polyhedron P is said to be an *equality subsystem of* P. Two valid inequalities for P are *equivalent* if one can be obtained from the other by multiplication with a positive scalar *and* then adding appropriate multiples of the equality subsystem. For our purpose it is sufficient to know that the flow constraints (2.1) are a subset of the equality subsystem. We will show that a valid inequality can be brought in such a form that the arcs corresponding to coefficients which are equal to zero contain a set $T \subseteq A$ whose induced underlying graph builds a spanning tree.

Definition 3.6. Let $D_n = (V, A)$ be the complete digraph on n nodes and T a subset of A with the properties

(a) $\forall\, u, v \in V$, $u \neq v$: $(u, v) \in T \Rightarrow (v, u) \notin T$,

(b) the underlying graph of $D' := (V(T), T)$ is a spanning tree. In particular, $V(T) = V$.

Then a valid inequality $b^T x \leq b_0$ for $P_C^L(D_n)$ is said to be in *T-rooted form* (or simply T-rooted) if

(i) $b_a = 0$ for all $a \in T$

(ii) the coefficients b_a, b_0 are relatively prime integers.

In particular, for $T = \delta^-(h)$ for some $h \in V$, $b^T x \leq b_0$ is said to be in *h-rooted form* (or simply h-rooted) if it satifies (i) and (ii) (see Balas and Oosten [6]).

Theorem 3.7. *Let $D_n = (V, A)$ be the complete digraph on n nodes and T a subset of A with the in Definition 3.6 required properties. Then, every valid inequality for $P_C^L(D_n)$ can be transformed to a T-rooted inequality.*

Proof. Let $T = \{a_1, \dots, a_{n-1}\}$, and let $b^T x \leq b_0$ be a valid inequality for $P_C(D_n)$. Since T is weakly connected, we can choose an ordering π of T such that the arcsets $T_k := \{a_{\pi(j)} \mid j = 1, \dots, k\}$ are weakly connected for all $k \in \{1, \dots, n-1\}$. Let v_1 be one endnode of $a_{\pi(1)}$ and denote by v_k the node incident with $a_{\pi(k)}$ not belonging to the nodeset of T_{k-1}, $k = 2, \dots, n-1$. (Note that this implies an ordering of the nodeset V.) Now, apply the following algorithm:

1. Set $b^0 := b$ and $b_0^0 := b_0$.
2. For $k = 1, \dots, n-1$ add to $(b^{k-1})^T x \leq b_0^{k-1}$ the equation

$$-b^{k-1}_{a_{pi(k)}}(x(\delta^+(v_k)) - x(\delta^-(v_k))) = 0$$

   if v_k is the tail of $a_{\pi(k)}$, and the equation

$$-b^{k-1}_{a_{\pi(k)}}(x(\delta^-(v_k)) - x(\delta^+(v_k)))$$

   if v_k is the head of $a_{\pi(k)}$, and denote the resulting inequality by $(b^k)^T x \leq b_0^k$.

Then $b_a^{n-1} = 0$ for all $a \in T$, $b_0^{n-1} = b_0$, and

$$\sum_{a \in A} b_a^{n-1} = \sum_{a \in A} = b_a.$$

By multiplying $(b^{n-1})^T x \leq b^0_{n-1}$ by an appropriate real number yields an inequality in T-rooted form. □

An interesting question is whether a face F of $P_C^L(D_n)$ can be defined by a unique T-rooted inequality or not. We will not discuss it here, but in the next section we will see, for example, that $\dim P_C^L(D_n) = \dim P_C(D_n)$ for $|L| \geq 2$. This implies that the equality subsystem of those polytopes consists only the flow constraints. Along the lines of the argumentation of Balas and Oosten in [6] one could show that then for a given tree T, the face F is defined by a unique T-rooted inequality if F is a facet.

3.3 Dimension

The dimension of the circuit polytope $P_C(D_n)$ is equal to $(n-1)^2$ (see Balas and Oosten [6]), while the dimension of $P_C^L(D_n)$ depends on both n and L. First we will study the dimension of the polytopes $P_C^3(D_n)$ and $P_C^{\{2,3\}}(D_n)$ in order to ascertain that the dimension of the polytope $P_C^{\leq k}(D_n)$, $3 \leq k < n$, is equal to $(n-1)^2$. After that we are prepared to specify the dimension of $P_C^k(D_n)$, $3 \leq k < n$. Finally we will investigate the general case, $\dim P_C^L(D_n)$, $L \subseteq \{2, \dots, n\}$. Clearly, $\dim P_C^L(D_n) \leq \dim P_C(D_n)$, since $P_C^L(D_n) \subseteq P_C(D_n)$.

In order to simplify the next proofs we introduce two useful lemmas.

Lemma 3.8. *Let D be a digraph on n nodes and $x_1, x_2, \dots, x_p \in P_C^k(D)$, $p \in \mathbb{N}$, affinely independent points. Then they are even linearly independent.*

Proof. Assume that the points are not linearly independent. Then exist a point, say x_p, and real numbers $\lambda_1, \lambda_2, \dots, \lambda_{p-1}$ such that

$$x_p = \sum_{i=1}^{p-1} \lambda_i x_i.$$

Then it follows:

$$k = \mathbf{1}^T x_p = \sum_{i=1}^{p-1} \lambda_i \underbrace{\mathbf{1}^T x_i}_{=k} = k \sum_{i=1}^{p-1} \lambda_i$$
$$\Rightarrow \quad \sum_{i=1}^{p-1} \lambda_i = 1. \qquad \text{Contradiction!}$$

□

Lemma 3.9. *Let $k, l \in L$, $k \neq l$, $x_1, x_2, \ldots, x_p \in P_C^k(D_n)$, $p \in \mathbb{N}$, and $y \in P_C^l(D_n)$. Then y is not in the affine hull of $\{x_1, x_2, \ldots, x_p\}$.*

Proof. Assume that $y = \sum_{i=1}^{p} \lambda_i x_i$ such that $\sum_{i=1}^{p} \lambda_i = 1$. Knowing $\mathbf{1}^T x_i = k$, $i = 1, 2, \ldots, p$, and $\mathbf{1}^T y = l$ yields the following contradiction:

$$\begin{aligned} y \quad &= \sum_{i=1}^{p} \lambda_i x_i \\ \Rightarrow \quad l = \mathbf{1}^T y \quad &= \sum_{i=1}^{p} \lambda_i \mathbf{1}^T x_i = k \sum_{i=1}^{p} \lambda_i = k, \end{aligned}$$

contrary to the assumption that $k \neq l$. □

Theorem 3.10.

(a) $\dim P_C^2(D_n) = \frac{1}{2}n(n-1) - 1 \ \forall\ n \geq 2.$

(b) $\dim P_C^3(D_n) = \begin{cases} 1 & \text{if } n = 3, \\ 6 & \text{if } n = 4, \\ \dim P_C(D_n) - 1 & \text{if } n \geq 5. \end{cases}$

(c) $\dim(P_C^{\{2,3\}}(D_n)) = \dim P_C(D_n)$ *for all* $n \geq 3$.

Proof.

(a) It exist precise $\frac{1}{2}n(n-1)$ 2-circuits whose incidence vectors are linearly independent. Thus $\dim P_C^2(D_n) = \frac{1}{2}n(n-1) - 1$ for all $n \geq 2$.

(b) Clearly, $\dim P_C^3(D_3) = 1$ For $n = 4$ assure yourself of the correctness of the statement.

Now we show that $\dim P_C^3(D_n) = \dim P_C(D_n) - 1$ for $n \geq 5$. It is an easy consequence of Lemma 3.9 that $\dim P_C^3(D_n) \leq \dim P_C(D_n) - 1$: suppose, for the sake of contradiction, that $\dim P_C^3(D_n) = \dim P_C(D_n)$. Since any k-circuit, with $k \neq 3$, is affinely independent of all 3-circuits, it follows that $\dim P_C^{\{3,k\}}(D_n) = \dim P_C(D_n) + 1$, but this contradicts $\dim P_C^{\{3,k\}}(D_n) \leq \dim P_C(D_n)$.

In order to show that $\dim P_C^3(D_n) \geq \dim P_C(D_n) - 1$ for $n \geq 5$, we prove per induction on the number of nodes n that the rank of the set $\mathcal{S}(D_n)$ of all incidence vectors of 3-circuits is at least $(n-1)^2$. Then the claim follows immediately.

For $n = 5$ assure yourself of the correctness of the statement! Let the assertion be true for $n \geq 5$. Then, we show it is also true for $n + 1$.

Let $\mathcal{S}(D_{n+1})$ be the set of all incidence vectors of 3-circuits of D_{n+1}, let $\mathcal{S}_{n+1}$ the set of all incidence vectors of 3-circuits entering node $n+1$, and $\bar{\mathcal{S}}_{n+1}$ the set of all incidence vectors of 3-circuits not entering node $n+1$. Further, let $D_n = (V', A')$ be the complete subdigraph of D_{n+1} induced by the n first nodes. From the assumption by induction we know that $\text{rank}(\bar{\mathcal{S}}_{n+1}) = (n-1)^2$.

From $\mathcal{S}_{n+1}$ we construct the set $\mathcal{R}_{n+1}$ by removing in each to an incidence vector corresponding circuit the arc $(i, j) \in A'$, i.e.,

$$\mathcal{R}_{n+1} := \{\chi^{\{(i,n+1),(n+1,j)\}} \mid (i,j) \in A'\}.$$

We will show in two steps that $\text{rank}(\mathcal{R}_{n+1}) = 2n - 1$.

Claim 1: $\text{rank}(\mathcal{R}_{n+1})$ $=\text{rank}(\tilde{\mathcal{R}}_{n+1})$, where $\tilde{\mathcal{R}}_{n+1} := \{\chi^{\{(i,n+1),(n+1,j)\}} \mid i, j \in V'\}$.

That can be seen as follows. Obviously is $\mathcal{R}_{n+1}$ a subset of $\tilde{\mathcal{R}}_{n+1}$. It remains to be shown that $\chi^{\{(i,n+1),(n+1,i)\}} \in \text{lin}(\mathcal{R}_{n+1})$ for all $i \in V'$. But this follows from the following equation for arbitrary nodes $i, j, k \in V'$, $i \neq j \neq k \neq i$:

$$\chi^{\{(i,n+1),(n+1,i)\}} = \chi^{\{(i,n+1),(n+1,j)\}} - \chi^{\{(k,n+1),(n+1,j)\}} + \chi^{\{(k,n+1),(n+1,i)\}}.$$

Claim 2: $\text{rank}(\mathcal{R}_{n+1}) = 2n - 1$.

$\tilde{\mathcal{R}}_{n+1}$ can be interpreted as the set of incidence vectors of the complete bipartite graph with nodeset $V' \cup V''$, where V'' is a copy of V' and the bipartition is $(V' : V'')$. It is well known that the rank of such a set is $2n - 1$. Together with Claim 1 we get $\text{rank}(\mathcal{R}_{n+1}) = 2n - 1$.

As can easily be seen, $\text{lin}(\bar{\mathcal{S}}_{n+1}) \cap \text{lin}(\mathcal{R}_{n+1}) = \{\mathbf{0}\}$, and thus $\text{rank}(\bar{\mathcal{S}}_{n+1} \cup \mathcal{R}_{n+1}) = n^2$ which completes the proof for (b).

(c) For $n = 3$ the claim follows, since $P_C^{\{2,3\}}(D_n) = P_C(D_3)$. For $n \geq 4$ the statement follows directly from (b) and Lemma 3.9.

□

Corollary 3.11. $\dim P_C^{\leq k}(D_n) = (n-1)^2$ *for all* $n \geq 3$, $3 \leq k < n$.

□

Theorem 3.12. *Let* $n \geq 5$ *and* $3 \leq k < n$. *Then the cardinality constraint*

$$x(A) \leq k \tag{3.25}$$

is facet defining for $P_C^{\leq k}(D_n)$.

Proof. For $k = 3$ the claim follows from Theorem 3.10 (b). If $k \geq 4$, we assume that there is an inequality $b^T x \leq b_0$, $b \in \mathbb{R}^A, b \neq \mathbf{0}$, which is valid for $P_C^{\leq k}(D_n)$ and satisfies $\{x \in P_C^{\leq k}(D_n) \mid x(A) = k\} \subseteq \{x \in P_C^{\leq k}(D_n) \mid b^T x = b_0\}$. Let w.l.o.g. $b^T \leq b_0$ in 1-rooted form, i.e., $b_{i1} = 0$ for all $i \in \{2, 3, \ldots, n\}$.

Let $u, v, w \in V \setminus \{1\}$, and let C be a circuit of cardinality k containing the arcs (u, v) and $(v, 1)$, but not the node w. With $\tilde{C} = (C \setminus \{(u, v), (v, 1)\}) \cup \{(u, w), (w, 1)\}$, we have $b^T \chi^C = b^T \chi^{\tilde{C}} = b_0$ and thus

$$\begin{array}{rlllll} & b_{uv} & + & b_{v1} & = & b_{uw} + b_{w1} \\ \Leftrightarrow & b_{uv} & & & = & b_{uw}. \end{array}$$

Since $u, v, w \in V \setminus \{1\}$ were arbitrary chosen, it follows

$$b_{u2} = b_{u3} = \cdots = b_{u,u-1} = b_{u,u+1} = \cdots = b_{un}$$

for all $u \in \{2, 3, \ldots, n\}$.

Next, let $p, q, r, s \in V \setminus \{1\}$ four arbitrary nodes such that (p, q) and (r, s) are not adjacent arcs. We show $b_{pq} = b_{rs}$. Consider the circuits $C := (1, v_1, v_2, \ldots, v_{k-3}, p, q)$ and $\hat{C} := (1, v_1, v_2, \ldots, v_{k-3}, q, r)$. C and $\hat{C}$ are circuits of cardinality k. Since $b_{q1} = b_{r1} = 0$ and $b_{k-3,p} = b_{k-3,q}$, we have $b_{pq} = b_{qr}$. Analogously follows $b_{qr} = b_{rs}$, and thus $b_{pq} = b_{rs}$. Also we have $b_{uv} = b_{vu}$ for all $u, v \in V \setminus \{1\}$.

This immediately yields $b_{ij} = \frac{b_0}{k}$ for all $i, j \in V \setminus \{1\}$ by considering a circuit of cardinality k which contains not the node 1. To prove $b_{1i} = \frac{2}{k} b_0$, $i = 2, 3, \ldots, n$ consider a circuit of cardinality k which contains the arc $(1, i)$. Assuming $b_0 = 0$ implies $b = \mathbf{0}$ which is a contradiction. Consequently $b_0 \neq 0$ and thus $b^T x \leq b_0$ is a positive multiple of $x(A) \leq k$ up till rooting. □

Theorem 3.13. *Let $D_n = (V, A)$, $n \geq 4$, be the complete digraph on n nodes. Then $\dim P_C^{\{2,n\}}(D_n) = \dim P_C(D_n)$.*

To prove the theorem requires a higher amount of technical detail, because the dimensions of the both polytopes $P_C^2(D_n)$ and $P_C^n(D_n)$ lie wide under $(n-1)^2$. We set $d_n := \dim P_C^n(D_n) = n^2 - 3n + 1$. The foundation of the proof builds the following lemma which is actually clear by the considerations in Chapter 2.

Lemma 3.14. *Let $P_0(D_n^0)$ be the circuit-and-loops polytope, and let $\begin{pmatrix} x^1 \\ y^1 \end{pmatrix}, \ldots, \begin{pmatrix} x^p \\ y^p \end{pmatrix} \in P_0(D_n^0), p \in \mathbb{N}$. Then the following statements are equivalent.*

(a) $x^1, x^2, \ldots, x^p$ are affinely independent.

(b) $\begin{pmatrix} x^1 \\ y^1 \end{pmatrix}, \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}, \ldots, \begin{pmatrix} x^p \\ y^p \end{pmatrix}$ are affinely independent.

Proof.

"(a) $\Rightarrow$ (b)": Trivial.

"(b) $\Rightarrow$ (a)": Suppose, for the sake of contradiction, that the points $x^1, \ldots, x^p$ are not affinely independent, say, e.g., $x^p = \sum_{i=1}^{p-1} \lambda_i x^i$, $\sum_{i=1}^{p-1} \lambda_i = 1$. Then follows

$$y^p = \begin{pmatrix} 1 - x^p(\delta^+(1)) \\ 1 - x^p(\delta^+(2)) \\ \vdots \\ 1 - x^p(\delta^+(n)) \end{pmatrix} = \begin{pmatrix} 1 - \sum_{i=1}^{p-1} x^i(\delta^+(1)) \\ 1 - \sum_{i=1}^{p-1} x^i(\delta^+(2)) \\ \vdots \\ 1 - \sum_{i=1}^{p-1} x^i(\delta^+(n)) \end{pmatrix} = \sum_{i=1}^{p-1} \lambda_i \begin{pmatrix} 1 - x^i(\delta^+(1)) \\ 1 - x^i(\delta^+(2)) \\ \vdots \\ 1 - x^i(\delta^+(n)) \end{pmatrix} = \sum_{i=1}^{p-1} \lambda_i y^i,$$

and hence

$$\begin{pmatrix} x^p \\ y^p \end{pmatrix} = \sum_{i=1}^{n-1} \lambda_i \begin{pmatrix} x^i \\ y^i \end{pmatrix}.$$

Contradiction! □

Proof of Theorem 3.13

Since $\dim P_C^n(D_n) = \dim P_C(D_n) - n$, there are $d_n + 1 = n^2 - 3n + 2$ linearly independent vectors $\begin{pmatrix} x^r \\ y^r \end{pmatrix} \in P_0(D_n^0)$ such that $y^r = \mathbf{0}$, $r = 1, 2, \ldots, d_n + 1$.

Next consider the point $\begin{pmatrix} x^{23} \\ y^{23} \end{pmatrix}$ where x^{23} is the incidence vector of the 2-circuit $C_{23} = \{(2,3),(3,2)\}$, and $n-1$ further points $\begin{pmatrix} x^{1i} \\ y^{1i} \end{pmatrix}$ where x^{1i} the incidence vector of the 2-circuit $C_{1i} = \{(1,i),(i,1)\}$, $i = 2, 3, \ldots, n$.

The incidence matrix Z whose rows are the vectors $((x^r)^T, (y^r)^T)$, $r = 1, 2, \ldots, d_n + 1$, $((x^{23})^T, (y^{23})^T)$, and $((x^{1i})^T, (y^{1i})^T)$, $i = 2, 3, \ldots, n$, is of the form

$$Z = \begin{pmatrix} X & \mathbf{0} \\ Y & L \end{pmatrix},$$

where

$$L = \begin{pmatrix} 1 & 0 \ 0 \ 1 \cdots 1 \\ \mathbf{0} & E - I \end{pmatrix}.$$

E is the $(n-1) \times (n-1)$ matrix of all ones and I the $(n-1) \times (n-1)$ identity matrix. $E - I$ is nonsingular, and thus L is of rank n. A is of rank $d_n + 1$, and hence rank $(Z) = d_n + 1 + n = n^2 - 2n + 2$. This yields together with Lemma 3.14 the desired result.

□

We summarise the results and add some easy conclusions.

Corollary 3.15.

(a) $\dim P_C^2(D_n) = \frac{1}{2} n(n-1) - 1 \ \forall \ n \geq 2$.

(b) $\dim P_C^3(D_n) = \begin{cases} 1 & \textit{if } n = 3, \\ 6 & \textit{if } n = 4, \\ \dim P_C(D_n) - 1 & \textit{if } n \geq 5. \end{cases}$

(c) $\dim P_C^k(D_n) = \begin{cases} \dim P_C(D_n) - n & \text{if } n = k, \\ \dim P_C(D_n) - 1 & \text{if } n > k, \end{cases} \qquad \forall\, k \in \mathbb{N}, k \geq 4.$

(d) If $|L| \geq 2$, *then* $\dim P_C^L(D_n) = \dim P_C(D_n)$ $\forall\, n \geq 3$.

Proof.

(i),(ii) See Theorem 3.10.

(iii) If $n = k$ then $P_C^k(D_n)$ is the asymmetric traveling salesman polytope, which has dimension $n^2 - 3n + 1 = dimP_C(D_n) - n$ (see Grötschel [17]). For $n > k$ apply Theorem 3.12. There we have shown that $P_C^k(D_n)$ is a facet of $P_C(D_n)$; hence, $\dim P_C^k(D_n) = \dim P_C(D_n) - 1$.

(iv) Because $|L| \geq 2$, it exist $i, j \in \{2, 3, \ldots, n\}$, $i < j$ with $i, j \in L$. If $i = 2$ and $j = 3$, we are ready. Otherwise let in the first instance $4 \leq j < n$. So we know that $\text{arank}(\mathcal{C}^j(D_n)) = (n-1)^2$. Then follows from Lemma 3.9 that $x \notin \text{aff}(\mathcal{C}^j(D_n))$ for any $x \in \mathcal{C}^i(D_n)$. This implies $\dim P_C^L(D_n) = \dim P_C(D_n)$.

Now let $j = n$. If $2 < i < j = n$ and $n \geq 5$, we can reverse the roles of i and j in the above proof of contradiction. If $i = 3$ and $n = 4$, i.e., $j = 4$, then assure yourself of the correctness of the statement. Otherwise is $i = 2$ and $j = n$, $n \geq 4$. But then follows the proposition from Theorem 3.13.

□

3.4 Inequalities from the circuit polytope $P_C(D_n)$

It would be nice to transform facet defining inequalities for the circuit polytope $P_C(D_n)$ into those for a length restricted circuit polytope $P_C^L(D_n)$. But this seems to be a hard problem, since $P_C^L(D_n)$ is in generally not a face of $P_C(D_n)$. In this thesis we will not discuss the problems, but only make some remarks.

For the symmetric counterpart Bauer [7] gives a criterion when a facet defining inequality for $P_C(K_n)$ is also facet defining for $P_C^{\leq k}(D_n)$, with $4 \leq k < n$, but I do not understand it. She states that

- if an inequality $b^T x \leq b_0$ is facet defining for $P_C(K_n)$,
- if there is a complete subgraph $G = (V', E')$ of K_n with at most k nodes such that the restriction of $b^T x \leq b_0$ is also facet defining for $P_C(G)$,
- and if for each edge $e \notin E'$ there is a circuit C with $e \in C$, $|C| \leq k$, and $b^T \chi^C x = b_0$,

then $b^T x \leq b_0$ is also facet defining for $P_C^{\leq k}(D_n)$. The proof of this statement is unfortunately deficient. Even though the criterion is true, we cannot be sure if a similar criterion is true for the directed case.

Hence we prefer it to investigate for each facet defining inequality for $P_C(D_n)$ separately on which conditions it is also facet defining for $P_C^L(D_n)$. We begin with the inequalities of the IP-formulation 2.1 and give a complete classification of these inequalities. Before starting let us make some easy observations:

Lemma 3.16. *Let $a^T x \leq a_0$ be a valid inequality with respect to a polyhedron $P \subseteq \mathbb{R}^m$ and $F := \{x \in P \mid a^T x = a_0\} \neq \emptyset$ the induced face. Then*

$$F = \mathit{aff}(F) \cap P.$$

Proof.

"$\subseteq$": Trivial.

"$\supseteq$":

$$\begin{aligned} & y \in \mathrm{aff}(F) \\ \Rightarrow\ & \exists\ x_1, \dots, x_k \in F, \lambda_1, \dots, \lambda_k \in \mathbb{R}: \quad y = \sum_{i=1}^{k} \lambda_i x_i, \sum_{i=1}^{k} \lambda_i = 1 \\ \Rightarrow\ & \exists\ x_1, \dots, x_k \in F, \lambda_1, \dots, \lambda_k \in \mathbb{R}: \quad a^T y = \sum_{i=1}^{k} \lambda_i \underbrace{a^T x_i}_{=a_0} = a_0 \underbrace{\sum_{i=1}^{k} \lambda_i}_{=1} = a_0 \\ \overset{y \in P}{\Rightarrow}\ & y \in F. \end{aligned}$$

□

Lemma 3.17. *Let $c^T x \leq c_0$ define a facet F of $P_C^L(D_n)$ which is not equivalent to a nonnegativity constraint. Then the following statements are valid.*

(a) To each arc $a \in A$ there is a point $x^* \in F$ with $x_a^* > 0$.

(b) For any two arcs $a, b \in A$, $a \neq b$, there are at least two points $x^a, x^b \in F$ such that $x_a^a > 0$, $x_b^a = 0$ and $x_b^b > 0$, $x_a^b = 0$.

Proof.

(a) Assume there is an arc a such that $x_a = 0$ for all $x \in F$. Then follows $F \subseteq \{x \in P_C^L(D_n) \mid x_j = 0\}$, contrary to the assumption that F is not induced by a nonnegativity constraint.

(b) Due to (i) there are points $x^a, x^b \in F$ with $x_a^a > 0$ and $x_b^b > 0$. Assume that $x_a = x_b$ for all $x \in F$. Then exist $\lambda > 0$ and $x^* \in P_C^L \setminus F$ such that

(i) $d^T x := c^T x + \lambda x_a - \lambda x_b \leq c_0$ is valid for $P_C^L(D_n)$ and

(ii) $c^T x^* + \lambda x_a^* - \lambda x_b^* = c_0$.

But since $d^T x = c_0$ for all $x \in F$, it follows $F \subsetneq \{x \in P_C^L(D_n) \mid d^T x = c_0\}$. Hence, F is not a facet of $P_C^L(D_n)$. Contradiction!

□

Nonnegativity constraints

First we investigate when a nonnegativity constraint defines a facet of $P_C^k(D_n)$. In order to simplify the proof of the next theorem we introduce a definition and a useful lemma.

Definition 3.18. Let $D = (V, A)$ be a digraph, $u, v \in V$, C_u a circuit containing u, and C_v a circuit containing v. Then we say that C_u and C_v are $[u, v]$-*adjacent* if $C_u \backslash \{u\} = C_v \backslash \{v\}$.

Lemma 3.19. *Let $D_n = (V, A)$, $n \geq 6$, be the complete digraph on n nodes, $a \in A$, and $k \in \{4, \ldots, n-1\}$. Let us denote by $\mathcal{C}_a^k(D_n)$ the set of circuits in D_n of length k containing arc a and by $\mathcal{C}_{\neg a}^k(D_n)$ the set of circuits in D_n of length k not containing arc a.*

(a) Let $p, q \in V$ and $C_p, C_q \in \mathcal{C}_a^k(D_n)$ be $[p, q]$-adjacent circuits. Then the incidence vector of C_p is in the affine hull of the incidence vectors generated by $\{C_q\} \cup \mathcal{C}_{\neg a}^k(D_n)$, and, vice versa, the incidence vector of C_q is in the affine hull of the incidence vectors generated by $\{C_p\} \cup \mathcal{C}_{\neg a}^k(D_n)$.

(b) Let $C_1, C_2, \ldots, C_p \in \mathcal{C}_a^k(D_n)$, $p \in \mathbb{N}$, $p \geq 2$, $v_i \in V(C_i)$ for $i = 1, \ldots, p$, and C_j and C_{j+1} be $[v_j, v_{j+1}]$-adjacent for $j = 1, \ldots, p-1$. Then the incidence vector of C_1 is in the affine hull of the incidence vectors generated by $\{C_p\} \cup \mathcal{C}_{\neg a}^k(D_n)$, and, vice versa, the incidence vector of C_p is in the affine hull of the incidence vectors generated by $\{C_1\} \cup \mathcal{C}_{\neg a}^k(D_n)$.

Proof.

(a) For $4 \leq k < n$, let

$$\begin{aligned} C_p &= (v_1, v_2, \ldots, v_{i-1}, p, v_{i+1}, \ldots, v_k, v_1) \\ C_q &= (v_1, v_2, \ldots, v_{i-1}, q, v_{i+1}, \ldots, v_k, v_1), \qquad 2 \leq i \leq k-1 \end{aligned}$$

and let w.l.o.g. $a = (v_k, v_1)$. We show $C_p \in \text{aff}(\{C_q\} \cup \mathcal{C}_{\neg a}^k(D_n))$. The other direction follows analogously. If $C_p = C_q$ it is nothing to show. So let $C_p \neq C_q$. For $k = 4$ exists at least one further node $z \in V \setminus (V(C_p) \cup V(C_q))$, since $n \geq 6$. C_p can be generated by adding to C_q the circuit $C^* = (v_{i-1}, p, v_{i+1}, z, v_{i-1})$ and subtracting the circuit $C' = (v_{i-1}, q, v_{i+1}, z, v_{i-1})$. For $5 \leq k < n$, C_p can be generated by adding to C_q the circuit $C^* = (v_{i-1}, p, v_{i+1}, v_{i-2}, v_{i-3}, \ldots, v_1, v_k, \ldots, v_{i+2}, v_{i-1})$ and subtracting the circuit $C' = (v_{i-1}, q, v_{i+1}, v_{i-2}, v_{i-3}, \ldots, v_1, v_k, \ldots, v_{i+2}, v_{i-1})$. This implies (a).

(b) This follows immediately from (a) by induction.

□

Theorem 3.20. *Let $D_n = (V, A)$, $n \geq 3$, be the complete digraph on n nodes and $k \in \{2, \ldots, n\}$. Then the inequality $x_a \geq 0$, $a \in A$, defines a facet of $P_C^k(D_n)$ if and only if $P_C^k(D_n) \neq P_C^4(D_4)$.*

Proof. Necessity. Suppose that $P_C^k(D_n) = P_C^4(D_4)$. If $x_a \geq 0$ would be a facet of $P_C^4(D_4)$, there had to be five linearly independent 4-circuits not containing a. But there are only four linearly independent 4-circuits not containing a.

Suffiency. The statement is obviously true if $k = 2$. For $n = k \neq 4$, Grötschel gave in [17] a constructive proof that the nonnegativity constraints x_a, $a \in A$, define facets.

Let now $k \in \{3, \ldots, n-1\}$. Denote by F the face induced by $x_a \geq 0$. As F is a proper face, we have $\dim F < \dim P_C^k(D_n)$. Let $C^* \in \mathcal{C}^k(D_n)$ be any circuit containing the arc a, and set as above $\mathcal{C}_a^k(D_n) := \{C \in \mathcal{C}^k(D_n) \mid a \in C\}$ and $\mathcal{C}_{\neg a}^k(D_n) := \{C \in \mathcal{C}^k(D_n) \mid a \notin C\}$. First we will show for $k \geq 4$ and $n \geq 6$ that the remaining circuits C containing the arc a are in the affine hull of $\{C^*\} \cup \mathcal{C}_{\neg a}^k(D_n)$. That proves $\dim F \geq \dim P_C^k(D_n) - 1$.

Let w.l.o.g. $a = (1,2)$ and C^* be the circuit $(1,2,3,\ldots,k,1)$. Further, let $C = (v_1, v_2, v_3, \ldots, v_k, v_1)$, $C \neq C^*$, be any circuit of length k with $v_1 = 1$ and $v_2 = 2$. In a first step we show that the incidence vector of C is in the affine hull of the incidence vectors generated by $\{C_\pi\} \cup \mathcal{C}_{\neg a}^k(D_n)$ where π is a permutation of $1, 2, \ldots, k$ such that the circuit $C_\pi := (v_{\pi(1)}, v_{\pi(2)}, v_{\pi(3)}, \ldots, v_{\pi(k)}, v_{\pi(1)})$ satisfies $v_{\pi(j)} < v_{\pi(j+1)}$, $j = 1, 2, \ldots, k-1$. (It follows immediately $v_{\pi(1)} = 1$, $v_{\pi(2)} = 2$ and $v_k > v_1$.) That can be seen as follows. Let $v_j > v_{j+1}$ for any $j \in \{1, 2, \ldots, k-1\}$. Since $k < n$, it exist a node $u \in V \setminus V(C)$. We set $v_{k+1} := v_1 = 1$. Consider the sequence of circuits generated by $C_1 := C$, $C_2 := (v_2, \ldots, v_{j-1}, u, v_{j+1}, \ldots, v_{k+1}, v_2)$, $C_3 := (v_2, \ldots, v_{j-1}, u, v_j, v_{j+2}, \ldots, v_{k+1}, v_2)$ and $C_4 := (v_2, \ldots, v_{j-1}, v_{j+1}, v_j, v_{j+2}, \ldots, v_{k+1}, v_2)$. This sequence fullfills the conditions of Lemma 3.19 (b). Thus, the incidence vector of C is in the affine hull of the incidence vectors generated by $\{C_4\} \cup \mathcal{C}_{\neg a}^k$. After at most $(k-2)!$ repetitions of node-exchanges we obtain the desired result.

Next, we show that χ^{C_π} is in the affine hull the incidence vectors generated by $\{C^*\} \cup \mathcal{C}_{\neg a}^k(D_n)$. Let us redefine C_π by $C_\pi := (1, 2, w_3, \ldots, w_k, 1)$. If $C_\pi = C^*$ it is nothing more to show. Otherwise we know especially that $w_k > k$. Let j^* be the first number with $w_{j^*} > j^*$. It follows $j^* \geq 3$. Generate for $j = j^*$ to k the sequences

$$[(2, \ldots, j-1, w_j, w_{j+1}, \ldots, w_{k+1}, 2), (1, 2, \ldots, j-1, j, w_{j+1}, \ldots, w_{k+1}, 1)],$$

where $w_{k+1} := 1$. By Lemma 3.19 (b) follows the claim, and thus holds Theorem 3.20 if $5 \leq k < n$.

In case $k = 4$ and $n = 5$ a computer verification with PORTA shows that $\dim F = 14 = \dim P_C^4(D_5) - 1$.

Finally, let $k = 3$. In case $n = 4$ and $n = 5$ assure yourself of the correctness of the claim. For $n \geq 6$ let $C = (1, 2, i, 1) \neq C^*$ be any circuit containing the arc $(1,2)$. By taking two further nodes $j, k \in V \setminus \{1, 2, 3, i\}$, $j \neq k$, the following equation completes the proof, where the incidence vectors are denoted by sums of unit vectors.

$$\begin{aligned} e_{12}+e_{2i}+e_{i1} = {} & e_{12}+e_{23}+e_{31} \\ & +e_{2i}+e_{ij}+e_{j2} \\ & +e_{i1}+e_{1k}+e_{ki} \\ & +e_{3j}+e_{k3}+e_{jk} \\ & -e_{23}-e_{31}-e_{ij}-e_{j2}-e_{1k}-e_{ki}-e_{3j}-e_{k3}-e_{jk} \end{aligned}$$

□

Theorem 3.21. *Let $D_n = (V, A)$, $n \geq 3$, be the complete digraph on n nodes, and let $|L| \geq 2$. Then the inequality $x_{uv} \geq 0$, $(u, v) \in A$, defines a facet of $P_C^L(D_n)$.*

Proof.
Case 1: $L \neq \{2, n\}$

Since $|L| \geq 2$ and $L \neq \{2, n\}$, it exist $k \in L \cap \{3, \ldots, n-1\}$, $l \in L \setminus \{k\}$. By Theorem 3.20, there are $n^2 - 2n$ linearly independent k-circuits not containing (u, v), and by Lemma 3.9, any l-circuit not containing (u, v) is affinely independent of them. This proves the claim.

Case 2: $L = \{2, n\}$

Let w.l.o.g. $(u, v) = (1, 2)$. The statement is evidently true for $n = 3$. Now let $n \geq 4$. By Theorem 3.20, it exist $n^2 - 3n + 1$ linearly independent incidence vectors x^r of tours satisfying $x^r_{uv} = 0$. Further, the incidence vectors x^{pq} of the 2-circuits on $\{2, 3\}$, $\{3, 4\}$, and $\{1, i\}$, $i = 3, \ldots, n$, satisfy $x^{pq}_{12} = 0$.

Let us consider now the corresponding circuit-and-loops polytope $P_0^{\{2,n\}}(D_n^0)$. Denote by y^r and y^{pq} the associated loops vectors. Then the matrix B whose rows are the vectors $\begin{pmatrix} x^r \\ y^r \end{pmatrix}^T$, $r = 1, \ldots, n^2 - 3n + 1$, $\begin{pmatrix} x^{23} \\ y^{23} \end{pmatrix}^T$, $\begin{pmatrix} x^{34} \\ y^{34} \end{pmatrix}^T$, $\begin{pmatrix} x^{1i} \\ y^{1i} \end{pmatrix}^T$, $i = 3, \ldots, n$, is of the form

$$B = \begin{pmatrix} X & \mathbf{0} \\ Y & Z \end{pmatrix}$$

where Z is of the form

$$Z = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 0 & 0 & 1 & \ldots & 1 \\ \mathbf{0} & \mathbf{1} & & & E - I & & \end{pmatrix}.$$

Here, E is the $(n-2) \times (n-2)$ matrix of all ones and I the $(n-2) \times (n-2)$ incidence matrix. It is easy to see that the affine rank of Z is n. Thus, the affine rank of B is $n^2 - 2n + 1$, and hence, by Lemma 3.14, $x_{12} \geq 0$ defines a facet of $P_C^{\{2,n\}}(D_n)$. $\square$

Degree constraints

Theorem 3.22. *Let $n \geq 5$, $D_n = (V, A)$ be the complete digraph on n nodes, $v \in V$, and $k \in \{2, \ldots, n\}$. Then the degree constraint*

$$x(\delta^+(v)) \leq 1$$

induces a facet of $P_C^k(D_n)$ if and only if $4 \leq k < n$.

Proof. Necessity. Suppose that $k \in \{2, 3\}$. The number of 2-circuits containing a chosen node v is $n - 1$; the number of 3-circuits containing a node v is $(n-1)(n-2)$. For these reasons, (2.2) is not facet defining for $P_C^k(D_n)$ for $k \in \{2, 3\}$.

Suppose that $k = n$. Then all $x \in P_C^n(D_n)$ satisfy (2.2) at equality, and thus (2.2) is not a facet defining inequality.

Suffiency. Let w.l.o.g. $v = 1$. We denote the face induced by $x(\delta^+(1)) \leq 1$ by F. Since it is a proper face of $P_C^k(D_n)$, it follows that $\dim F < n^2 - 2n$. We set

$$\mathcal{C}_1^k(D_n) := \{C \in \mathcal{C}^k(D_n) \mid 1 \in V(C)\}$$

and prepare the main step of the proof by two claims.

Claim 1: Let $4 \leq k < n$, $u, v \in V$, and $C_u, C_v \in \mathcal{C}^k(D_n)$ be $[u,v]$-adjacent circuits. Then

$$\chi^{C_u} \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}$$

if and only if

$$\chi^{C_v} \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}.$$

Let us denote by $\Pi = (w_1, w_2, \ldots, w_{k-1})$ the path $C_u \setminus \{u\} = C_v \setminus \{v\}$. Clearly, the claim holds if $1 \in V(\Pi)$. Otherwise holds the equation

$$\chi^{C_v} = \chi^{C_u} + \chi^{(1,w_{k-1},v,w_1,\ldots,w_{k-3},1)} - \chi^{(1,w_{k-1},u,w_1,\ldots,w_{k-3},1)},$$

which proves just so the assertion.

Claim 2: Let $n \geq 6$, $4 \leq k < n$, Π be any path of length $k-3$, p the origin and q the terminus of Π, and $u, v \in V \setminus V(\Pi)$. Further, set $C_{uv} := \{(u,v),(v,p),(q,u)\} \cup \Pi$ and $C_{vu} := \{(v,u),(u,p),(q,v)\} \cup \Pi$. Then

$$\chi^{C_{uv}} \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}$$

if and only if

$$\chi^{C_{vu}} \in \text{aff}\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}.$$

Let $\Pi = (w_1, w_2, \ldots, w_{k-2})$. Clearly, the claim holds if $1 \in \{u,v\} \cup V(\Pi)$. Otherwise we have for $k \geq 5$

$$\chi^{(v,u,\Pi,v)} = \chi^{(u,v,\Pi,u)} + \chi^{(1,w_{k-2},v,u,w_1,\ldots,w_{k-4},1)} - \chi^{(1,w_{k-2},u,v,w_1,\ldots,w_{k-4},1)}.$$

Thus, the claim holds if $k \geq 5$. If $k = 4$ then we proceed as depicted in Figure 3.1.

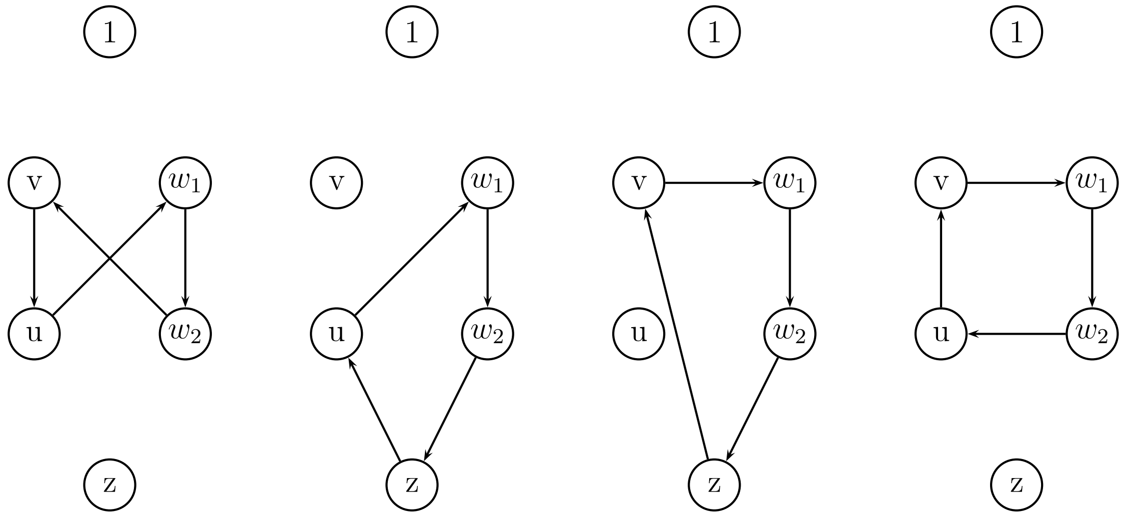


Figure 3.1

We prove now the statement of the Theorem by showing

$$\text{aff}\{\chi^C \mid C \in \mathcal{C}^k(D_n)\} = \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,4,\ldots,k+1,2)\}\}.$$

This proves $\dim F \geq n^2 - 2n - 1$, and hence it follows the statement.
"$\supseteq$": trivial.
"$\subseteq$": In case $k = 4, n = 5$ a verification with PORTA shows that the statement is true. Otherwise let $C' = (v_1, v_2, \ldots, v_k, v_1)$ be any circuit of length k, and in order to avoid trivial cases we assume that $1 \notin V(C')$ and $C' \neq (2,3,\ldots,k+1,2)$. By an iterated application of the node-exchange in Claim 2 it follows that

$$\begin{aligned} & \chi^{C'} \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\} \\ \Leftrightarrow \quad & \chi^{\{(v_{\pi(1)},v_{\pi(2)},\ldots,v_{\pi(n)},v_{\pi(1)})\}} \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}. \end{aligned}$$

for any permutation π, in particular for the permutation with

$$v_{\pi(j)} < v_{\pi(j+1)}, \qquad j = 1,2,\ldots,k-1.$$

Finally, for $i = 1,2,\ldots,k$ exchange $v_{\pi(i)}$ with $i+1$, i.e.,

$$(2,\ldots,i,v_{\pi(i)},v_{\pi(i+1)},\ldots,v_{\pi(k)},2) \longrightarrow (2,\ldots,i,i+1,v_{\pi(i+1)},\ldots,v_{\pi(k)},2).$$

This yields

$$\begin{aligned} & \chi^{C'} && \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\} \\ \Leftrightarrow \quad & \chi^{(v_{\pi(1)},v_{\pi(2)},\ldots,v_{\pi(n)},v_{\pi(1)})} && \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\} \\ \Leftrightarrow \quad & \chi^{(2,3,\ldots,k+1,2)} && \in \text{aff}\{\chi^C \mid C \in \mathcal{C}_1^k(D_n) \cup \{(2,3,\ldots,k+1,2)\}\}, \end{aligned}$$

which is a true statement. □

Theorem 3.23. *For* $|L| \geq 2$ *the degree constraint*

$$x(\delta^+(v)) \leq 1$$

induces a facet of $P_C^L(D_n)$.

Proof. We distinguish four cases.

(1) Let $L = \{k, l\}$ with $4 \leq l < n$ and $2 \leq k < l$.

Constraint (2.2) defines a facet of $P_C^l(D_n)$ and $\dim P_C^l(D_n) = \dim P_C^L(D_n) - 1$. Hence, with Lemma 3.9 follows the statement.

(2) Let $L = \{2, 3\}$.

Assume that there is an inequality $b^T x \leq b_0$, $b \in \mathbb{R}^A, b \neq \mathbf{0}$, which is valid for $P_C^{\{2,3\}}(D_n)$ and satisfies $\{x \in P_C^{\{2,3\}}(D_n) \mid x(\delta^+(v)) = 1\} \subseteq \{x \in P_C^{\{2,3\}} \mid b^T x = b_0\}$. Let w.l.o.g. $v = 1$, and let $b^T \leq b_0$ in 1-rooted form, i.e., $b_{i1} = 0$ for all $i \in \{2, 3, \ldots, n\}$. From the 2-circuits on $\{1, i\}$ we derive $b_{1i} = b_0$, $i = 2, 3, \ldots, n$. This implies immediately $b_{ij} = 0$ for all $(i, j) \in A$, $i \neq 1 \neq j$, by considering the 3-circuits $(1, i, j, 1)$. Further, $b \neq \mathbf{0}$ implies $b_0 \neq 0$, and together with the fact that $b^T x \leq b_0$ must be valid for $\chi^{\{(2,3,2)\}}$ it follows the claim.

(3) Let $L = \{2, n\}$, $n \geq 4$.

Let w.l.o.g. $v = 1$. We consider the corresponding circuit-and-loops polytope $P_0^{\{2,n\}}(D_n^0)$.

Since all tours satisfy (2.2) at equality, there are $n^2 - 3n + 2$ linearly independent vectors $\begin{pmatrix} x^r \\ y^r \end{pmatrix} \in P_0^{\{2,n\}}(D_n^0)$, with $y^r = \mathbf{0}$ for all r.

Further, the 2-circuits on $\{1, i\}$, $i \in \{2, \ldots, n\}$, satisfy (2.2) at equality. Thus, there are $n - 1$ linearly independent vectors $\begin{pmatrix} x^{1i} \\ y^{1i} \end{pmatrix} \in P_0^{\{2,n\}}(D_n^0)$, with $x^{1i} = \chi^{\{(1,i,1)\}}$.

It is easy to see now that the matrix whose columns are the vectors $\begin{pmatrix} x^r \\ y^r \end{pmatrix}$, $r = 1, \ldots, n^2 - 3n + 2$, $\begin{pmatrix} x^{1i} \\ y^{1i} \end{pmatrix}$, $i = 2, \ldots, n$, is fulldimensional, i.e., by Lemma 3.14, the $n^2 - 2n + 1$ vectors x^r, $r = 1, \ldots, n^2 - 3n + 2$, x^{1i}, $i = 2, \ldots, n$, are affinely independent.

(4) Let $|L| \geq 3$.

Then the claim follows immediately by (1), (2), and (3).

□

These results imply that the subtour elimination constraint $x_{ij} + x_{ji} \leq 1$, which is valid for all circuit polytopes $P_C^L(D_n)$ with $2 \notin L$, does in generally not define a facet of $P_C^L(D_n)$, since each circuit $C \in \mathcal{C}^L(D_n)$ containing (i, j) or (j, i) satisfies $x(\delta^+(i)) = 1$ as well as $x(\delta^+(j)) = 1$. They are only facet defining for the asymmetric traveling salesman polytope $P_C^n(D_n)$ (see Grötschel [17]). This is possible, since the degree constraints (2.2) are satisfied at equality by all tours in D_n.

Linear ordering constraints

Theorem 3.24. *Let $D_n = (V, A)$ be the complete digraph on n nodes. For any permutation π of the nodeset V, the inequality (2.7)*

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i),\pi(j)} \geq 1$$

defines a facet of $P_C^L(D_n)$ if and only if one of the following conditions holds:

(i) $P_C^L(D_n) = P_C(D_3)$.

(ii) $k \geq 4$, $L = \{2, k\}$, *and* $n \geq 2k - 2$.

(iii) $2 \in L$, $|L| \geq 3$, *and* $n \geq 2k - 3$ *where* $k := \min\{l \in L \mid l > 2\}$.

Proof. Assume w.l.o.g. that $(\pi(1), \dots, \pi(n)) = (1, \dots, n)$, that is, that (2.7) is (2.3). Further, set $A^* := \{(3, 2), \dots, (n, n-1)\}$.

Sufficiency.

(i): Constraint (2.3) defines obviously a facet of $P_C(D_3)$.

(ii) and (iii): Every 2-circuit satisfies (2.3) at equality, .i.e., $b_{uv} + b_{vu} = b_0$ for all $(u, v) \in A$; hence, (2.3) defines a nonempty face F of $P_C^L(D_n)$. In order to show that F is a facet of $P_C^{\{2,k\}}(D_n)$, assume that there is a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, in 1-rooted form satisfying

$$\{x \in P_C^L(D_n) \mid \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{ij} = 1\} \subseteq \{x \in P_C^L(D_n) \mid b^T x = b_0\}.$$

First, consider the coefficients b_{1v}, $v = 2, \dots, n$. Since $b_{1v} + b_{v1} = 0$ and $b_{v1} = 0$, we get $b_{1v} = b_0$ for all $v \in \{2, \dots, n\}$.

Next consider the coefficients b_a, $a \in A^*$.

If (ii) is true, we consider the k-circuits $(n-1, n-2, \dots, n-k+3, v+1, v, 1, n-1)$ and $(n, n-1, \dots, n-k+3, v+1, 1, n)$ for $v = 2, \dots, k-1$. We obtain $b_{v+1,v} = b_{n,n-1}$, $v = 2, \dots, k-1$, and thus $b_{32} = \dots = b_{k,k-1}$. The circuit $(k, k-1, \dots, 1, k)$ yields then $b_{32} = \dots = b_{k,k-1} = 0$, since $b_{1k} = b_0$ and $b_{21} = 0$, and with the circuits $(v, v-1, \dots, v-k+2, 1, v)$, $v = k+1, \dots, n$, we can conclude successive $b_{v,v-1} = 0$, $v = k+1, \dots, n$.

If (iii) is true, we consider the k-circuits $(n-1, n-2, \dots, n-k+3, v+1, v, 1, n-1)$ and $(n, n-1, \dots, n-k+3, v+1, 1, n)$ for $v = 2, \dots, k-2$. We obtain

$$b_{32} = \dots = b_{k-1,k-2} = b_{n,n-1}. \tag{3.26}$$

Further, we derive from the circuits $(w, w-1, \dots, w-k+4, 3, 2, 1, w)$ and $(w+1, w, \dots, w-k+4, 3, 1, w+1)$ for $w = k, \dots, n-1$

$$b_{32} = b_{k+1,k} = \dots = b_{n,n-1}. \tag{3.27}$$

By (3.26) and (3.27), it follows that

$$b_{32} = \cdots = b_{k-1,k-2} = b_{k+1,k} = \cdots = b_{n,n-1}. \tag{3.28}$$

Since there is $m \in L$ with $m > k$, we derive from the circuits $(k, k-1, \ldots, 1, k)$ and $(m, m-1, \ldots, 1, m)$

$$\begin{aligned} \sum_{i=k}^{m-1} b_{i+1,i} &= 0 \\ \overset{(3.27)}{\Rightarrow} \quad b_{i+1,i} &= 0, \qquad i = k, \ldots, m-1 \\ \overset{(3.28)}{\Rightarrow} \quad b_{i+1,i} &= 0, \qquad i = 2, \ldots, k-2, k, \ldots, m-1, \end{aligned}$$

and again the circuit $(k, k-1, \ldots, 1, k)$ yields also $b_{k,k-1} = 0$.

Next, consider the coefficients b_{uv} and b_{vu} for $1 < u < v \leq n$ such that $2 \leq v - u \leq n - k + 2$. Since the nodeset $\{1, \ldots, u, v, \ldots, n\}$ is of cardinality at least k, there is a k-circuit containing (v, u) whose remaining arcs are in

$$(A^* \setminus \{(v, v-1), \ldots, (u+1, u)\}) \cup \{(2, 1), \ldots, (u, 1)\} \cup \{(1, v), \ldots, (1, n)\}.$$

Clearly, it follows immediately that $b_{vu} = 0$, and hence we conclude $b_{uv} = b_0$.

Finally, consider the coefficients b_{uv} and b_{vu} for $1 < u < v \leq n$ such that $n - k + 3 \leq v - u \leq n$. The nodeset $\{u, u+1, \ldots, v\}$ is of cardinality at least $k+1$. From the k-circuit $(v, v-1, \ldots, v-k+2, u, v)$ we obtain $b_{uv} = b_0$, since $b_{v,v-1} = \cdots = b_{v-k+3,v-k+2} = b_{v-k+2,u} = 0$. Moreover, this implies $b_{vu} = 0$.

Clearly, $b_0 > 0$, and hence we have shown that the inequality $b^T x \geq b_0$ is equivalent to (2.3) up to multiplication with a positive scalar. This proves that (2.3), and hence, (2.7), is facet defining.

Necessity. For $2 \leq n \leq 3$ the statement is obviously true. Hence let $n \geq 4$, and let us suppose, for the sake of contradiction, that (ii) or (iii) is not true.

a) Assume that $2 \notin L$.
Then there is no point $x \in F$ with $x_{12} > 0$, contrary to Lemma 3.17 (a).

b) Assume that $|L| = 1$.
Since $2 \in L$, it follows $L = \{2\}$. But then (2.3) is satisfied at equality by all $x \in P_C^2(D_n)$. Thus it is not a facet defining inequality and hence $|L| \geq 2$.

c) Assume that $k \leq n \leq 2k - 4$.
Consider the arcs (u, v) and (v, u) given by $u := 2$ and $v := n - k + 4$. Since $2 \leq v - u = n - k + 2 \leq k - 2$, there is for any $l, m \in L$, with $l, m \geq k$, neither a l-circuit containing (u, v) nor a m-circuit containing (v, u) whose incidence vectors satisfy (2.3) at equality, that is, only the 2-circuit on $\{(u, v, u)\}$ satisfies (2.3) at equality. But this is a contradiction to Lemma 3.17 (b).

d) Suppose that $n = 2k - 3$ and $L = \{2, k\}$.

We consider the polytope P^* defined by

$$P^* := \operatorname{conv}(P_C^L(D_n) \cup C^*)$$

where C^* is the triangle $(1,3,2,1)$. Note that $\dim P^* = \dim P_C^L(D_n)$. We will first show that inequality (2.3) defines a facet F^* of P^* and then that it is not facet defining for $P_C^{\{2,k\}}(D_n)$.

Let us assume that $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, is a valid inequality for P^*, with

$$\{x \in P^* \mid x \text{ satisfies (2.3) at equality}\} \subseteq \{x \in P^* \mid b^T x = b_0\}.$$

As is easily seen, it follows $b_{1v} = b_0$, $v = 2, \ldots, n$. In order to show $b_{v+1,v} = 0$, $v = 2, \ldots, n-1$, consider the k-circuits

$$\begin{array}{ll} (n-1, n-2, \ldots, n-k+3, v+1, v, 1, n-1), & \\ (n, n-1, \ldots, n-k+3, v+1, 1, n), & \\ & v = 2, \ldots, k-2. \end{array}$$

We obtain $b_{n,n-1} = b_{32} = b_{43} = \cdots = b_{k-1,k-2}$. Now the triangle $(1,3,2,1)$ yields $b_{32} = 0$, and hence $b_{n,n-1} = b_{32} = b_{43} = \cdots = b_{k-1,k-2} = 0$. Further, the circuit $(k, k-1, \ldots, 1, k)$ yields $b_{k,k-1} = 0$.

The remaining coefficients can be determined as in the part *Sufficiency*, (ii) and (iii), since all arguments hold also for $n = 2k-3$ and $L = \{2, k\}$. Hence, F^* is a facet of P^*.

Now we will prove that (2.3) is not facet defining for $P_C^L(D_n)$, $n = 2k-3$, by showing that $C^* \notin \operatorname{aff}(F)$. The crucial point is, that for $n = 2k-3$ the k-circuits satisfying (2.3) at equality are linearly independent of the 2-circuits. For $n \geq 2k-2$ this is no longer true.

Let us denote by $F(k)$ the k-circuits whose incidence vectors satisfying (2.3) at equality. Suppose, for the sake of contradiction, that

$$\begin{array}{rcl} \chi^{(1,3,2,1)} & = & \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_{ij} \chi^{\{(i,j,i)\}} + \sum_{C \in F(k)} \mu_C \chi^C \\ & & \text{and } \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \lambda_{ij} + \sum_{C \in F(k)} \mu_C = 1. \end{array}$$

Since the component 13 is contained only in the incidence vectors of the circuits $(1,3,2,1)$ and $\{(1,3,1)\}$, it follows $\lambda_{13} = 1$. However, $\chi_{ij}^{(1,3,2,1)} = 0$ and $\{C \in F(k) \mid (i,j) \in C\} = \emptyset$ for all $1 \leq i < j \leq n$, $(i,j) \neq (1,3)$, with $j - i \leq k-1$, and hence $\lambda_{ij} = 0$ for those components ij. Analogous follows $\lambda_{ij} = 0$ for all $1 \leq i < j \leq n$ with $j - i \geq k$, since $\chi_{ij}^{(1,3,2,1)} = 0$ and $\{C \in F(k) \mid (i,j) \in C\} = \emptyset$ for those components ij. Thus,

$$\begin{array}{rcl} \chi^{(1,3,2,1)} & = & \chi^{\{(1,3,1)\}} + \sum_{C \in F(k)} \mu_C \chi^C \\ \Leftrightarrow \quad e_{32} + e_{21} - e_{31} & = & \sum_{C \in F(k)} \mu_C \chi^C \end{array}$$

$$\begin{aligned}
\Rightarrow\ 1 &= \mathbf{1}^T \sum_{C\in F(k)} \mu_C \chi^C\\
\Rightarrow\ 1 &= \sum_{C\in F(k)} \mu_C \underbrace{\mathbf{1}^T\chi^C}_{=k}\\
\Rightarrow\ 1 &= k \underbrace{\sum_{C\in F(k)} \mu_C}_{=0}\\
\Rightarrow\ 1 &= 0. \qquad \text{Contradiction!}
\end{aligned}$$

□

Disjoint circuits elimination constraints

As we have already mentioned in Chapter 2, the subtour elimination constraints (2.4)

$$x(\delta^+(p)) + x(\delta^+(q)) - x((S:T)) \ \le 1 \quad S \subseteq V, T = V \setminus S, p \in S, q \in T$$

are related to the subtour elimination constraints for the asymmetric traveling salesman polytope, and equivalent to them if $P_C^L(D_n) = P_C^n(D_n)$. Moreover, the subtour elimination constraints are known to be facet defining for $P_C^n(D_n)$ if $2 \le |S| \le n-2$ (see Grötschel [17]).

First we investigate in which cases constraints (2.4) define facets of $P_C^k(D_n)$. If $S = \emptyset$ or $S = \{p\}$ then (2.4) is equivalent to $x(\delta^+(q)) \le 1$ (respectively $x(\delta^+(p)) \le 1$ for $T = \emptyset$ or $T = \{q\}$). Hence we assume for this section $2 \le |S| \le n-2$ and thus $n \ge 4$.

Theorem 3.25. *Let $D_n = (V, A)$, $n \ge 4$, be the complete digraph on n nodes, $k \in \{2, \ldots, n\}$, S, T be a bipartition of V, $2 \le |S| \le n-2$, $p \in S$, and $q \in T$. Then the inequalities* (2.4) *define facets of $P_C^k(D_n)$ if and only if*

(i) $k = n$, *or*

(ii) $4 \le k < n$ *and* $|S|, |T| \ge k$.

Proof. Set

$$\begin{aligned}
X_p^k &:= \{\chi^C \mid C \in \mathcal{C}^k(D_n), p \in V(C) \subseteq S\},\\
X_q^k &:= \{\chi^C \mid C \in \mathcal{C}^k(D_n), q \in V(C) \subseteq T\}, \text{ and}\\
X_{pq}^k &:= \{\chi^C \mid C \in \mathcal{C}^k(D_n), p, q \in V(C), |C \cap (S:T)| = 1\}.
\end{aligned}$$

A vector χ^C, $C \in \mathcal{C}^k(D_n)$, satisfies (2.4) at equality if $\chi^C \in X_p^k \cup X_q^k \cup X_{pq}^k$.

Necessity. Suppose that $k = 2$. Then the inequalities (2.4) are not facet defining for $P_C^2(D_n)$, $n \ge 4$, since the cardinality of $X_p^2 \cup X_q^2 \cup X_{pq}^2$ is only $n-1$. Contradiction.

Similar is the situation for $k = 3$. The cardinality of $X_p^3 \cup X_q^3 \cup X_{pq}^3$ is less than $\dim P_C^3(D_n)$. Contradiction.

Next, suppose, for the sake of contradiction, that $4 \le k \le n-1$ and $2 \le |S| \le k-1$ or $2 \le |T| \le k-1$. Let w.l.o.g. $2 \le |T| \le k-1$. Then follows that $X_q^k = \emptyset$, and thus

$$F := \operatorname{conv}(X_p^k \cup X_{pq}^k) \subseteq \{x \in P_C^k(D_n) \mid x(\delta^+(p)) = 1\}.$$

However, $x \in P_C^k(D_n)$, with $x(\delta^+(p)) = 1$, $x(\delta^+(q)) = 0$, and $x((S:T)) = 1$, is not in the affine hull of F. Hence, by Lemma 3.16, $\dim F < \dim\{x \in P_C^k(D_n) \mid x(\delta^+(p)) = 1\}$ and thus not a facet.

Suffiency. If $k = n$, then the theorem holds obviously. So let (ii) be true. Note that (ii) implies $n \geq 2k$.

Set $S' := S \setminus \{p\}$ and $T' := T \setminus \{q\}$. In order to show that F is a facet of $P_C^k(D_n)$ we consider the polytope P^* defined by

$$P^* := \text{conv}(P_C^k(D_n) \cup \{C^*\})$$

where C^* is a triangle (p, q, t^*, p) for some $t^* \in T'$. First we will show that the face F^* which is the convex hull of all incidence vectors satisfying (2.4) is a facet of P^*. This implies immediately that F is a facet of $P_C^k(D_n)$.

Suppose that we have a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, such that

$$\{x \in P^* \mid x \text{ satisfies (2.4) at equality}\} \subseteq \{x \in P^* \mid b^T x = b_0\},$$

and we may assume that $b^T x \geq b_0$ is in p-rooted form. The p-rooted form of (2.4) is illustrated in Figure 3.2.

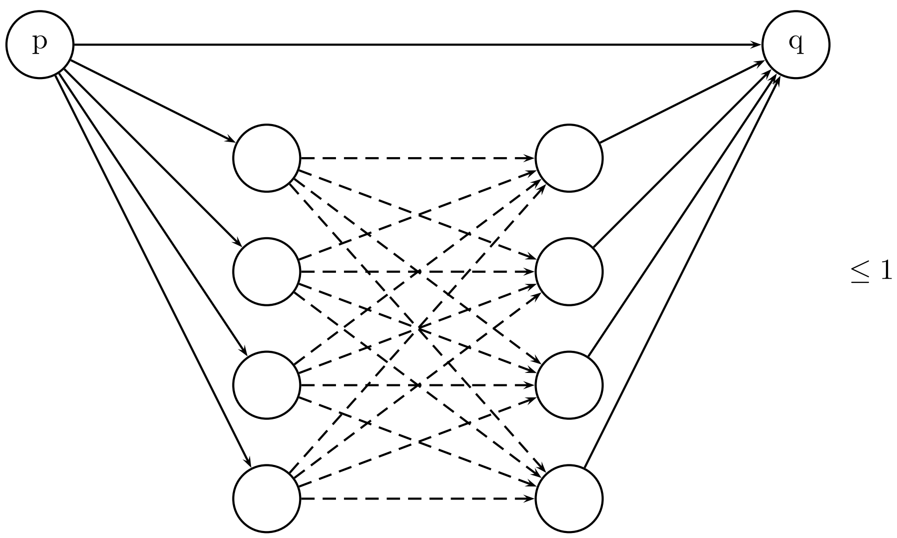


Figure 3.2

First we show that

$$b_{pq} + b_{qi} = b_0 \qquad \forall\, i \in S' \cup T'. \tag{3.29}$$

Let $i \in S' \cup T'$, and consider any circuit $C \in \mathcal{C}^k(D_n)$, with $(q,i), (i,p) \in C$ and $t^* \notin V(C)$, whose incidence vector satisfies (2.4) at equality. Such C obviously exists for all $k \in \{4, \ldots, n-1\}$.

Then the circuit $C' := (C \setminus \{(q,i),(i,p)\}) \cup \{(q,t^*),(t^*,p)\}$ is in $\mathcal{C}^k(D_n)$ and satisfies also (2.4) at equality. Thus,

$$\begin{array}{rrcl} & b_{qi} + \underbrace{b_{ip}}_{=0} & = & b_{qt^*} + \underbrace{b_{t^*p}}_{=0} \\ \Leftrightarrow & b_{qi} & = & b_{qt^*}. \end{array}$$

Since $b^T\chi^{C^*} = b_0$ and $b_{t^*p} = 0$, we obtain $b_{pq} + b_{qt^*} = b_0$ and consequently the desired result.

Next we show that

$$b_a = 0 \qquad \forall\, a \in A^* := A(S') \cup A(T') \cup (T' : S'). \tag{3.30}$$

Let (i,j), $(g,h) \in A^*$, and let $C \in \mathcal{C}^l(D_n)$ be any circuit satisfying (2.4) at equality, with $(q,i),(i,j),(j,p) \in C$ and $u,v \notin V(C)\setminus\{i,j\}$. We may assume that $(j,g) \notin (S' : T')$. (Otherwise $(g,j) \notin (S' : T')$, and we reverse the roles of (i,j) and (g,h).) If $g = i$ then the circuit $C_i := (C \setminus \{(i,j),(j,p)\}) \cup \{(i,h),(h,p)\}$ satisfies (2.4) at equality, and thus

$$\begin{array}{rrcl} & b_{ij} + b_{jp} & = & b_{ih} + b_{hp} \\ \overset{3.31}{\Leftrightarrow} & b_{ij} & = & b_{gh}. \end{array}$$

If $g = j$ then the circuit $C_j := (C \setminus \{(q,i),(i,j),(j,p)\}) \cup \{(q,j),(j,h),(h,p)\}$ yields $b_{gh} = b_{ij}$. Finally, if $g \notin \{i,j\}$ then the circuit $C' := (C \setminus \{(q,i),(i,j),(j,p)\}) \cup \{(q,j),(j,g),(g,p)\}$ yields $b_{jg} = b_{ij}$. Analogous follows $b_{gh} = b_{gj}$, and thus, $b_{gh} = b_{ij}$. Hence, a circuit $\tilde{C}$ satisfying (2.4) at equality, with $(s,p),(p,q),(p,t) \in \tilde{C}$ for some $s \in S', t \in T'$, and all other arcs in A^* yields $b_a = 0$ for all $a \in A^*$, since $b_{sp} = 0$, $b_{pq} + b_{qt} = b_0$, and $b_{ij} = b_{gh}$ for all $(i,j),(g,h) \in A^*$.

The remaining coefficients can be easily determined. One can show by considering appropriate circuits $C \in \mathcal{C}^k(D_n)$ satisfying (2.4) at equality (in brackets)

$$\begin{array}{rcllll} b_{pi} & = & b_0 & \forall\, i \in S', & & ((p,i) \in C, V(C) \subseteq S) \\ b_{pq} & = & 0 & \forall\, i \in S', & & ((i,q),(q,p) \in C, V(C) \setminus \{q\} \subseteq S) \\ b_{qi} & = & 0 & \forall\, i \in S' \cup T', & & ((q,i),(i,p) \in C, V(C) \setminus \{q,i\} \subseteq S) \\ b_{iq} & = & b_0 & \forall\, i \in T' \cup \{p\}, & & ((i,k+1) \in C, V(C) \subseteq T \cup \{p\}) \\ b_{pi} & = & 0 & \forall i \in T', & & ((p,i),(i,q) \in C, V(C) \subseteq T \cup \{p\}) \\ b_{ij} & = & -b_0 & \forall\, i \in S', j \in T'. & & ((p,i),(i,j),(j,q) \in C, V(C) \setminus \{p,i\} \subseteq T) \end{array}$$

Finally, it follows immediately that $b_0 > 0$, and thus, $b^Tx \le b_0$ is equivalent to (2.4). This proves that F^* is a facet of the polytope P^*. That is, F^* contains $n^2 - 2n + 1$ affinely independent vertices and exactly one of them is χ^{C*}. Thus, the other vertices in F^* are incidence vectors of k-circuits. This implies that F is a facet of $P_C^k(D_n)$. □

Next, let us consider the case $|L| \ge 2$.

Theorem 3.26. *Let $D_n = (V, A)$, $n \ge 4$, be the complete digraph on n nodes, S, T be a bipartition of V, $2 \le |S| \le n-2$, $p \in S$, $q \in T$, and $|L| \ge 2$. Further, set $k := \min\{l \in L\}$. Then the inequalities* (2.4) *define facets of $P_C^L(D_n)$ if and only if*

(a) $k = 2$ and $\{2,3\} \ne L \ne \{2,n\}$, or

(b) $k \ge 3$ and $|S|, |T| \ge k$.

Proof.

Necessity. Suppose that $L = \{2,3\}$. Then it is easy to see that the face F induced by (2.4) is not a facet, since for any $a \in A$ not incident with p and q we have

$$F \subseteq \{x \in P_C^{\{2,3\}} \mid x_a = 0\}.$$

Further, suppose that $L = \{2,n\}$. Then an inequality (2.4), with $2 \leq |S| \leq n-2$, defines not a facet of $P_C^{\{2,n\}}(D_n)$, since the number of affinely independent tours in $X_p^n \cup X_q^n \cup X_{pq}^n$ is equal to $n^2 - 3n + 1$, the number of 2-circuits in $X_p^2 \cup X_q^2 \cup X_{pq}^2$ is equal to $n-1$, and thus we have only $n^2 - 2n$ affinely independent circuits in F.

Finally, suppose that $k \geq 3$ and $|S| \leq k-1$ or $|T| \leq k-1$. Then an analogous argument holds as in the proof to Theorem 3.25.

Suffiency. Let (a) be true. Since $\{2,3\} \neq L \neq \{2,n\}$, there is $l \in L, 4 \leq l \leq n-1$. Set $S' := S \setminus \{p\}$ and $T' := T \setminus \{q\}$. Further, since $n \geq 5$, $|S'|$ or $|T'|$ is of cardinality greater than or equal to 2, say $|T'|$. In order to show that (2.4) defines a facet of $P_C^L(D_n)$ we suppose that we have a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, such that

$$\{x \in P_C^L(D_n) \mid x \text{ satisfies (2.4) at equality}\} \subseteq \{x \in P_C^L(D_n) \mid b^T x = b_0\},$$

and we may assume that $b^T x \geq b_0$ is in p-rooted form, i.e.,

$$b_{ip} = 0 \qquad \forall\, i \in V \setminus \{p\}. \tag{3.31}$$

Since $b_{pi} + b_{ip} = b_0$ and $b_{ip} = 0$ for all $i \in S' \cup \{q\}$, we conclude $b_{pi} = b_0$.

Now we show

$$b_{qi} = b_{qj} \qquad \forall\, i,j \in S' \cup T'. \tag{3.32}$$

Let $i,j \in S' \cup T'$, and consider any circuit $C \in \mathcal{C}^l(D_n)$, with $(q,i),(i,p) \in C$ and $j \notin V(C)$, whose incidence vector satisfies (2.4) at equality. Such C obviously exists for all $l \in \{4,\ldots,n-1\}$. Then the circuit $C' := (C \setminus \{(q,i),(i,p)\}) \cup \{(q,j),(j,p)\}$ is in $\mathcal{C}^l(D_n)$ and satisfies also (2.4) at equality. Thus,

$$\begin{aligned} b_{qi} + \underbrace{b_{ip}}_{=0} &= b_{qj} + \underbrace{b_{jp}}_{=0} \\ \Leftrightarrow \qquad b_{qi} &= b_{qj}. \end{aligned}$$

Further, since $b_{hq} + b_{qh} = b_0$ for all $h \in T'$ and $b_{qi} = b_{qj}$ for all $i,j \in S' \cup T'$ it follows that $b_{uq} + b_{qv} = b_0$ for all $u \in T'$, $v \in S' \cup T'$. Moreover, we obtain

$$b_{iq} = b_{jq} \qquad \forall\, i,j \in T'. \tag{3.33}$$

Now one can show

$$b_{gh} = b_{ij} \qquad \forall\, (g,h),(i,j) \in A^* := A(S') \cup A(T') \cup (T' : S') \tag{3.34}$$

as in the proof to Theorem 3.25.

In order to show $b_a = 0$ for all $a \in A^*$ consider a circuit $C \in \mathcal{C}^l(D_n)$ satisfying (2.4) at equality, with $(t,q),(q,u),(u,p) \in C$ for some $t,u \in T'$. Then the circuit $C' := (C \setminus \{(t,q),(q,u),(u,p)\}) \cup \{(t,u),(u,q),(q,p)\}$ satisfies also (2.4) at equality, and thus

$$\begin{aligned} b_{tq} + b_{qu} + b_{up} &= b_{tu} + b_{uq} + b_{qp} \\ \overset{3.31}{\Leftrightarrow} \qquad b_{tq} + b_{qu} &= b_{tu} + b_{uq} \\ \overset{3.33}{\Leftrightarrow} \qquad b_{qu} &= b_{tu}. \end{aligned}$$

Hence, a circuit C^* satisfying (2.4) at equality, with $(p,q) \in C$, $(s,p) \in C$ for some $s \in S'$, and all other arcs in A^* yields $b_a = 0$ for all $a \in A^*$.

The remaining coefficients can be easily determined. Then it follows immediately that (2.4) defines a facet of $P_C^L(D_n)$.

Now let (b) be true.

Case 1: $L = \{3, n\}$

Let w.l.o.g. $\{1,2\} \subseteq S$ and $p \notin \{1,2\}$. Since (2.4) defines a facet of $P_C^n(D_n)$ and $\dim P_C^n(D_n) = n^2 - 3n + 1$, there are $n^2 - 3n + 1$ linearly independent vectors $x^u \in P_C^{\{3,n\}}(D_n)$ satisfying (2.4) at equality.

Next consider any $x^p \in X_p^3$, $x^q \in X_q^3$, and $n-2$ further points $x^v \in X_{pq}^2$, one for each $v \in V \setminus \{p,q,1,2\}$, such that x^v is the incidence vector of the triangle (p,v,q,p). Then x^p, x^q, and all x^v satisfy (2.4) at equality.

We bear that in mind that to each point $x \in P_C(D_n)$ exists a point $\begin{pmatrix} x \\ y \end{pmatrix} \in P_0(D_n^0)$. It is now easy to see that the matrix whose rows are the extended incidence vectors is of the form

$$Z = \begin{pmatrix} T & \mathbf{0} \\ X & Y \end{pmatrix},$$

where T is the matrix whose rows are the $n^2 - 3n + 1$ linearly independent tours, (X, Y) the matrix whose rows are the vectors $\begin{pmatrix} x^q \\ y^q \end{pmatrix}^T$, $\begin{pmatrix} x^p \\ y^p \end{pmatrix}^T$, and $\begin{pmatrix} x^v \\ y^v \end{pmatrix}^T$, $v \in V \setminus \{p,q\}$. Y is of the form

$$Y = \begin{pmatrix} \overset{q}{1} & \overset{p}{0} & \\ 0 & 1 & * \\ & & \\ \mathbf{0} & \mathbf{0} & (E-I)_{(n-k+1)\times(n-k+1)} \end{pmatrix}$$

after a suitable arrangement of the columns, where E is the matrix of all ones and I is the identity matrix. Y is of rank n and thus Z of rank $n^2 - 2n + 1$. Thus, by Lemma 3.16, (2.4) defines a facet of $P_C^{\{3,n\}}(D_n)$.

Case 2: $L \neq \{3, n\}$

If $k \geq 4$ then the claim follows by Theorem 3.25 and Lemma 3.9. If $k = 3$ and $L \neq \{3, n\}$ then we know that there is $m \in L$, $4 \leq m \leq n-1$. Thus we can apply Theorem 3.25 and Lemma 3.9 again. □

From the remaining classes of inequalities considered in 2.1 we investigate only the lifted primitive SD and the clique tree inequalities.

Primitive SD inequalities

The next theorem is easily obtained by looking for the shortest circuit satisfying a primitive SD inequality at equality.

Theorem 3.27. *Let $n \geq 7$, $||S| - |D|| \leq \max\{0, s-3\}$, and*

$$l := 2\min\{|S|, |D|\} + 3||S| - |D|| + 4\,\frac{s - 1 - ||S| - |D||}{2}.$$

Then the primitive SD inequality (2.13)

$$x((S : H \cup D)) + \sum_{j=1}^{s} x(A(T_j)) - x((H : V \setminus (D \cup H))) \leq (|S| + |D| + s - 1)/2$$

defines a facet of $P_C^{\geq k}(D_n)$ for all $k \in \{2, \ldots, l\}$.

Proof. By Theorem 2.8, (2.13) is facet defining for $P_C(D_n)$. To complete the proof we show that each circuit satisfying (2.13) at equality is of length at least l.

A circuit of minimum length satisfying (2.13) at equality contains exactly $(|S|+|D|+s-1)/2$ arcs whose associated coefficients are equal to 1, no arcs with coefficient -1, and as less as possible arcs with coefficients equal to zero. Let us denote by G the digraph obtained from D_n by removing all arcs with coefficient equal to -1.

Since the arcs with coefficient equal to 1 are only the arcs in $T := \{(v_j, u_j) | j = 1, \ldots, s\}$ and the arcs in $(S : H \cup D)$, the arcset $C \cap (S : H \cup D) \cap T$ is disconnected for any circuit C.

To join two disconnected arcs $(u, v) \in (S : D), (w, z) \in (S : D \cup H) \cup T$ in G requires at least one arc and joining two disconnected arcs $(u, v) \in (S : H) \cup T, (w, z) \in (S : H) \cup T$ in G requires at least two arcs. However, if a circuit C contains more than $|D|$ arcs in $(S : D) \cup T$, then joining two arcs $(u, v), (w, z) \in T$ or $(u, v) \in (S : D \cup H)$, $(w, z) \in T$ requires for some of such pairs at least three arcs.

Case 1: $|S| \geq |D|$

Let C be a circuit in G with $|D|$ arcs in $(S : D)$, $|S| - |D|$ arcs in $(S : H)$, and $(s - 1 + |D| - |S|)/2$ arcs in T. (Note that $(s - 1 + |D| - |S|)/2$ is nonnegative.) Then C contains exactly $(|S| + |D| + s - 1)/2$ arcs whose associated coefficients are equal to 1 and its length is at least $2|D| + 3(|S| - |D|) + 4\frac{s-1+|D|-|S|}{2} = l$.

Case 2: $|D| \geq |S|$

Then there is a circuit C in G with $|S|$ arcs in $(S : D)$ and $(s - 1 + |D| - |S|)/2$ arcs in T. Again, C contains exactly $(|S| + |D| + s - 1)/2$ arcs whose associated coefficients are equal to 1 and its length is at least $2|S| + 3(|D| - |S|) + 4\frac{s-1+|S|-|D|}{2} = l$. ☐

Clique tree inequalities

Theorem 3.28. *Let $n \geq 7$ and $l = 3\frac{t-1}{2} + 1$. Then the primitive clique tree inequality*

$$\sum_{j=1}^{t} x(A(T_j)) - \sum_{i=1}^{h} \sum_{v \in H_i} x((v : V \setminus H_i)) \leq \frac{t-1}{2} \tag{3.35}$$

defines a facet of $P_C^{\geq k}(D_n)$ for all $k \in \{2, \ldots, l\}$.

Proof. Recall, each tooth T_j of a primitive clique tree has exactly one node r_j not contained in any handle and every nonempty intersection of a handle H_i and T_j contains exactly one node s_{ij}. Now it is not hard to see that a coefficient of the inequality (3.35) associated to an arc a is equal to one if and only if $a = (r_j, s_{ij})$ for some $i \in \{1, \dots, h\}$, $j \in \{1, \dots, t\}$.

Clearly, a circuit C of minimum length satisfying (3.35) at equality contains exactly $(t-1)/2$ arcs of the form (r_j, s_{ij}), no arcs whose associated coefficients are equal to -1, and a minimum number of arcs whose associated coefficients are equal to zero. In particular, such a circuit C covers no node which is not contained in any tooth. Further, since a circuit cannot contain an arc $(r_j, s_{i_1 j})$ and at the same time an arc $(r_j, s_{i_2 j})$, C covers exactly $(t-1)/2$ nodes r_j, say r_j, $j = 1, \dots, (t-1)/2$, such that $(r_j, s_{ij}) \in C$ for an appropriate i.

Now define the subgraph $D = (V', A')$ of D_n by removing all nodes not contained in any tooth and all arcs whose associated coefficients are equal to -1. In this digraph two disconnected arcs $(r_{j_1}, s_{i_1,j_1}), (r_{j_2}, s_{i_2,j_2})$ can be always joined to a directed path by three arcs $(s_{i_1,j_1}, s_{i_1,j_3}), (s_{i_1,j_3}, r_{j_3})$, and (r_{j_3}, r_{j_2}) for some $r_{j_3} \neq r_{j_2}$. But they can be joined to a directed path by only two arcs if $i_1 \neq i_2$ and $T_{j_2} \cap H_{i_1} \neq \emptyset$.

Thus, if there are $\frac{t-1}{2}$ arcs (r_{j_p}, s_{ij}) which can be joined to a directed path P in D of length $3\frac{t-1}{2} - 2$, then there is a circuit satisfying (3.35) at equality which contains only $3\frac{t-1}{2} + 1$ arcs. Clearly, this result can be strengthened, since in generally there are not so many arcs (r_{j_p}, s_{ij}) which can be joined to a directed path in D of length $3\frac{t-1}{2} - 2$. $\square$

3.5 The connection between directed and undirected circuit polytopes

To study the relation between directed and undirected circuit polytopes is an important matter, since it could be useful to generate facet defining inequalities for a circuit polytope of a directed graph from facet defining inequalities for the corresponding circuit polytope of an undirected graph, and conversely. For valid inequalities there is an easy connection which we will elaborate first of all.

A valid inequality $a^T x \leq a_0$ for $P_C^L(D_n)$ is called *symmetric* when $a_{ij} = a_{ji}$ for all $i, j \in V$, $i \neq j$. Symmetric inequalities can be thought of as derived from valid inequalities for $P_C^L(K_n)$. Given a valid inequality $b^T y \leq b_0$ for $P_C^L(K_n)$, it can be transformed into a valid inequality for $P_C^L(D_n)$ by simply replacing y_e by $x_{ij} + x_{ji}$ and setting for all $e = [i, j] \in E$. (Note $2 \notin L$.) This produces the symmetric inequality $\tilde{b}^T x \leq \tilde{b}_0$, where $\tilde{b}_{ij} = \tilde{b}_{ji} = b_{[i,j]}$ for all $i, j \in V$, $i \neq j$, and $\tilde{b}_0 = b_0$. Conversely, every symmetric inequality $b^T x \leq b_0$ which is valid for $P_C^L(D_n)$ corresponds to the inequality $\sum_{[i,j] \in E} b_{ij} y_{[i,j]} \leq b_0$. Moreover, each inequality which is valid for $P_C^L(D_n)$, $2 \notin L$, can be transformed into a valid inequality for $P_C^L(K_n)$:

Theorem 3.29. *Let $2 \notin L$ and $b^T x \leq b_0$ be a valid inequality for $P_C^L(D_n)$. Then the inequality*

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (b_{ij} + b_{ji}) y_{[i,j]} \leq 2b_0$$

is valid for $P_C^L(K_n)$.

Proof.

$$
\begin{array}{rrlll}
& \sum_{(i,j)\in A} b_{ij}x_{ij} & \le b_0 & \text{valid for } P_C^L(D_n) & \text{(i)}\\
\Leftrightarrow & \sum_{(i,j)\in A} b_{ji}x_{ij} & \le b_0 & \text{valid for } P_C^L(D_n) & \text{(ii)}\\
\overset{\text{(i)+(ii)}}{\Rightarrow} & \sum_{(i,j)\in A} [b_{ij}+b_{ji}]x_{ij} & \le 2b_0 & \text{valid for } P_C^L(D_n) & \text{(iii)}\\
\overset{\text{sym.}}{\Leftrightarrow} & \sum_{[i,j]\in E} [b_{ij}+b_{ji}]y_{[i,j]} & \le 2b_0 & \text{valid for } P_C^L(K_n) & \text{(iv)}
\end{array}
$$

□

However, if an inequality $b^T y \le b_0$ is facet defining for $P_C^L(K_n)$, then the corresponding symmetric inequality $\tilde{b}^T x \le \tilde{b}_0$ is not necessary facet defining. For example, the matching inequality (see 3.6)

$$y(E) - y_{12} - y_{34} - y_{56} \ge 2$$

is facet defining for $P_C(K_n)$, but the corresponding symmetric inequality

$$x(A) - x_{12} - x_{21} - x_{34} - x_{43} - x_{56} - x_{65} \ge 2$$

is not facet defining for $P_C^{\ge 3}(D_n)$. Conversely the same: the inequality

$$2x_{21} - 2x_{32} + 2x_{25} - 2x_{52} + x_{43} + x_{35} + 2x_{45} - x_{54} \le 2$$

is facet inducing for $P_C^{\ge 3}(D_5)$ but not the corresponding undirected inequality

$$2x_{12} - 2x_{23} + x_{34} + x_{35} + x_{45} \le 4.$$

On the other hand, there are a lot of facet defining inequalities for $P_C^(K_n)$ that can be derived from those for $P_C^{\ge 3}(D_n)$. For example, the sum of the degree constraints $x(\delta^+(v)) \le 1$ and $x(\delta^-(v)) \le 1$ corresponds to the degree constraint $y(\delta(v)) \le 2$, which is known to be facet defining for $P_C(K_n)$. Or, the sum of the dce constraints $x(\delta^+(p)) + x(\delta^+(q)) - x((S : V \setminus S)) \le 1$ and $x(\delta^-(p)) + x(\delta^-(q)) - x((V \setminus S : S)) \le 1$ corresponds to the dce constraint $y(\delta(p)) + y(\delta(q)) - y((S : V \setminus S)) \le 2$, which is also facet defining for $P_C(K_n)$ if $|S|, |V \setminus S| \ge 3$. So it is an interesting issue if there is a procedure (or only a criterion) in order to derive facet defining inequalities for $P_C^L(D_n)$ from those for $P_C^L(K_n)$, and conversely. For one direction (directed to undirected) we can respond partly the question.

The answer is based on three observations: Firstly, the facets of $P_C^L(K_n)$ $(2 \notin L)$ not induced by a nonnegativity constraint are in 1-1 correspondence with those of $P_W^L(D_n)$. Secondly, facets of $P_C^L(D_n)$ can be lifted to polyhedra $P \subseteq P_W(D_n)$ by relaxing the flow constraints. Thirdly, facet defining inequalities for $P_C^L(D_n)$ which are equivalent to symmetric inequalities can be lifted to facet defining inequalities for $P_W^L(D_n)$.

In the following we will discuss the three items. Since we search a procedure to derive facet defining inequalities for $P_C^L(D_n)$ from those for $P_C^L(K_n)$, let in the following considerations $2 \notin L$.

The connection between $P_C^L(K_n)$ and $P_W^L(D_n)$

Our proof of the statement that the facets of $P_C^L(K_n)$ not induced by a nonnegativity constraint are in 1-1 correspondence with those of $P_W^L(D_n)$ is based on two easy facts: The facet defining inequalities for $P_W^L(D_n)$ are symmetric, with the exception of the nonnegativity constraints, and $\dim P_W^L(D_n) = |E| + \dim P_C^L(K_n)$.

In order to simplify the next statements we stipulate that x, y, z are vectors in $\mathbb{R}^E$ such that the components of x correspond to the edges of E, that of y to the arcs $(i, j) \in A$, and that of z to the arcs $(j, i) \in A$, with $i < j$. Let us first show the statement about the dimension.

Lemma 3.30. *Let $K_n = (V, E)$ and $D_n = (V, A)$ be the complete graph and complete digraph on nodeset V, respectively, and let $L \subseteq \{3, \dots, n\}$ where $n = |V|$. Then*

$$\dim P_W^L(D_n) = |E| + \dim P_C^L(K_n).$$

Proof. We first show that each equation of the equality subsystem of $P_W^L(D_n)$ is symmetric. Let $b^T \begin{pmatrix} y \\ z \end{pmatrix} = b_0$ be such an equation and let $W \in \mathcal{W}^L(D_n)$ be any walk containing $(i, j) \in A$. Since $2 \notin L$, the walk W' obtained by substituting (i, j) by (j, i) is also in $\mathcal{W}^L(D_n)$. Moreover, since $b^T\chi^W = b_0$ and $b^T\chi^{W'} = b_0$ it follows immediately that $b_{ij} = b_{ji}$. Hence the equation $b^T \begin{pmatrix} y \\ z \end{pmatrix} = b_0$ is symmetric.

Thus, the equation $a^T x = a_0$ defined by $a_e := (b_{ij} + b_{ji})/2$ for all $e = ij \in E$ is satisfied by all $x \in P_C^L(K_n)$, i.e., $a^T x = a_0$ is a linear combination of equations of the equality subsystem of $P_C^L(n)$.

Conversely, if $b^T x = b_0$ is an equation of the equality subsystem of $P_C^L(K_n)$, then the equation $b^T(y + z) = b_0$ is satisfied by all $\begin{pmatrix} y \\ z \end{pmatrix} \in P_W^L(D_n)$, since each walk $W \in \mathcal{W}^L(D_n)$ contains at most one of the arcs (i, j) and (j, i).

Hence, the system $Cx = d$ of linear equations is an equality subsystem of $P_C^L(K_n)$ if and only if the system $(C, C) \begin{pmatrix} y \\ z \end{pmatrix} = C(y + z) = d$ is an equality subsystem of $P_W^L(D_n)$. Now follows from Linear Algebra that $\text{rank}(C) = \text{rank}(C, C)$, and thus $\dim P_W^L(D_n) = |A| - \text{rank}(C) = |E| + \dim P_C^L(K_n)$. □

In particular, as the circuit polytope $P_C(K_n)$ is full-dimensional, the polytope $P_W^L(D_n)$ is full-dimensional.

Lemma 3.31. *Let $D_n = (V, A)$ be the complete digraph on n nodes and $L \subseteq \{3, \dots, n\}$. Then every nontrivial facet defining inequality for $P_W^L(D_n)$ is symmetric.*

Proof. Let $b^T u \le b_0$ be a nontrivial facet defining inequality for $P_W^L(D_n)$. Since $b^T u \le b_0$ is not equivalent to a nonnegativity constraint, there is for every arc $(i, j) \in A$ a closed walk $W \in \mathcal{W}^L(D_n)$ containing (i, j) whose incidence vector χ^W satisfies $b^T\chi^W = b_0$. As the length of W is at least three, substituting (i, j) by (j, i) yields a closed walk $W' = (W \setminus \{(i, j)\}) \cup \{(j, i)\}$ of the same length, and hence $b^T\chi^{W'} \le b_0$. This implies immediately $b_{ij} \ge b_{ji}$. Analogous one shows $b_{ji} \ge b_{ij}$. Thus the inequality $b^T u \le b_0$ is symmetric. □

Theorem 3.32. *Let $K_n = (V, E)$ and $D_n = (V, A)$ be the complete graph and the complete digraph on nodeset V, respectively, and let $L \subseteq \{3, \dots, n\}$ where $n = |V|$. Moreover, let $b^T x \le b_0$, $b \in \mathbb{R}^E$, be any inequality which is not equivalent to a nonnegativity constraint. Then the inequality $b^T x \le b_0$ defines a facet of $P_C^L(K_n)$ if and only if the inequality $(b^T, b^T)\begin{pmatrix} y \\ z \end{pmatrix} \le b_0$ defines a facet of $P_W^L(D_n)$.*

Proof. Necessity. The validity of the inequality $(b^T, b^T)\begin{pmatrix} y \\ z \end{pmatrix} = b^T(y+z) \le b_0$ is easily checked, since any closed walk $W \in \mathcal{W}^L(D_n)$ contains at most one of the arcs (i,j) and (j,i).

Since $b^T x \le b_0$ induces a facet of $P_C^L(K_n)$, there are $\dim P_C^L(K_n)$ affinely independent points $\begin{pmatrix} y \\ z \end{pmatrix}^w \in P_W^L(D_n)$, with $z = \mathbf{0}$, satisfying $b^T(y^w + z^w) = b_0$.

Moreover, since for each edge $e = ij \in E$, $i < j$, there is a circuit C^e with $ij \in C$ and $b^T \chi^{C^{ij}} = b_0$, there are $|E|$ integer points $\begin{pmatrix} y \\ z \end{pmatrix}^{ij} \in P_W^L(D_n)$ with $z = \mathbf{0}$, $y_{ij} = 1$, and $b^T(y+z) = b_0$. Then the $|E|$ points $\begin{pmatrix} y \\ z \end{pmatrix}^{ji} := \begin{pmatrix} y \\ z \end{pmatrix}^{ij} - e_{ij} + e_{ji}$ are linearly independent (e_{ij} and e_{ji} are the ij-th and ji-th unit vectors in $\mathbb{R}^A$, respectively) and they are also affinely independent of the $\dim P_C^L(D_n)$ points $\begin{pmatrix} y \\ z \end{pmatrix}^w$. Since they also satisfy $b^T(y+z) \le b_0$ at equality, this proves that $b^T(y+z) \le b_0$ defines a facet of $P_W^L(D_n)$.

Suffiency. Clearly, $b^T x \le b_0$ is valid for $P_C^L(K_n)$.

Let $b^T(y+z) \le b_0$ be a nontrivial facet defining inequality for $P_W^L(D_n)$ and suppose, for the sake of contradiction, that $b^T x \le b_0$ is not facet defining for $P_C^L(K_n)$. Then the inequality $b^T x \le b_0$ is a conical combination of two valid and linearly independent inequalities $c^T x \le c_0$ and $d^T x \le d_0$, that is, there are $\lambda, \mu \ge 0$ with $(b^T, b_0) = \lambda(c^T, c_0) + \mu(d^T, d_0)$.

Now the inequalities $c^T(y+z) \le c_0$ and $d^T(y+z) \le d_0$ are valid for $P_W^L(D_n)$, since each walk $W \in \mathcal{W}^L(D_n)$ contains at most one of the arcs (i,j) and (j,i) for all $i, j \in V$, $i \ne j$, and they are linearly independent. Thus, $b^T(y+z) \le b_0$ is a conical combination of linearly independent and valid inequalities with respect to $P_W^L(D_n)$. □

The connection between $P_C^L(D_n)$ and $P_W^L(D_n)$

As can be easily seen that the circuit polytope $P_C^L(D_n)$ is a subset of the restriction of $P_W^L(D_n)$ to the hyperspace defined by the flow constraints (2.1), i.e.,

$$P_C^L(D_n) \subset \{x \in P_W^L(D_n) \mid x(\delta^+(v)) - x(\delta^-(v)) = 0 \quad \forall v \in V\},$$

but the sets are generally not identical. To see this consider the point

$$x^* := \frac{1}{3}(e_{12} + e_{31} + e_{32}) + \frac{1}{3}(e_{24} + e_{25} + e_{45}) + \frac{1}{3}(e_{53} + e_{63} + e_{56}) \in \mathbb{R}^{30}$$

whose support graph is shown in Figure 3.3. Now, x^* satisfies the flow constraints (2.1) and is obviously in $P_W(D_6)$ but not in $P_C(D_6)$. Needless to say that

$$\{x \in P_W^L(D_n) \mid x(\delta^+(v)) - x(\delta^-(v)) = 0 \quad \forall v \in V\} \subseteq C_C(D_n).$$

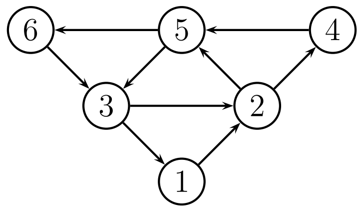
$\frac{1}{3}\cdot$


Figure 3.3

Clearly, we can not expect to have a procedure which lifts facet defining inequalities for $P_C^L(D_n)$ into those for $P_W^L(D_n)$, since the former polytope is not a face of the latter polytope.

However, $P_C^L(D_n)$ is a face of polytopes of the form

$$P_W^L(D_n)|_{RST} := \left\{ \quad x \in P_W^L(D_n) \;\middle|\; \begin{array}{l} x(\delta^+(v)) - x(\delta^-(v)) = 0 \,\forall\, v \in R \\ x(\delta^+(v)) - x(\delta^-(v)) \geq 0 \,\forall\, v \in S \\ x(\delta^+(v)) - x(\delta^-(v)) \leq 0 \,\forall\, v \in T \end{array} \right\},$$

where $R \cup S \cup T$ is a partition of the nodeset V of D_n. Hence we could use standard sequential lifting to obtain facet defining inequalities for $P_W^L(D_n)|_{RST}$ from those for $P_C^L(D_n)$.

Since the basic theorem on sequential lifting is usually stated for full-dimensional polyhedra (see Nemhauser and Trotter [23] and Padberg [24]), and neither $P_C^L(D_n)$ nor the polytopes P are in generally full-dimensional, we restate it in an appropriate form.

Procedure 3.33. *Given polytopes $P_i \subset \mathbb{R}^n$, $i = 0, \ldots, m$, such that P_i is a facet of P_{i+1} for $i = 0, \ldots, m-1$. Then a facet defining inequality for P_0 can be lifted into a facet defining inequality for P_m by applying the following procedure:*

Input: *A facet defining inequality $c^T x \leq \gamma$ for P_0, a matrix $A \in \mathbb{R}^{m \times n}$, and a vector $b \in \mathbb{R}^m$, such that $A_{i\cdot}x \leq b_i$ is facet defining for P_i and*

$$P_{i-1} = \{x \in P_i \mid A_{i\cdot}x = b_i\}, \qquad i = 1, \ldots, m.$$

Output: *A facet defining inequality for P_m.*

1. *SET $c_0 := c$ and $\gamma_0 := \gamma$.*
2. *FOR $i = 1, \ldots, m$ SET*

$$\begin{array}{rcl} \lambda_i^* & := & \min\{\dfrac{\gamma_{i-1} - c_{i-1}^T x}{b_i - A_{i\cdot}x} \mid x \text{ is a vertex of } P_i, A_{i\cdot}x < b_i\}, \\ c_i & := & c_{i-1} - \lambda_i^* A_{i\cdot}^T, \\ \gamma_i & := & \gamma_{i-1} - \lambda_i^* b_i. \end{array}$$

□

Let us call λ_i^* the *i-th lifting number* and in case $m = 1$, $\lambda^* := \lambda_1^*$ simply *lifting number.*

Theorem 3.34. *Procedure 3.33 works correct.*

Proof. Let $i \in \{1,\ldots,m\}$ and $c_{i-1}^T x \leq \gamma_{i-1}$ a facet defining inequality for P_{i-1}. Since $A_{i\cdot}x = b_i$ for all $x \in P_{i-1}$, it follows that for all $\lambda \in \mathbb{R}$ the inequality $c_{i-1}^T x - \lambda A_{i\cdot}x \leq \gamma_{i-1} - \lambda b_i$ is facet defining for P_{i-1}, in particular for λ_i^*, i.e., there are $\dim P_i - 1$ affinely independent points x^r in P_i satisfying $c_i^T x = c_i$, with $A_{i\cdot}x^r = b_i$.

By choice of λ^*, for any vertex x of P_i, with $A_{i\cdot}x < b_i$, we have:

$$\begin{aligned} c_i^T x &= c_{i-1}^T x - \lambda_i^* A_{i\cdot}x \\ &\leq \gamma_{i-1} - \lambda_i^* b_i + \lambda_i^* b_i - \lambda_i^* A_{i\cdot}x \\ &= \gamma_i + \lambda_i^*(b_i - A_{i\cdot}x) \\ &\leq \gamma_i. \end{aligned}$$

Moreover, there is at least one vertex x^* satisfying $c_i x^* = \gamma_i$ which is affinely independent of the points x^r. □

In our context, the polytopes $P_W^L(D_n)|_{RST}$ correspond to the polytopes P_i, in particular, $P_C^L(D_n)$ corresponds to P_0 and a polytope of the form $P_W^L(D_n)|_{RST}$ with $R = \emptyset$ to P_m. Further, $n-1$ flow constraints (2.1) correspond to the matrix A. But this means that the output inequality is obtained from $c^T x \leq \gamma$ by adding appropriate multiples of some flow constraints (2.1). Thus, $c^T x \leq \gamma$ can be lifted into a facet defining inequality for $P_W^L(D_n)$ - by applying Procedure 3.33 - only if the resulting inequality is symmetric. In other words, $c^T x \leq \gamma$ must, from the first, be equivalent to a symmetric inequality with respect to $P_C^L(D_n)$. This can we prove directly:

Theorem 3.35. *Let $c^T x \leq \gamma$ be a symmetric facet defining inequality for $P_C^L(D_n)$, with $2 \notin L$. Then it is also facet defining for $P_W^L(D_n)$, i.e., the inequality*

$$\sum_{[i,j]\in E} c_{ij} y_{[i,j]} \leq \gamma$$

is facet defining for $P_C^L(K_n)$.

Proof. Clearly, $c^T x \leq \gamma$ is valid for $P_W^L(D_n)$. Suppose, for the sake of contradiction, that $c^T x \leq \gamma$ is not facet defining for $P_W^L(D_n)$. Then there is a facet defining inequality $d^T x \leq \delta$ for $P_W^L(D_n)$ with $\{x \in P_W^L(D_n) \mid c^T x = \gamma\} \subsetneq \{x \in P_W^L(D_n) \mid d^T x = \delta\}$. The inequality $d^T x \leq \delta$ is valid for $P_C^L(D_n)$ due to $P_C^L(D_n) \subseteq P_W^L(D_n)$. If $d^T x \leq \delta$ is equivalent to a nonnegativity constraint (with respect to $P_W^L(D_n)$), then it follows that $c^T x \leq \gamma$ is equivalent to a nonnegativity constraint (with respect to $P_C^L(D_n)$) due to $\{x \in P_C^L(D_n) \mid c^T x = \gamma\} \subseteq \{x \in P_C^L(D_n) \mid d^T x = \delta\}$. However, a nonnegativity constraint is not equivalent to a symmetric inequality, a contradiction. Thus, $d^T x \leq \delta$ is symmetric. By our assumption, there is an integer point $x^* \in P_W^L(D_n)$ such that $d^T x^* = \delta$ and $c^T x^* \neq \gamma$. But then the point $\tilde{x}^*$ defined by

$$\tilde{x}_{ij}^* := \frac{x_{ij}^* + x_{ji}^*}{2} \qquad \forall\ (i,j) \in A$$

is in $P_C^L(D_n)$ and satisfies $d^T \tilde{x}^* = \delta$ and $c^T \tilde{x}^* \neq \gamma$. The latter holds, since both inequalities are symmetric. Hence, $\{x \in P_C^L(D_n) \mid c^T x = \gamma\} \subsetneq \{x \in P_C^L(D_n) \mid d^T x = \delta\}$, contrary to the assumption that $c^T x \leq \gamma$ is facet defining for $P_C^L(D_n)$. □

The degree and the dce constraints are equivalent to symmetric inequalities. Thus, the corresponding undirected inequalities are facet defining for $P_C^L(K_n)$, if the originally inequalities are facet defining for $P_C^L(D_n)$. In the following section we will consider some further classes of symmetric inequalities, but we will not point out this fact.

3.6 Inequalities from undirected circuit polytopes

Bauer studied in her dissertation [7] the polytope $P_C(K_n)$, the circuit polytope of the complete graph on n nodes, and introduced several families of facet defining inequalities for $P_C(K_n)$. Further, there are some known facet defining inequalities for the polytope $P_C^{\leq k}(K_n)$ (see Bauer [7] and Bauer, Linderoth, and Savelsbergh [8]). Based on these results we derive some classes of facet defining inequalities for $P_C^L(D_n)$. Thereby we are mainly interested in inequalities which defines facets of $P_C^k(D_n)$, $3 \leq k < n$, because then we can apply the lifting procedure 3.33 in order to generate facet defining inequalities for $P_C^{\leq k}(D_n)$ as well as for $P_C^{\geq k}(D_n)$. As $P_C^k(D_n)$ is a facet of these both polytopes, the lifting procedure requires only one step.

In order to avoid confused definitions and terms we introduce a map dir : $P(E) \to P(A)$, $\operatorname{dir}(F) := \{(i,j) \in A \mid ij \in E\}$ from the powerset of the edgeset E of the complete graph K_n defined on a nodeset V ($n = |V|$) to the power set of the arcset A of the complete digraph D_n defined on the same nodeset V. Note that $(i,j) \in \operatorname{dir}(F)$ if and only if $(j,i) \in \operatorname{dir}(F)$. Such two graphs K_n and D_n defined on the same nodeset V are called *associated*, and we denote $D_n := \operatorname{dir}(K_n)$ (for $D_n = (V,A) = (V, \operatorname{dir}(E))$).

Bipartition inequalities

Bauer [7] introduced the class of *bipartition inequalities*

$$y((S:T)) + 2y(E(S)) + 2y(E(T)) \geq 4$$

and showed them to be facet defining for $P_C(K_n)$, $n \geq 5$. Here, S, T is a bipartition of V with $|S|, |T| \geq 2$. (Recall that $(S:T)$ is the set $\{uv \in E \mid u \in S, v \in T\}$ if we consider an undirected graph.)

Let us start with the investigation whether the bipartition inequalities are facet defining for $P_C^k(D_n)$ or not. Note that we present them in an adjusted form.

Theorem 3.36. *Let $k \in \mathbb{N}$, $k \geq 3$ odd, let $D_n = (V,A)$, $n \geq k+1$, be the complete digraph on n nodes, and let S, T be a bipartition of V with $|S|, |T| \geq \frac{k+1}{2}$. Then a facet of $P_C^k(D_n)$ is given by the bipartition inequality*

$$x(A(S)) + x(A(T)) + x((S:T)) \geq \frac{k+1}{2}. \tag{3.36}$$

Proof. The validity of the bipartition inequality is obvious.

Let us show the facet defining property: Define $l := \frac{k+1}{2}$, let w.l.o.g. $\{1, \ldots, l\} \subseteq S$, $\{l+1, l+2, \ldots, k+1\} \subseteq T$, and set $S' := S \setminus \{1\}$. Then the 1-rooted form of (3.36) is the inequality

$$2x(\delta_S^+(1)) + x(A(S')) + x(A(T)) + x(\delta_T^+(1)) + x((T:S')).$$

For an illustration see Figures 3.4 (a) and (b).

In order to show that (3.36) defines a facet of $P_C^k(D_n)$, we first show that (3.36) induces a facet of

$$P^* := \operatorname{conv}(P_C^k(D_n)) \cup C^*$$

where

$$C^* := \bigcup_{i=1}^{l} \{(i, l+i), (l+i, i+1)\} \cup \{(k+1, 1)\}.$$

Since C^* is a circuit of length $k+1$, $\dim P^* = \dim P_C^k(D_n) + 1$. Moreover, since χ^{C^*} satisfies (3.36) at equality, we can conclude that (3.36) defines a facet of $P_C^k(D_n)$ if it defines a facet of P^*.

Now, assume that we have a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, such that $\{x \in P^* \mid x \text{ satisfies (3.36) at equality}\} \subseteq \{x \in P^* \mid b^T x = b_0\}$, and we may assume that $b^T x \geq b_0$ is in 1-rooted form. We will show that $b^T x \geq b_0$ is equivalent to the 1-rooted form of (3.36) up to multiplication with a positive scalar.

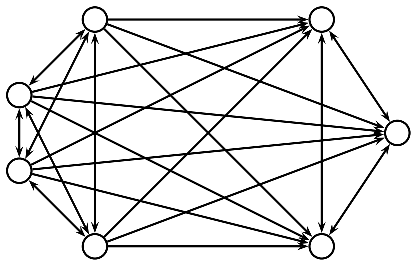


Figure 3.4 (a)

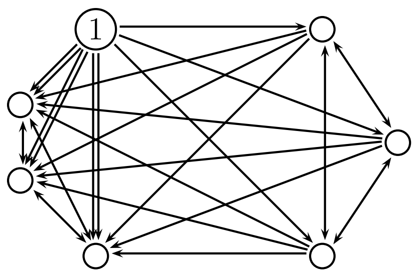


Figure 3.4 (b): 1-rooted form

Let us consider the coefficients b_a, $a \in A$.

Claim 1: $b_{st} = 0$ for all $s \in S'$, and all $t \in T$.

First we show that $b_{l,j} = 0$ for all $j \in \{l+1, \ldots, k+1\}$: From C^* and $C := (C^* \setminus \{(l, k+1), (k+1, 1)\}) \cup (l, 1)$ we derive the equations

$$\left.\begin{array}{rl} \sum_{i=1}^{l-1} b_{i,l+i} + \sum_{i=1}^{l-1} b_{l+i,i+1} + b_{l1} & = b_0 \\ \wedge \quad \sum_{i=1}^{l-1} b_{i,l+i} + \sum_{i=1}^{l-1} b_{l+i,i+1} + b_{l,k+1} + b_{k+1,1} & = b_0 \end{array}\right\} \overset{b_{l1}=b_{k+1,1}=0}{\Longrightarrow} \quad b_{l,k+1} = 0.$$

Further we have for each $j \in \{l+1, \ldots, k\}$ the equations

$$\left.\begin{array}{rl} \sum_{\substack{i=1\\ i \neq j^*}}^{l-1} b_{i,l+i} + \sum_{\substack{i=1\\ i \neq j^*}}^{l-1} b_{l+i,i+1} + b_{j^*,j^*+1} + b_{l,j} + b_{j,1} & = b_0 \\ \wedge \quad \sum_{\substack{i=1\\ i \neq j^*}}^{l-1} b_{i,l+i} + \sum_{\substack{i=1\\ i \neq j^*}}^{l-1} b_{l+i,i+1} + b_{j^*,j^*+1} + b_{l,k+1} + b_{k+1,1} & = b_0 \end{array}\right\} \overset{b_{l,k+1}=0}{\Longrightarrow} \quad b_{l,j} = 0$$

where $j^* := j - l$.

Now consider a coefficient b_{st}, $s \in S' \setminus \{l\}, t \in T$. Let $u \in T$, $u \neq t$. Then there is a circuit $C \in \mathcal{C}^k(D_n)$, with $b^T\chi^C = b_0$, containing the arcs $(u,t),(t,1)$ but not the node s. Then the circuit $C' := (C \setminus \{(u,t),(t,1)\}) \cup \{(u,s),(s,1)\}$ satisfies $b^T\chi^{C'} = b_0$, and hence it follows that $b_{ut} = b_{us}$. (Note that $|C \cap (A(T))| = |C' \cap A(S)| = l$.) Analogous one can show that $b_{ut} = b_{ul}$, and thus $b_{us} = b_{ul}$. Since there is a circuit $\tilde{C} \in \mathcal{C}^k(D_n)$, with $b^T\chi^{\tilde{C}} = b_0$, containing the arcs $(u,s),(s,t)$ but not the node l, the circuit $\hat{C} := (\tilde{C} \setminus \{(u,s),(s,t)\}) \cup \{(u,l),(l,t)\}$ satisfies also $b^Tx \geq b_0$ at equality, and thus we obtain $b_{st} = b_{lt} = 0$.

Claim 2: $b_a = \frac{b_0}{k+1}$ for all $a \in (1:T) \cup (T:S') \cup A(S') \cup A(T)$.

Let again $\{2,3,\dots,l\} \subseteq S'$ and $\{l+1,l+2,\dots,k+1\} \subseteq T$. If one can show $b_{ts} = b_{uv}$ for all $(t,s),(u,v) \in (1:T) \cup (T:S') \cup A(S') \cup A(T)$, then the claim follows for example from the equation

$$\sum_{i=1}^{l} b_{i,l+i} + \sum_{i=1}^{l-1} b_{l+i,i+1} + b_{k+1,1} = b_0,$$

since $b_{k+1,1} = 0$ and $b_a = 0$ for all $a \in (S':T)$. We will show exemplarily $b_{ts} = b_{uv}$ for all $(t,s),(u,v) \in (T:S')$. The rest can be done as an exercise. We can assume that two given arcs $(t,s),(u,v) \in (T:S')$ are contained in the cut $(\{2,3,\dots,l\} : \{l+1,l+2,\dots,k+1\})$.

Case 1: $s = v$.

W.l.o.g. let $s = v = l$, $t = k$, and $u = k+1$. Consider the equations

$$\begin{aligned} &\sum_{i=1}^{l-2} b_{i,l+i} + \sum_{i=1}^{l-2} b_{l+i,i+1} + b_{l-1,k} + b_{kl} + b_{l1} = b_0 \\ \wedge\ &\sum_{i=1}^{l-2} b_{i,l+i} + \sum_{i=1}^{l-2} b_{l+i,i+1} + b_{l-1,k+1} + b_{k+1,l} + b_{l1} = b_0. \end{aligned}$$

Since $b_{l-1} = b_{l-1,k+1} = b_{l1} = 0$, we get the desired result.

Case 2: $t = u$.

Let w.l.o.g. $t = u = k$, $s = l-1$, and $v = l$. From the equations

$$\begin{aligned} &\sum_{i=1}^{l-1} b_{i,l+i} + \sum_{i=1}^{l-2} b_{l+i,i+1} + b_{kl} + b_{l1} = b_0 \\ \wedge\ &\sum_{i=1}^{l-1} b_{i,l+i} + \sum_{i=1}^{l-2} b_{l+i,i+1} + b_{k,k+1} + b_{k+1,1} = b_0, \end{aligned}$$

we get $b_{kl} = b_{k,k+1}$, since $b_{l1} = b_{k+1,1} = 0$. Analogously one can affiliate $b_{k,l-1} = b_{k,k+1}$, and hence $b_{kl} = b_{k,l-1}$.

Case 3: The arcs (t,s) and (u,v) are nonincident.

Let w.l.o.g. $(t,s) = (k,l-1)$ and $(u,v) = (k+1,l)$. From the steps before follows $b_{k,l-1} = b_{kl} = b_{k+1,l}$.

Thus $b_{ts} = b_{uv}$ for all $(t,s),(u,v) \in (T:S')$.

Claim 3: $b_{1s} = 2\frac{b_0}{k+1}$ for all $s \in S'$.

Exercise!

So we have shown that (3.36) defines a facet of P^*, and thus it induces also a facet of $P_C^k(D_n)$. ☐

Corollary 3.37. *(a) Let $k \in \mathbb{N}$, $k \geq 3$ odd, $L \subseteq \{k, \dots, n\}$, $|L| \geq 2$, and $k \in L$. Let $D_n = (V, A)$ be the complete digraph on n nodes and S, T be a bipartition of V.*

(i) If L contains only odd numbers and $|S| \geq \frac{p-1}{2}, |T| \geq \frac{p+1}{2}$ (or conversely) where $p = \min\{l \in L \mid l > k\}$, then the bipartition inequality

$$x(A) - 2x((T : S)) \geq 1$$

defines a facet of $P_C^L(D_n)$.

(ii) If $k+1 \in L$ and $|S|, |T| \geq \frac{k+1}{2}$ then the bipartition inequality

$$x(A) - x((T : S)) \geq \frac{k+1}{2}$$

$$(1 + \frac{1-p+k}{2(p-k)})x(A) - x((T : S)) \geq \frac{1}{2} + \frac{k}{2(p-k)}$$

defines a facet of $P_C^L(D_n)$.

(iii) If L contains even numbers but not $k+1$ and $|S|, |T| \geq (k+3)/2$, then the bipartition inequality

$$(1 + \frac{1-p+k}{2(p-k)})x(A) - x((T : S)) \geq \frac{1}{2} + \frac{k}{2(p-k)}$$

defines a facet of $P_C^L(D_n)$ where $p = \max\{l \in L \mid l \text{ even}, |S|, |T| \geq l/2\}$.

(b) Let $k \in \mathbb{N}$, $k \geq 3$ odd, $L \subseteq \{2, \dots, k\}$, $k \in L$, and $|L| \geq 2$. Let $D_n = (V, A)$ be the complete digraph on n nodes and S, T be a bipartition of V with $|S|, |T| \geq \frac{k+1}{2}$.

(i) If L contains only odd numbers then the bipartition inequality

$$x(A) - 2x((T : S)) \geq 1$$

defines a facet of $P_C^L(D_n)$.

(ii) If L contains even numbers and $p = \max\{l \in L \mid l \text{ even }\}$, then the bipartition inequality

$$(\frac{1}{2} - \frac{1}{2(k-p)})x(A) - x((T : S)) \geq \frac{1}{2} - \frac{k}{2(k-p)}$$

defines a facet of $P_C^L(D_n)$.

Proof. (a) First note that (3.36) is equivalent to the inequality

$$x(A) - x(T : S) \geq \frac{k+1}{2}.$$

Now we apply the procedure 3.33, i.e., we add an adequate multiple of the equation $x(A) = k$ to the bipartition inequality:

$$x(A) - x((T : S)) + \lambda^* x(A) \geq \frac{k+1}{2} + \lambda^* k.$$

For each circuit of length $l \in L$, $l > k$, it follows that

$$\begin{array}{rrl} & x(A) - x((T:S)) + \lambda x(A) & \geq \dfrac{k+1}{2} + \lambda k \\ \Rightarrow & \lambda(x(A) - k) & \geq \dfrac{k+1}{2} + x((T:S)) - x(A) \\ \overset{x(A)-k>0}{\Rightarrow} & \lambda & \geq \dfrac{\frac{k+1}{2} + x((T:S)) - x(A)}{x(A) - k}, \end{array}$$

that is, we have to determine a circuit C^* with $|C^*| \in L \setminus \{k\}$ which maximizes

$$\frac{\frac{k+1}{2} + x((T:S)) - x(A)}{x(A) - k}. \tag{3.37}$$

That can be done by determining for each $l \in L$, $l > k$, a l-circuit which maximizes (3.37) and choosing the circuit among these circuits with the maximum weight.

For $l > k$ define $p_l := l - k$, i.e., $l = k + p_l$. A l-circuit contains at most $\frac{l-1}{2}$ arcs in $(T : S)$ if l is odd, and since $|S| \geq \frac{p-1}{2}, |T| \geq \frac{p+1}{2}$ (or conversely), there is a l-circuit C_l containing $\frac{l-1}{2}$ arcs in $(T : S)$ for $k < l \leq p$. Thus

$$\begin{array}{rcl} \dfrac{\frac{k+1}{2} + \chi^{C_l}((T:S)) - \chi^{C_l}(A)}{\chi^{C_l}(A) - k} & = & \dfrac{\frac{k+1}{2} - (k + p_l) + \frac{k+p_l-1}{2}}{p_l} \\ & = & -\dfrac{1}{2} \qquad\qquad \forall\, l \in L, l > k. \end{array}$$

This implies $\lambda^* = -\frac{1}{2}$ if L contains no even numbers, and hence we get

$$\begin{array}{rrl} & \frac{1}{2}x(A) - x((T:S)) & \geq \dfrac{1}{2} \\ \Leftrightarrow & x(A) - 2x((T:S)) & \geq 1. \end{array}$$

If l is even then we can find a circuit C_l with $|(S : T) \cap C_l| = l/2$. Thus

$$\begin{array}{rcl} \dfrac{\frac{k+1}{2} + \chi^{C_l}((T:S)) - \chi^{C_l}(A)}{\chi^{C_l}(A) - k} & = & \dfrac{\frac{k+1}{2} - (k + p_l) + \frac{k+p_l}{2}}{p_l} \\ & = & \dfrac{1 - p_l}{2p_l}. \end{array}$$

Hence, if L contains even numbers then it follows that $\lambda^* = 0$ if $k + 1 \in L$ and otherwise $\lambda^* = \frac{1-q}{2q}$ where q is the greatest number in L.

(b) Analogous.

□

In particular, with the lifting procedure we have obtained a facet defining inequality for $P_C{}^{\text{odd}}(D_n)$, namely the inequality

$$x(A) - 2x((T:S)) \geq 1.$$

The class of bipartition inequalities (3.36) can be generalized in case $k = 3$ (see Bauer [7]). We first introduce some terminology.

Definition 3.38. Let $K_n = (V, E)$, $n \geq 3$, be the complete graph on n nodes, $e = (p, q) \in E$, and $T \subseteq V \setminus \{p, q\}$. We define an *e-cover of T* to be an edgeset $H_e \subseteq (\{p, q\} : T)$ such that H_e covers every node of T exactly once.

If $M \subseteq E$ is a matching, $T \subseteq V \setminus V(M)$, and H_e is an e-cover of T for all $e \in M$, we say that $H = \cup_{e \in M} H_e$ is a *M-cover of T*.

Theorem 3.39. *Let $2 \notin L$, $3, 4 \in L$, $D_n = (V, A)$, $n \geq 6$, be the complete digraph on n nodes, and S, T be a bipartition of V with $|S| \geq 3$. Further, let $K_n = (V, E)$ be the associated complete graph on nodeset V, that is, $\operatorname{dir}(K_n) = D_n$, let $M \subseteq E(S)$ be a matching, and let $H = \cup_{e \in M} H_e$ be a M-cover of T. Then a facet of $P_C^L(D_n)$ is given by the generalized bipartition inequality*

$$c^T x \geq 4, a \in \mathbb{R}^A, \text{ with } c_a = \begin{cases} 0, & \text{if } a \in \operatorname{dir} M, \\ 3, & \text{if } a \in \operatorname{dir} H, \\ 1, & \text{if } a \in \operatorname{dir}((S : T)) \setminus \operatorname{dir}(H), \\ 2, & \text{otherwise, i.e. if } a \in A(S) \setminus \operatorname{dir}(M) \\ & \text{or } a \in A(T), \end{cases} \tag{3.38}$$

as long as we do not have one of the following two cases:

(i) $M = \{(p, q), (q, p)\}$ and all arcs of H are either incident with p or with q;

(ii) $n = 6$, $|S| = 4$ and $|M| = 2$, i.e., $|\operatorname{dir}(M)| = 4$.

A sketch of the proof of the Theorem is given in Appendix A.

Parity and cut constraints

The following inequalities are derived from the linear description of the circuit cone $C_C(K_n)$ of the complete undirected graph K_n (see Bauer [7]).

Theorem 3.40.

(a) Let $3 \leq k \leq n - 2$, $n \geq 6$, and $v, w \in V$, $v \neq w$. Then the parity constraint

$$x(\delta^+(v) \setminus (v, w)) - x_{wv} \geq 0 \tag{3.39}$$

is facet defining for $P_C^k(D_n)$.

(b) Let $k \geq 5$ be an integer and $V = S \cup T$ be a bipartition of V with $|S| \geq k + 1$, $|T| \geq k$. Further, let $i \in S$ and $j \in T$. Then the cut inequality

$$x((S : T) \setminus (i, j)) - x_{ji} \geq 0 \tag{3.40}$$

defines a facet of $P_C^k(D_n)$.

Proof. The validity of the parity and cut inequalities is easily checked. To prove that the inequalities are facet inducing we construct for each inequality $n^2 - 2n$ affinely independent vectors satisfying it at equality.

(a) Let us denote the complete subgraph of D_n induced by the nodeset $V \setminus \{v\}$ by $D' = (V', A')$. Due to the suppositions $3 \leq k \leq n-2$ and $n \geq 6$ the dimension of $P_C^k(D')$ is equal to $(n-1)^2 - 2(n-1) = n^2 - 4n + 3$. Hence, we have $n^2 - 4n + 4$ linearly independent vectors $y^r \in P_C^k(D_n)$, with $y^r_{uv} = y^r_{vu} = 0$ for all $u \in V'$, satisfying (3.39) at equality.

In order to add $2n-4$ further vectors, we consider for each node $u \in V' \setminus \{w\}$ two k-circuits, namely a circuit C_{uv} containing the arcs (u,v) and (v,w), and a circuit C_{vu} containing the arcs (v,u) and (w,v). That are $2n-4$ k-circuits whose incidence vectors also satisfy (3.39) at equality. Moreover, the incidence vectors are linearly independent, and they are also linearly independent of the $n^2 - 4n + 4$ vectors y^r, since each of them contains positive entries for two arcs incident with node v, whereas all such entries are equal to zero in the points y^r.

(b) Let F be the face induced by (3.40). Set $\delta^+_T(j) := \delta^+(j) \cap A(T)$, $\delta^-_T(j) := \delta^-(j) \cap A(T)$, $s := |S|$, and $t := |T|$. We first construct five sets of linearly independent points, and show then that all points are linearly independent.

1. S contains at least $k+1$ nodes. Thus there are $s^2 - 2s + 1$ linearly independent points $x^\sigma \in F$, with $x^\sigma_{uv} = 0$ for all $(u,v) \in A \setminus A(S)$.

2. For each $(l,m) \in A^1 := (T \setminus \{j\} : S \setminus \{i\})$ there is an integer point $x^{lm} \in F$, with $x^{lm}_{jl} = x^{lm}_{lm} = 1$ and $x^{lm}_a = 0$ for all $a \in A(T) \setminus \{(j,l),(l,m)\}$.

3. For each arc $(u,v) \in A(T \setminus \{j\})$ there is an integer point $x^{uv} \in F$, with $x^{uv}_{ij} = x^{uv}_{ju} = x^{uv}_{uv} = x^{uv}_{vw} = 1$ for some $w \in S \setminus \{i\}$ and $x^{uv}_a = 0$ for all $a \in A(T) \setminus \{(u,v),(j,u)\}$. This are $(t-1)(t-2)$ linearly independent points.

4. Clearly, $|\delta^-_T(j)| = t-1$. Hence there exist $t-1$ linearly independent points $x^\tau \in F$, one for each node $\tau \in T \setminus \{j\}$, with $x^\tau_{\tau j} = 1$ and $x^\tau_a = 0$ for all $a \in A \setminus A(T)$.

5. For each $(y,z) \in A^2 := (S : T) \setminus \{(i,j)\} \cup (j : S \setminus \{i\}) \cup (T \setminus \{j\} : i)$ there is an integer point $x^{yz} \in F$, with $x^{yz} = 1$ and $x^{yz}_a =$ for all $a \in (S : T) \cup (T : S) \setminus \{(i,j),(j,i),(y,z)\}$. That are $st + n - 3$ linearly independent points.

We have constructed a total of $n^2 - 2n$ points. It remains to be shown that these points are linearly independent. Suppose that

$$\begin{aligned} \sum_{\sigma=1}^{s^2-2s+1} \alpha_\sigma x^\sigma + \sum_{(l,m)\in A^1} \beta_{lm} x^{lm} + \sum_{(u,v)\in A(T\setminus\{j\})} \gamma_{uv} x^{uv} & \\ + \sum_{\tau \in T\setminus\{j\}} \delta_\tau x^\tau + \sum_{(y,z)\in A^2} \eta_{yz} x^{yz} &= \mathbf{0}. \end{aligned} \tag{3.41}$$

It follows that $\eta_{yz} = 0$ for $(y,z) \in A^2$, since $x^{yz}_{yz} = 1$, but the entry yz is equal to zero for all other vectors in (3.41). Thus (3.41) can be reduced to

$$\sum_{\sigma=1}^{s^2-2s+1} \alpha_\sigma x^\sigma + \sum_{(l,m)\in A^1} \beta_{lm} x^{lm} + \sum_{(u,v)\in A(T\setminus\{j\})} \gamma_{uv} x^{uv} + \sum_{\tau\in T\setminus\{j\}} \delta_\tau x^\tau = \mathbf{0}. \tag{3.42}$$

Of the remaining vectors only the vectors x^τ have positive components (τj), whereas all such components are zero in all other points. Hence, $\delta_\tau = 0$ for all $\tau \in T \setminus \{j\}$ and thus

$$\sum_{\sigma=1}^{s^2-2s+1} \alpha_\sigma x^\sigma + \sum_{(l,m)\in A^1} \beta_{lm} x^{lm} + \sum_{(u,v)\in A(T\setminus\{j\})} \gamma_{uv} x^{uv} \quad = \quad \mathbf{0}. \tag{3.43}$$

With similar arguments follows that the remaining scalars are zero. □

A cut inequality (3.40) is not facet defining when for example $|S| \le k-2$, because then the face F induced by (3.40) is a proper subset of the face induced by the nonnegativity constraint $x_{jv} \ge 0$ for any $v \in S \setminus \{i\}$. If $|S|, |T| \le k$ then it is also not facet defining, since then the face F induced by (3.40) is a proper subset of the face induced by the dce-constraint $x(\delta^+(i)) + x(\delta^+(j)) - x((S:T)) \ge 1$.

The following theorem is immediate.

Theorem 3.41. *Let S, T be a bipartition of V with $|S|, |T| \ge 5$. Further, let $i \in S$ and $j \in T$. Then the inequality*

$$x((S:T) \setminus (i,j)) - x_{ji} \ge 0 \tag{3.44}$$

defines a facet of $P_C^4(D_n)$.

Proof. Set $s := |S|$ and $t := |T|$ and denote the face induced by 3.44 by F. Since $|S|, |T| \ge 5$, there are $s^2 - 2s + 1$ linearly independent points $x^\sigma \in F$, with $x^\sigma_{uv} = 0$ for all $(u,v) \in A \setminus A(S)$ and $t^2 - 2t + 1$ linearly independent points $x^\tau \in F$, with $x^\tau_{uv} = 0$ for all $(u,v) \in A \setminus A(T)$.

Next consider $2st - 2$ integer points $x^\rho \in F$, one for each $\rho \in A' := (S:T) \cup (T:S) \setminus \{(i,j),(j,i)\}$, such that $x^\rho_\rho = 1$ and $x^\rho_a = 0$ for all $A' \setminus \{\rho\}$.

It is not hard to see that these $n^2 - 2n$ points are linearly independent, which completes the proof. □

We turn now to the case $|L| \ge 2$.

Corollary 3.42.

(i) Let $n \ge 6$, $v, w \in V$, $v \ne w$, $L \subseteq \{3,\dots,n\}$, $|L| \ge 2$, and $L \cap \{3,\dots,n-2\} \ne \emptyset$. Then the inequality (3.39) *defines a facet of $P_C^L(D_n)$.*

(ii) Let $n \ge 6$, $v, w \in V$, $v \ne w$, $L \subseteq \{2,\dots,n\}$, $|L| \ge 2$, $2 \in L$, $L \cap \{3,\dots,n-2\} \ne \emptyset$ and $k := \min\{l \in L \mid l > 2\}$. Then the inequality

$$x(A) + (k-2)x_{wv} - (k-2)x(\delta^+(v) \setminus \{(v,w)\}) \le k$$

defines a facet of $P_C^L(D_n)$.

Proof. (i) We have to construct $n^2 - 2n + 1$ affinely independent vectors satisfying (3.39) at equality. By the conditions, there is $k \in L \cap \{3,\dots,n-2\}$. Thus, by Theorem 3.40, there are $n^2 - 2n$ linearly independent vectors $x^r \in P_C^L(D_n)$, with $\mathbf{1}^T x^r = k$, satisfying (3.39) at equality.

Since $|L| \geq 2$ there is another feasible length l and a vector y^l, with $\mathbf{1}^T y^l = l$, satisfying (3.39) at equality. By Lemma 3.9, the points y^l, x^r, $r = 1, \dots, n^2 - 2n$, are affinely independent, and thus (3.39) defines a facet of $P_C^L(D_n)$.

The statement can be also proved by applying Procedure 3.33.

(b) Let $k := \min\{l \in L \mid l > 2\}$. Note that k satisfies $2 < k < n-1$, and thus (3.39) defines a facet of $P_C^k(D_n)$. Applying Procedure 3.33 yields the lifting number $\lambda^* = \frac{-1}{k-2}$ and hence the inequality

$$\begin{aligned} & x(\delta^+(v) \setminus (v,w)) - x_{wv} - \tfrac{1}{k-2} x(A) \geq \tfrac{-k}{k-2} \\ \Leftrightarrow \quad & x(A) + (k-2)x_{wv} - (k-2)x(\delta^+(v) \setminus \{(v,w)\}) \leq k. \end{aligned}$$

□

An statement analogous to the above can be obtained by applying Procedure 3.33 to (3.40).

Maximal set inequalities

Wang [28] introduced a class of valid inequalities for the circuit polytope $P_C(K_n)$ he called the multipartition inequalities. This class generalizes the *$\bar{S}\bar{T}$-inequalities* introduced by Bauer [7].

Let $V = \cup_{i=1}^s V_i$ be a partition of the nodeset V with $|V_i| \geq 2$ for all $i \in \{1, \dots, s\}$. Moreover, let $T_i \subseteq E(V_i)$, $i \in \{1, \dots, s\}$, be a spanning tree of the complete graph induced by V_i and $\bar{T}_i$ be its complement with respect to $E(V_i)$. Then the inequality

$$2 \sum_{i=1}^{s} y(\bar{T}_i) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} y((V_i : V_j)) \geq 2.$$

is called *multipartition inequality* and is facet defining for $P_C(K_n)$ if $s \geq 2$.

These inequalities can be strengthened[1] to facet defining inequalities of $P_C^{\leq k}(K_n)$ by replacing the sets $\bar{T}_i$, which are the complements of maximal sets in $E(V_i)$ not containing any circuit, by complements of maximal sets in $E(V_i)$ not containing any circuit of length less than or equal to k.

Theorem 3.43 ([8]). *Let $K_n = (V, E)$, $4 \leq k < n$, and a partition of V be given by*

$$V = \bigcup_{i=1}^{s} V_i$$

where $s \geq 2$ and $|V_i| \geq 2$ for all $i \in \{1, \dots, s\}$.[2] *Moreover, let $M_i \subseteq E(V_i)$, $i \in \{1, \dots, s\}$, be a maximal edge set with respect to $E(V_i)$ not containing any circuit of length less than*

[1] Note that the statement of Bauer in [7], p. 67, is false that the $\bar{S}\bar{T}$-inequalities would define facets of $P_C(K_n)$.

[2] In their theorem they do not postulate explicit $s \geq 2$, but it is not hard to see that the statement is wrong for $s = 1$.

or equal to k. Let $\bar{M}_i = E(V_i) \setminus M_i$ be its complement with respect to $E(V_i)$. Then a facet of $P_C^{\leq k}(K_n)$ is defined by the maximal set inequality

$$2\sum_{i=1}^{s} y(\bar{M}_i) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} y((V_i : V_j)) \geq 2.$$

We will show that an asymmetric version of these inequalities are also facet defining for the polytopes $P_C^{\{3,\dots,k\}}(D_n)$, $4 \leq k \leq n$.

Theorem 3.44. *Let $D_n = (V, A)$ be the complete digraph on n nodes and $K_n = (V, E)$ be the associated complete graph on n nodes, i.e., $\text{dir}(K_n) = D_n$. Next, let $4 \leq k \leq n$ and a partition of V be given by*

$$V = \bigcup_{i=1}^{s} V_i$$

where $s \geq 2$ and $|V_i| \geq 2$ for all $i \in \{1, \dots, s\}$. Moreover, let $M_i \subseteq E(V_i)$, $i \in \{1, \dots, s\}$, be a maximal edge set with respect to $E(V_i)$ not containing any undirected circuit of length less than or equal to k, in particular, M_i contains no undirected circuit if $k = n$. Let $\bar{M}_i = E(V_i) \setminus M_i$ be its complement with respect to $E(V_i)$. Then a facet of $P_C^{\{3,\dots,k\}}(D_n)$ is defined by the asymmetric maximal set inequality

$$\sum_{i=1}^{s} x(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s} x((V_i : V_j)) \geq 1. \tag{3.45}$$

Proof. The inequality is valid, as every feasible circuit either uses at least one arc in $\cup_{i=1}^{s}\text{dir}(\bar{M}_i)$ or one arc in $\cup_{i=1}^{s-1}\cup_{j=i+1}^{s}(V_i : V_j)$.

Now we show that the face F induced by (3.45) is a facet. Let ARB_1 be a spanning arborescence of $\text{dir}(M_1)$, let w.l.o.g 1 be the root of ARB_1, and $(1, 2) \in \text{ARB}_1$.

Let F^* be a facet of $P_C^L(D_n)$ containing F, and let $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, be an inequality defining F^* with $b_{v1} = 0$ for all $v \in V_i$,. $i = 2, \dots, s$ and $b_a = 0$ for all $a \in \text{ARB}_1$ (The inequality is in T-rooted form). We show that $b^T x \geq b_0$ is equivalent to the asymmetric maximal set inequality up to multiplication with a positive scalar.

Now, let us consider the coefficients b_a, $a \in A$.

1. $b_{2v} = b_0$ for all $v \in V_i$, $i = 2, \dots, s$.

   Let $v \in V_i$ for some $i \in \{2, \dots, s\}$. Then the triangle $(1, 2, v, 1)$ yields $b_{2v} = b_0$, since $b_{12} = b_{v1} = 0$.

2. $b_{uv} = b_0$ for all $(u, v) \in \text{dir}(M_i)$, $i = 2, \dots, s$.

   Let $(u, v) \in \text{dir}(M_i)$ for some $i \in \{2, \dots, s\}$. Since $b_{2u} = b_0$ and $b_{12} = b_{v1} = 0$, the circuit $(1, 2, u, v, 1)$ yields $b_{uv} = 0$.

3. $b_{1v} = b_0$ for all $v \in V_i$, $i = 2, \dots, s$. To each $v \in V_i$ there is a node $w \in V_i$ with $(v, w) \in \text{dir}(M_i)$. Thus $b_{vw} = 0$ and hence the triangle $(1, v, w, 1)$ yields $b_{1v} = b_0$.

4. $b_{uv} = b_0$ for all $(u, v) \in \text{dir}(\bar{M}_i)$, $i = 2, \dots, s$.

   Let $i \in \{2, \dots, s\}$ and $(u, v) \in \text{dir}(\bar{M}_i)$. Since $(u, v) \notin \text{dir}(M_i)$, there must be a circuit $C \subseteq M_1 \cup \{(u, v)\}$ with $|C| \leq k$. This circuit satisfies $b_0 = b^T \chi^C = b_{uv}$ and thus we conclude $b_{uv} = b_0$.

5. $b_{v2} = 0$ for all $v \in V_i$, $i = 2, \ldots, s$.

   Let $v \in V_i$ for some $i \in \{2, \ldots, s\}$. Since there is a node $w \in V_i$ with $b_{vw} = 0$ and since $b_{2w} = b_0$, we derive from the circuit $(2, v, w, 2)$, $b_{v2} = 0$.

6. $b_{21} = 0$.

   Clear.

7. $b_{uv} = 0$ for all $(u, v) \in (V_j : V_i)$, $2 \le i < j \le s$.

   Let $(u, v) \in (V_j : V_i)$, $1 \le i < j \le s$. From the triangle $(u, v, 1, u)$ we derive the equation $b_{uv} + b_{v1} + b_{1u} = b_0$. Since $b_{v1} = 0$ and $b_{1u} = b_0$, we obtain $b_{uv} = 0$.

8. $b_{uv} = b_0$ for all $(u, v) \in (V_i : V_j)$, $2 \le i < j \le s$.

   Let $2 \le i < j \le s$ and $(u, v) \in (V_i : V_j)$. Then a circuit (t, u, v, t) with $(t, u) \in \text{dir}(M_i)$ yields $b_{uv} = b_0$.

9. $b_{uv} = b_0$ for all $(u, v) \in (V_1 : V_i)$, $b_{v,u} = 0$ for all $(v, u) \in (V_i : V_1)$, $i = 2, \ldots, s$, and $b_{wu} = 0$ for all $(w, u) \in A$ with $(u, w) \in \text{ARB}_1$.

   This can be shown per induction. In V_1 we define a distance function $\delta : V_1 \to \mathbb{Z}_+$ by $\delta(v) := |P|$ where P is the unique $(1, v)$-path in ARB_1. Now suppose that the statement is true for all $v \in V_1$ with $\delta(v) \le d$, then we will show that it is also true for $d + 1$. For $d \in \{0, 1\}$ the statement is true (see the steps before). So let $1 \le d$, $u \in V_1$ with $\delta(u) = d + 1$, and $w \in V_i$ for some $i \in \{2, \ldots, n\}$. Then there is a predecessor of u, say z, in ARB_1. Since $\delta(z) = d$, the above statement is true for z; in particular $b_{zw} = b_0$ and $b_{wz} = 0$. Since also $b_{zu} = 0$, we derive from the circuit (z, u, w, z), $b_{uw} = b_0$. Further, it exists a node $p \in V_i$ with $(p, w) \in \text{dir}(M_i)$, i.e., $b_{pw} = 0$. Clearly, $b_{up} = b_0$, and thus the circuit (u, p, w, u) yields $b_{wu} = b_0$. Hence, we get from the circuit (z, w, u, z), $b_{uz} = 0$. (Note that $d \le |V_1| - 1$.)

10. $b_a = 0$ for all $a \in \text{dir}(M_1) \setminus \text{ARB}_1$ and $b_a = b_0$ for all $a \in \text{dir}(\bar{M}_1)$.

    That is clear now.

Since $b^T x \ge b_0$ has to be valid also for the circuit $(1, v, 2, w, 1)$ for some $v, w \in V_2$, $v \ne w$, we know that $b_0 > 0$, which completes the proof. □

The perfect matching inequalities

If M is a matching in the complete graph $K_n = (V, E)$, every (undirected) circuit contains at least two edges not belonging to the matching. Bauer showed, if $n \ge 6$ is even and M a perfect matching, then the *perfect matching inequality*

$$y(E \setminus M) \ge 2$$

is facet defining for $P_C(K_n)$ [7].

The symmetric version of these inequalities are not facet inducing for $P_C^{\ge 3}(D_n)$ but can be strengthened. Let $m := |M|/2$ and $M := \{e_1, e_2, \ldots, e_m\}$. The perfect matching

M induces a partition of V in sets of cardinality 2, namely $V = \cup_{i=1}^m V_i$, where $V_i := V_{e_i} := \{u \in V \mid e_i \text{ covers } u\}$. Then the *asymmetric perfect matching inequality*

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) \geq 1 \tag{3.46}$$

is facet defining for $P_C^{\geq 3}(D_n)$.

The statement will be proved in the next but one paragraph. Adding to (3.46) the analogous inequality for the ordering $(m, m-1, \ldots, 1)$ yields precisely the symmetric version of the perfect matching inequality.

k-partition inequalities

The k-partition inequalities were introduced by Bauer et al. [8] and they are specific to the polytope $P_C^{\leq k}(K_n)$. If the nodesset V is partitioned into sets V_i, $i = 1, \ldots, m$, of size $k-1$ (V_m is of cardinality at most $k-1$), then the *k-partition inequality*

$$2\sum_{i=1}^{m} y(E(V_i)) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} y((V_i : V_j) \leq 2(k-1)$$

says that a circuit of length at most k contains enough edges across the partition of V. The k-partition inequality defines a facet of $P_C^{\leq k}(K_n)$ for $4 \leq k < n$ (see Bauer et al. [8]).

The symmetric version

$$2\sum_{i=1}^{m} x(A(V_i)) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} [x((V_i : V_j) + x((V_j : V_i))] \leq 2(k-1)$$

is not facet defining for $P_C^{\leq k}(D_n)$, since it is the sum of the valid inequalities

$$\begin{aligned}
\sum_{i=1}^{m} x(A(V_i)) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j) &\leq (k-1), \\
\sum_{i=1}^{m} x(A(V_i)) + \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_j : V_i)) &\leq (k-1).
\end{aligned}$$

The last both inequalities are called *asymmetric k-partition inequalities*, and we will show that they are facet defining for $P_C^{\leq k}(D_n)$. We will defer the proof to the next paragraph.

A common generalization of the perfect matching inequalities and the linear ordering constraints (2.3) is the next family of inequalities.

The generalized linear ordering constraints inequalities

If C is a directed circuit of length at least k and $V = \cup_i V_i$ a partition of V such that the subsets V_i are of cardinality at most $k-1$, then the *generalized linear ordering constraint* says that C uses at least one arc in $(V_i : V_j)$, $i < j$. Let us first study the case $L = \{k\}$.

Theorem 3.45. *Let* $4 \leq k < n$*,* $n \geq 5$*,* $m \in \mathbb{N}$*,* $m \geq 2$*, and* $V = \cup_{i=1}^{m} V_i$ *be a partition of* V *with* $|V_i| \leq k-1$ *for* $1 \leq i \leq m$*. Then the following statements are equivalent.*

(i) The inequality

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) \geq 1 \tag{3.47}$$

defines a facet of $P_C^k(D_n)$*.*

(ii) $|V_i| + |V_{i+1}| \geq k$ *for* $i = 1, \ldots, m-1$*, and* $|V_1| + |V_i| \geq k$ *for* $i = 2, \ldots, m$*.*

Proof.
"$(i) \Rightarrow (ii)$:" For $m = 2$, statement (ii) is true independent of (i).

Let $m \geq 3$. Assume that there are two sets V_i, V_{i+1} with $1 \leq i \leq m-1$ such that $|V_i| + |V_{i+1}| < k$. Then exists no feasible circuit containing an arc $a \in (V_i : V_{i+1})$ whose incidence vector satisfies (3.47) at equality, and thus, by Lemma (3.17), inequality (3.47) is not facet inducing for $P_C^L(D_n)$, a contradiction.

If we assume $|V_1| + |V_i| < k$ for an $i \in \{2, \ldots, m\}$, then there is no circuit of length k containing an arc $a \in (V_i : V_1)$ whose incidence vector satisfies (3.47) at equality, a contradiction.
"$(ii) \Rightarrow (i)$:" The validity of the generalized linear ordering constraint is easily checked.

For showing the facet defining property, we consider the polytope

$$P^* := \text{conv}(P_C^k(D_n) \cup \{C^*\}),$$

where C^* is a circuit of length $k+1$ which satisfies (3.47) at equality. We first show that constraint (3.47) induces a facet of P^*. Then it follows immediately that (3.47) defines also a facet of $P_C^k(D_n)$. Let us assume that $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, is a valid inequality for P^*, with

$$\{x \in P^* \mid x \text{ satisfies (3.47) at equality}\} \subseteq \{x \in P^* \mid b^T x = b_0\}.$$

Let w.l.o.g. $1 \in V_1 \cap V(C^*)$ and $b^T x \geq b_0$ in 1-rooted form.

Case 1: $m = 2$

Since $k < n$, we have $|V_1| + |V_2| \geq k+1$ and $|V_1|, |V_2| \geq 2$. Let $u, v \in V$ such that $a^* := (u^*, v^*), (v^*, 1) \in C^*$. The circuit $C := (C^* \setminus \{(u^*, v^*), (v^*, 1)\}) \cup \{(u^*, 1)\}$ is of length k and satisfies $b^T \chi^C = b_0$, since $u^*, v^* \in V_2$ or $u^* \in V_2, v^* \in V_1$. We have $b_{u^*1} = b_{v^*1} = 0$, and thus we get $b_{(u^* v^*)} = 0$.

Next we show $b_a = 0$ for all $a \in A(V_1 \setminus \{1\}) \cup A(V_2) \cup (V_2 : V_1 \setminus \{1\})$. Let $(p, q) \in (V_2 : V_1 \setminus \{1\})$ and $(r, s) \in A(V_2)$.

Case 1.1: $p \neq s$

There is a circuit C of length k, with $b^T \chi^C = b_0$, which contains not the node q, such that $(r, s), (s, 1) \in C$. The circuit C' defined by $C' := (C \setminus \{(r, s), (s, 1)\}) \cup \{(r, q), (q, 1)\}$ satisfies $b^T \chi^{C'} = b_0$. Since $b_{s1} = b_{q1} = 0$, we get $b_{rs} = b_{rq}$. For $p = r$ this proves $b_{pq} = b_{rs}$.

Otherwise one can show analogously $b_{sr} = b_{sp}$. Further, let $\tilde{C}$ a circuit of length k such that $p \notin V(\tilde{C})$, $(s, r), (r, q) \in \tilde{C}$ and $b^T \chi^{\tilde{C}} = b_0$. Then the circuit $\hat{C} := (\tilde{C} \setminus \{(s, r), (r, q)\}) \cup \{(s, p), (p, q)\}$ yields $b_{rq} = b_{pq}$. So we get $b_{rs} = b_{pq}$.

Case 1.2: $p = s$

If $|V_2| \geq 3$ then there is a node $s' \in V_2 \setminus \{r,p\}$. Substituting s by s' in the above arguments yields $b_{rs'} = b_{pq}$, and since $b_{rs'} = b_{rp}$, we get $b_{rp} = b_{pq}$.

If $|V_2| = 2$, then there is a node $v' \in V_1 \setminus \{1,q\}$. One can show $b_{rp} = b_{rq} = b_{rv'}$, $b_{pr} = b_{pv'}$, $b_{v'q} = b_{pq}$, $b_{rq} = b_{v'q}$ and thus $b_{rp} = b_{pq}$.

Analogously one shows $b_e = b_f$ for all $e \in (V_2 : V_1 \setminus \{1\})$ and all $f \in A(V_1 \setminus \{1\})$. Hence, all coefficients b_a, $a \in A(V_1 \setminus \{1\}) \cup A(V_2) \cup (V_2 : V_1 \setminus \{1\})$, are equal, and they are all equal to zero, as $b_{a^*} = 0$.

As is easily seen, it follows immediately $b_{1v} = 0$ for all $v \in V_1 \setminus \{1\}$ and $b_a = b_0$ for all $a \in (V_1 : V_2)$.

Case 2: $m \geq 3$

With similar arguments as in the first case one can show that $b_a = 0$ for all

- $a \in A(V_1) \setminus \{1\}$,
- $a \in A(V_i)$, $i = 2, \ldots, m-1$,
- $a \in (V_j : V_i)$, $1 \leq i < j \leq m-1$ and
- $a \in (V_m : V_i)$, $i = 2, \ldots, m-1$.

Next let $a \in (V_i : V_j)$, $2 \leq i < j \leq m-1$. Then we get $b_a = b_0$ from any circuit C of length k containing a and satisfying $b^T \chi^C = b_0$, since the coefficients b_f, $f \in C \setminus \{a\}$, are equal to zero.

Further, let $a = (u,v) \in (V_1 : V_i)$, $i = 3, \ldots, m$. For $u \neq 1$ note that $|V_1 \setminus \{1\}| + |V_2| + |\{v\}| \geq k$. Hence there is a circuit C of length k containing a with $b_f = 0$ for all $f \in C \setminus \{a\}$. For $u = 1$ such a circuit exists more than ever. This yields $b_a = b_0$. Analogously one shows $b_a = b_0$ for all $a \in (V_j : V_m)$, $j = 2, \ldots, m-2$.

It remains to be shown that $b_{1v} = 0$ for all $v \in V_1 \setminus \{1\}$, $b_a = b_0$ for all $a \in (V_1 : V_2) \cup (V_{m-1} : V_m)$ and $b_a = 0$ for all $a \in A(V_m)$. This we commit for the reader as an exercise.

We have determined all coefficients b_a, $a \in A$. Clearly, $b_0 > 0$, and thus inequality (3.47) defines a facet F of P^*. F contains $n^2 - 2n$ linearly independent incidence vectors of k-circuits, and thus (3.47) defines also a facet of $P_C^k(D_n)$.

□

Corollary 3.46. *Let $4 \leq k < n$, $2, \ldots, k-1 \notin L$, $k \in L$ and $|L| \geq 2$. Let $m \in \mathbb{N}$, $m \geq 2$, and $V = \cup_{i=1}^m V_i$ be a partition of V with $|V_i| \leq k-1$ for $1 \leq i \leq m$. Then the following statements are equivalent.*

(i) The inequality (3.47) defines a facet of $P_C^L(D_n)$.

(ii) $|V_i| + |V_{i+1}| \geq k$ for $i = 1, \ldots, m-1$, and $|V_1| + |V_i| \geq k$ for $i = 2, \ldots, m$.

Proof.
"$(i) \Rightarrow (ii)$:" Clear.

"$(ii) \Rightarrow (i)$:" For any $l \in L$, $l > k$, there is a circuit whose incidence vector satisfies (3.47) at equality. By Lemma 3.9, it follows the statement. □

For $m \geq 3$, one can show that (3.47) is also facet defining for $P_C^3(D_n)$. Applying Procedure 3.33 to lift (3.47) (as a facet defining inequality for $P_C^k(D_n)$) into a facet defining inequality for $P_C^L(D_n)$ where $L \subseteq \{k, \dots, n\}$, $|L| \geq 2$, $k \in L$, yields by the way no new inequality: Adding a multiple of the equation $x(A) = k$ to (3.47) leads to

$$\begin{aligned} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) + \lambda x(A) &\geq 1 + \lambda k \\ \Rightarrow \qquad \lambda &\geq \frac{1 - \sum\limits_{i=1}^{m-1} \sum\limits_{j=i+1}^{m} x((V_i : V_j))}{x(A) - k}. \end{aligned}$$

Thus $\lambda^* = \max\{\frac{1 - \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \chi^C((V_i:V_j))}{|C|-k} \mid C \in \mathcal{C}^{L \setminus \{k\}}(D_n)\}$. Since $|C| - k > 0$ and $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \chi^C((V_i : V_j)) \geq 1$ for all $C \in \mathcal{C}^{L \setminus \{k\}}(D_n)$, it follows that $\lambda^* \leq 0$. Further, to each $l \in L$, $l > k$ there is a circuit C^l whose incidence vector satisfies (3.47) at equality, and hence $\lambda^* = 0$.

If in contrast $L \subseteq \{2, \dots, k\}$, $k \in L$, $|L| \geq 2$, we can derive:

$$\begin{aligned} \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) - \lambda x(A) &\geq 1 - \lambda k \\ \Rightarrow \qquad \lambda &\geq \frac{1 - \sum\limits_{i=1}^{m-1} \sum\limits_{j=i+1}^{m} x((V_i : V_j))}{k - x(A)}. \end{aligned}$$

Thus, if $L' := \{l \in L \mid l < k, \exists\, i : |V_i| \geq l\}$ is nonempty, then it follows that $\lambda^* = \frac{1}{k-p}$ where $p = \max\{l \in L'\}$. Otherwise, we obtain $\lambda^* = 0$.

Corollary 3.47. *Let $D_n = (V, A)$ be the complete digraph on n nodes, $4 \leq k < n$, $n \geq 5$, $m \in \mathbb{N}$, and $V = \cup_{i=1}^{m} V_i$ be a partition of V with $|V_i| \leq k - 1$ $(1 \leq i \leq m)$ satisfying the conditions (ii) in* 3.45*. Moreover, let $L \subseteq \{2, \dots, k\}$, $k \in L$, $|L| \geq 2$, and $L' := \{l \in L \mid l < k, \exists\, i : l \leq |V_i|\}$ be nonempty. Then the inequality*

$$x(A) - (k - p) \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) \leq p \tag{3.48}$$

defines a facet of $P_C^L(D_n)$ where $p = \max\{l \in L'\}$. □

In particular, if $p = k - 1$ then (3.48) is a k-partition inequality.

The cardinality-path inequalities

Let P be a k-path in an undirected graph $G = (V, E)$ and C be a circuit of cardinality at most k. Then the cardinality - path inequality corresponding to P ensures that C never uses more edges of P than inner nodes of P and can be expressed as

$$y(P) \leq \frac{1}{2} \sum_{v \in \dot{P}} y(\delta(v))$$

where $y \in P_C^{\leq k}(G)$.

The next theorem shows that the corresponding symmetric inequality is facet defining for $P_C^{\{3,\ldots,k\}}(D_n)$.

Theorem 3.48. *Let $D_n = (V, A)$ be the complete digraph on n nodes, $K_n = (V, E)$ the associated complete graph, i.e., $\text{dir}(K_n) = D_n$, $4 \leq k < n$, and P a path in K_n consisting of k edges. Then the cardinality-path inequality*

$$x(\text{dir}(P)) - \sum_{v \in \dot{P}} x(\delta^-(v)) \leq 0 \tag{3.49}$$

defines a facet of $P_C^{\{3,\ldots,k\}}(D_n)$.

Proof. Let the inequality be denoted by $d^T x \leq 0$, $d \in \mathbb{R}^A$. Bauer showed the validity of the cardinality-path inequality in the undirected case [8]. Hence, constraint (3.49) is valid.

In order to show that (3.49) defines a facet of $P_C^{\{3,\ldots,k\}}(D_n)$, assume that there is a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, satisfying

$$\{x \in P_C^{\{3,\ldots,k\}}(D_n) \mid x \text{ satisfies (3.49) at equality}\} \subseteq \{x \in P_C^{\{3,\ldots,k\}}(D_n) \mid b^T x = b_0\}.$$

Assume w.l.o.g. that $P = (1, \ldots, k+1)$ and that

$$b_{i,i+1} = 0 \quad \forall\, i \in \{1, \ldots, k-1\}, \tag{3.50}$$

$$b_{k+1,k} = 0, \tag{3.51}$$

$$b_{3,i} = 0 \quad \forall\, i \in V \setminus V(P). \tag{3.52}$$

(See Figure 3.5 for an illustration.)

The argumentation is built-on so that the proof is correct if $n = k+1$ or $n = k+2$, i.e., if $V^* := V \setminus V(P) = \emptyset$ or $V^* = \{k+2\}$. Set

$$\begin{aligned} A^0 &:= \{a \in A \mid d_a = 0\}, \\ A^+ &:= \{a \in A \mid d_a = 1\}, \text{ and} \\ A^- &:= \{a \in A \mid d_a = -1\}, \end{aligned}$$

in particular, $A^+ = \{(2,1), (k, k+1)\}$.

(i) Let us first show that $b_0 = 0$ and $b_a = 0$ for all $a \in A^0$.

To each arc $(i, 1)$, $i = 3, \ldots, k$, consider the circuit $C = (1, \ldots, i, 1)$. Since $3 \leq |C| \leq k$ and $b^T \chi^C = b_0$, we obtain

$$b_{i1} \overset{(3.50)}{=} b_0 \qquad \forall\, i \in \{3, \ldots, k\}. \tag{3.53}$$

Next, from the circuit $(k+1, k, 1, k)$ we derive

$$b_{1,k+1} \overset{(3.52),(3.53)}{=} 0. \tag{3.54}$$

Now, the circuits $(k+1,\ldots,i,1,k+1)$, $(k+1,\ldots,i,k+1)$, $i=k-1,\ldots,3$, $(1,2,3,k+1,1)$, $(1,2,k+1,1)$, $(k+1,\ldots,2,k+2)$ and the equations (3.50)-(3.54) yield (gradual)

$$\begin{array}{rcll} b_{i+1,i} &=& 0 & \text{for } i=k-1,\ldots,3;\\ b_{i,k+1} &=& b_0 & \text{for } i=k-1,\ldots,3;\\ b_{k+1,1} &=& 0; & \\ b_{2,k+1} &=& b_0; & \\ b_{3,2} &=& 0. & \end{array} \tag{3.55}$$

Furthermore, the circuit $(1,2,k+1,k,1)$ yields

$$\begin{array}{lrcl} & b_{12}+b_{2,k+1}+b_{k+1,k}+b_{k1} &=& b_0\\ \Leftrightarrow & 2b_0 &=& b_0\\ \Leftrightarrow & b_0 &=& 0. \end{array}$$

In particular, $b_a=0$ for all $a\in A^0\cap A(V\setminus V^*)$.

If $V^*=\emptyset$ go to (ii). Otherwise consider the circuits $(1,2,3,i,1)$ and $(i,k+1,\ldots,3,i)$, $i=k+2,\ldots,n$. Since $b_{12}=b_{23}=b_{3i}=0$ and $b_{j+1,j}=0$, $j=3,\ldots,k$, we obtain

$$b_{i1}=b_{i,k+1}=0, \qquad i=k+2,\ldots,n. \tag{3.56}$$

Further, the circuits $(1,2,k+1,i,1)$ and $(k+1,k,1,i,k+1)$, $i=k+2,\ldots,n$, yield

$$b_{1i}=b_{k+1,i}\overset{(3.56)}{=}0, \qquad i=k+2,\ldots,n. \tag{3.57}$$

From (3.56) follows also that

$$b_{2i}=b_{ji}=0, \qquad i=k+2,\ldots,n, j=4,\ldots,k, \tag{3.58}$$

by consideration of the circuits $(1,2,i,1)$ and $(i,k+1,\ldots,j,i)$.

Finally, if $A(V^*)\neq\emptyset$, consider any arc $(u,v)\in A(V^*)$. Then, from the triangle $(1,u,v,1)$ and the equations (3.56), (3.57) we derive

$$b_{uv}=0, \qquad (u,v)\in A(V^*).$$

Thus, $b_a=0$ for all $a\in A^0$.

(ii) Let us show now

$$\begin{array}{rl} b_a=b_{\tilde a} & \forall\, a,\tilde a\in A^+,\\ b_a=b_{\tilde a} & \forall\, a,\tilde a\in A^-, \text{ and}\\ b_a+b_{\tilde a}=0 & \forall\, a\in A^+,\tilde a\in A^-. \end{array} \tag{3.59}$$

From the circuits $(2,\ldots,k+1,2)$ and $(k+1,2,1,k+1)$ we obtain

$$\begin{array}{lrcl} & b_{k,k+1}+b_{k+1,2} &=& b_{k+1,2}+b_{21}\\ \Leftrightarrow & b_{k,k+1} &=& b_{21}. \end{array}$$

It is now easy to see that (3.59) holds. Consider the circuits

$$\begin{array}{rl} (i,\ldots,1,i) & \text{for } i=3,\ldots,k,\\ (i,\ldots,k+1,i) & \text{for } i=2,\ldots,k-1,\\ (1,\ldots,i,j,\ldots,k+1,1) & \text{for } 2\le i<j\le k, j-i\ge 2,\\ (k+1,\ldots,i,j,\ldots,1,k+1) & \text{for } 2\le j<i\le k, i-j\ge 2, \text{ and maybe}\\ (i,2,1,i) & \text{for } i=k+2,\ldots,n,\\ (i,j,\ldots,k+1,i) & \text{for } i=k+2,\ldots,n, j=3,\ldots,k. \end{array}$$

It follows immediately that

$$\begin{aligned} b_a = b_{\tilde{a}} & \quad \forall\, a, \tilde{a} \in A^-, \text{ and} \\ b_a + b_{\tilde{a}} = 0 & \quad \forall\, a \in A^+, \tilde{a} \in A^-, \end{aligned}$$

since $b_{21} = b_{k,k+1}$. Hence, (3.59) holds.

Suppose that $b_a = 0$ for some $a \in A^+ \cup A^-$. That implies $b = \mathbf{0}$, a contradiction. Further, $b^T x \leq b_0$ is a valid inequality with respect to $P_C^{\{3,\ldots,k\}}(D_n)$, and hence, $b_a > 0$ for all $a \in A^1$ and $b_a < 0$ for all $a \in A^2$.

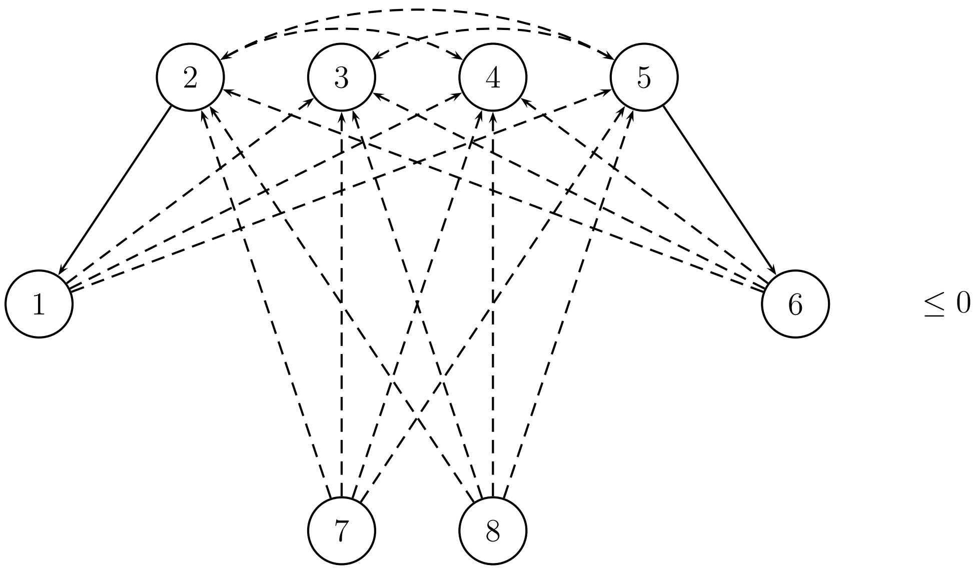


Figure 3.5: $P = (1, 2, 3, 4, 5, 6)$

□

Some experiments with PORTA for small instances imply that the cardinality-path inequality (3.49) is also facet defining for $P_C^k(D_n)$ if the number of nodes not covered by the path P is not to small ($|V \setminus V(P)| \approx k$). Applying the Procedure 3.33 yields:

(a) For $P_C^L(D_n)$, $L \subseteq \{2, \ldots, k\}$, $k \in L$, $|L| \geq 2$:

(i) If $2 \in L$, then the lifting number λ^* is equal to $\frac{-1}{k-2}$. Thus we get the inequality

$$\begin{array}{rrcl} & x(\text{dir}(P)) - \sum_{v \in \dot{P}} x(\delta^-(v)) - \frac{1}{k-2} x(A) & \leq & \frac{-k}{k-2} \\ \Leftrightarrow & (k-2) \sum_{v \in \dot{P}} x(\delta^-(v)) + x(A) - (k-2) x(\text{dir}(P)) & \geq & k. \end{array}$$

(ii) If $2 \notin L$, then it follows that $\lambda^* = 0$.

(b) For $P_C^L(D_n)$ with $L \subseteq \{k, \ldots, n\}$, $k \in L$, and $|L| \geq 2$ we obtain $\lambda^* = \frac{-1}{p-k}$, where $p := \min\{l \in L \mid l > k\}$. This leads to the inequality

$$(p-k) \sum_{v \in \dot{P}} x(\delta^-(v)) + x(A) - (p-k) x(\text{dir}(P)) \geq k$$

These inequalities are valid for the respective polytopes and even facet defining if the above conjecture is true.

3.7 The connection between s-t path polytopes and circuit polytopes

In the sequel we present two transformations of s-t paths in circuits. The first is obvious, while the second admits the interpretation that a s-t-path polytope is essentially the restriction of a circuit polytope to the hyperplane defined by

$$x(\delta^+(v)) = 1$$

for an appropriate node v. Hence, we can apply the lifting procedure 3.33 to lift facets of a s-t path polytope to the associated circuit polytope.

Let $D = (V, A)$ be a digraph on n nodes containing two special nodes s and t. We may assume that the arcset A does not contain arcs entering s and arcs leaving t. Further, let the digraph $D' = (V', A')$ defined by $V' := V$ and $A' := A \cup \{(t, s)\}$. Then a (s, t)-path $P \in \mathcal{P}^L(D)$ is equivalent to a circuit $C = P \cup \{(t, s)\} \in \mathcal{C}^{L+1}(D')$ where $L+1 := \{k+1 \,|\, k \in L\}$. The convex hull of the incidence vectors of circuits $C \in \mathcal{C}^{L+1}(D')$ containing (t, s) is the restriction of $P_C^{L+1}(D')$ to the hyperplane defined by

$$x_{ts} = 1.$$

Hence, if $d^T x \le d_0$ is a valid inequality for $P_C^{L+1}(D')$, then the inequality $\tilde{d}^T \tilde{x} \le \tilde{d}_0$ defined by

$$\begin{aligned} \tilde{d}_a &:= d_a && \forall\, a \in A \\ \tilde{d}_0 &:= d_0 - d_{ts} \end{aligned}$$

is valid with respect to $P^L_{s-t \text{ path}}(D)$.

We illustrate now the second transformation. Let again $D = (V, A)$ be a digraph on n nodes containing two special nodes s and t where the arcset A does not contain arcs entering s and arcs leaving t. We denote by $D' = (V', A')$ the digraph obtained by contracting $\{s, t\}$ to a single new node, say w, and define $\tilde{D} := D' \cup \mathcal{L}$ where

$$\mathcal{L} := \begin{cases} \{(w, w)\} & \text{if } \ (s, t) \in A \\ \emptyset & \text{otherwise.} \end{cases}$$

Then it is easy to see that a (s, t)-path $P \in \mathcal{P}^L(D)$ is equivalent to a circuit or loop $C \in \mathcal{C}^L(\tilde{D})$ containing w. Thus, the polytope $P_{s-t \text{ path}}(D)$ and the face $\tilde{F}$ of $\tilde{D}$ induced by the degree constraint

$$x(\delta^+(w)) = 1$$

are obviously isomorphic. Moreover, the facets of $P_C^L(\tilde{D})$ are in 1-1 correspondence with those of $P_C^L(D')$. This is clear for $D' = \tilde{D}$. Otherwise is $\dim P_C^L(\tilde{D}) = \dim P_C^L(D') + 1$, and thus, an inequality

$$b^T x \le b_0$$

is facet defining for $P_C^L(D')$ if and only if

$$b^T x + b_0 x_{ww} \leq b_0$$

is facet defining for $P_C^L(\tilde{D})$.

Based on these results we can apply the lifting procedure 3.33 in order to generate facets of $P_C^L(D')$ from facets of $P_{s-t \text{ path}}^L(D)$ if $x(\delta^+(w)) \leq 1$ is facet defining for $P_C^L(D')$; or more generally, valid inequalities for $P_{s-t \text{ path}}^L(D)$ can be lifted into valid inequalities for $P_C^L(D')$.

Theorem 3.49. *Let $D = (V, A)$ be a digraph on n nodes, $s, t \in V$, and $L \subseteq \{1, \ldots, n-1\}$. Suppose that A does not contain arcs entering s and arcs leaving t. Denote by $D' = (V', A')$ the digraph obtained by contracting $\{s, t\}$ to a single new node, say w. Further, let*

$$d^T x \leq d_0$$

be a valid inequality for $P_{s-t \text{ path}}^L(D)$ and

$$\lambda^* := \min\{d_0 - d^T \chi^C \mid C \in \mathcal{C}^L(D), s, t \notin V(C)\}.$$

Then the inequality

$$d'^T x' \leq d'_0$$

defined by

$$\begin{array}{rcll} d'_{ij} & := & d_{ij} & \forall\, (i,j) \in A(V \setminus \{s,t\}), \\ d'_{wi} & := & d_{si} - \lambda^* & \forall\, i \in V \setminus \{s\}, \\ d'_{iw} & := & d_{it} & \forall\, i \in V \setminus \{t\}, \text{ and} \\ d'_0 & := & d_0 - \lambda^* & \end{array}$$

is valid for $P_C^L(D')$. Moreover, if $d^T x \leq d_0$ is facet defining for $P_{s-t \text{ path}}^L(D)$, but not equivalent to the nonnegativity constraint $x_{st} \geq 0$, D chosen such that D' is a complete digraph on $n-1$ nodes, and $2 \neq L \setminus \{1\} \neq 3$, then $d'^T x' \leq d'_0$ defines a facet of $P_C^L(D')$.

Proof. Set $\tilde{D} := D' \cup \mathcal{L}$ where

$$\mathcal{L} := \begin{cases} \{(w,w)\} & \text{if } (s,t) \in A \\ \emptyset & \text{otherwise.} \end{cases},$$

and identify $P_{s-t \text{ path}}^L(D)$ with the face $\tilde{F}$ of $\tilde{D}$ induced by the degree constraint

$$x(\delta^+(z)) = 1.$$

Now let $d^T x + d_{ww} x_{ww} \leq d_0$ be a valid inequality with respect to $\tilde{F}$. Then the inequality $d^T x \leq d_0$ is valid for

$$F := \{x \in P_C^L(D') \mid x(\delta^+(w)) = 1\}.$$

The remainder follows immediately from Procedure 3.33 and the Theorems 3.22 and 3.23. $\square$

Dahl and Gouveia [12] introduced a class of inequalities for the length restricted $s-t$ path polytope $P^{\leq k}_{s-t \text{ path}}(D)$ they called *jump and lifted jump inequalities.* Let $V = \bigcup_{i=0}^{k+1} V_i$ be a partition of V, with $V_0 = \{s\}$ and $V_{k+1} = \{t\}$. Then the set

$$J := J(V_0, \ldots, V_{k+1}) := \bigcup_{i=0}^{k-1} \bigcup_{j=i+2}^{k+1} (V_i : V_j)$$

is called a $(s-t,k)$-*jump* and the associated inequality

$$\sum_{(i,j)\in J} y_{ij} \geq 1$$

jump inequality, i.e., any (s,t)-path P of length at most k must make at least one "jump" from an nodeset V_i to an nodeset V_j, with $j-i \geq 2$.

If

$$\bigcup_{i=0}^{k} \bigcup_{j=i+1}^{k+1} (V_j : V_i) \neq \emptyset,$$

then, under certain conditions, the path P must make more than one "jump". This suggests that the following *lifted jump inequality* may be valid for adequate choices of arcsets $B \subseteq A$:

$$\sum_{(i,j)\in J} y_{ij} \geq 1 + \sum_{(i,j)\in B} y_{ij}.$$

Dahl and Gouveia showed that one valid instantiation of B is given by the arcset

$$(V_{k-1} \cup V_k : V_1 \cup V_2),$$

that is,

$$\begin{array}{lll} \sum_{(i,j)\in J} y_{ij} - y((V_{k-1} \cup V_k : V_1 \cup V_2)) & \geq 1 & \text{for all } (s-t,k)\text{-jumps} \\ & & J = J(\{s\}, V_1, \ldots, V_k, \{t\}) \end{array} \tag{3.60}$$

are valid inequalities for $P^{\leq k}_{s-t \text{ path}}(D)$. Furthermore, in case $k=3$ they proved

Theorem 3.50 ([12]). *A complete linear description of $P^{\leq 3}_{s-t \text{ path}}(D)$ is given by the following system of equations and inequalities:*

$$y(\delta^+(v)) - y(\delta^-(v)) = \begin{cases} 1 & \text{if } v = s, \\ 0 & \text{if } v \in V \setminus \{s,t\}, \\ -1 & \text{if } v = t. \end{cases} \tag{3.61}$$

$$\sum_{(i,j)\in J} y_{ij} - \sum_{i\in V_3 \cup \{p\}} \sum_{j \in V_1 \cup \{p\}} y_{ij} \geq 1 \quad \textit{for all } (s-t,3)\textit{-jumps} \quad J = J(\{0\}, V_1, \{p\}, V_3, \{t\}) \tag{3.62}$$

$$y_a \geq 0 \quad \forall\, a \in A. \tag{3.63}$$

□

By using the flow conservation constraints (3.61) for all $v \in V_3 \cup \{t\}$ on the left-hand side of (3.62) and canceling equal terms we obtain the following equivalent inequality:

$$y_{si} - \sum_{j \in V \setminus \{s,t\}} y_{ij} \geq 0 \qquad i \in V \setminus \{s,t\}. \tag{3.64}$$

We will sketch the proof of Theorem 3.50 in an appropriate form in order to draw later a further conclusion. The set of all (s,t)-paths of length at most 3 can be represented by the following binary variables: For a given internal arc (i,j), variable z_{ij} is associated with the 3-path $\{(s,i),(i,j),(j,t)\}$. For a given node $j \in V \setminus \{s,t\}$, variable z_j is associated with the 2-path $\{(s,j),(j,t)\}$. Finally, variable z_{st} is associated with the 1-path $\{(s,t)\}$. An extended formulation is now given by the following system:

$$z_{ij} = y_{ij} \qquad \forall\, i,j \in V, i \neq s, j \neq t \tag{3.65}$$

$$z_{st} = y_{st} \tag{3.66}$$

$$\sum_{j \in V \setminus \{s,t,i\}} z_{ij} + z_i = y_{si} \qquad \forall\, i \in V \setminus \{s,t\} \tag{3.67}$$

$$\sum_{i \in V \setminus \{s,t,j\}} z_{ij} + z_j = y_{jt} \qquad \forall\, j \in V \setminus \{s,t\} \tag{3.68}$$

$$\sum_{i,j \in V \setminus \{s,t\}} z_{ij} + \sum_{j \in V \setminus \{s,t\}} z_j + z_{st} = 1 \tag{3.69}$$

$$z_{ij} \geq 0 \qquad \forall\, i,j \in V, i \neq s, j \neq t \tag{3.70}$$

$$z_j \geq 0 \qquad \forall\, j \in V \setminus \{s,t\} \tag{3.71}$$

$$z_{st} \geq 0. \tag{3.72}$$

Dahl and Gouveia showed that $P^{\leq 3}_{s-t \text{ path}}(D)$ is the projection of (3.65)-(3.72) into the space of y_{ij} variables, and the system (3.61)-(3.63) can be obtained by using Fourier-Motzkin elimination.

For $k = 4$ it can be shown that the lifted jump inequalities (3.60) are equivalent to the following inequalities:

$$y((s:V_1)) + y((V_2:t)) - y((V_1:V_2)) \geq 0 \quad \forall\, V_1, V_2 \subseteq V \setminus \{s,t\}, \; |V_1|,|V_2| \geq 1, V_1 \cap V_2 = \emptyset. \tag{3.73}$$

Moreover, Dahl and Gouveia showed

Theorem 3.51 ([12])**.** *An inequality (3.73) is facet defining for $P^{\leq 4}_{s-t \text{ path}}(D)$ if and only if either*

(i) $V \setminus (\{s,t\} \cup V_1 \cup V_2)$ is nonempty or

(ii) $|V_1| \geq 2$ and $|V_2| \geq 2$.

□

By applying Theorem 3.49 we can lift the inequalities (3.60) which are valid (facet inducing) for $P^{\leq k}_{s-t \text{ path}}(D)$ into valid (facet inducing) inequalities for $P^{\leq k}_C(D')$. We illustrate the lifting procedure for the inequalities (3.64) and (3.73):

Example 3.52 (Inequality (3.64))**.** The inequality

$$y_{si} - \sum_{j\in V\setminus\{s,t\}} y_{ij} \geq 0 \qquad i \in V \setminus \{s,t\},$$

which is valid for $P^{\leq 3}_{s-t \text{ path}}(D)$ corresponds to the inequality

$$x_{wi} - \sum_{j\in V'\setminus\{w\}} x_{ij} \geq 0 \qquad i \in V' \setminus \{w\}.$$

Here V' is the nodeset of the digraph D' which is obtained by contracting $\{s,t\}$ to the new node w. This inequality is valid for the polytope $\{x \in P^{\leq 3}_C(D') \mid x(\delta^+(w)) = 1\}$. Subtracting the equation $x(\delta^+(w)) = 1$ and multiplying then with -1 we obtain the inequality

$$\sum_{j\in V'\setminus\{w,i\}} [x_{wj} + x_{ij}] \leq 1 \qquad i \in V' \setminus \{w\}$$

which is valid for $P^{\leq 3}_C(D')$.

Corollary 3.53. *Let $D_n = (V,A)$ be the complete digraph on n nodes, let $w \in V$, and define $P^L_C(D_n)_{\delta^+(w)} := \{x \in P^L_C(D_n) \mid x(\delta^+(w)) = 1\}$.*

(a) The polytope $P^{\leq 3}_C(D_n)|_{\delta^+(w)}$ is determined by the system

$$\begin{array}{rcll} x(\delta^+(v)) - x(\delta^-(v)) & = & 0 & \forall\, v \in V \\ x(\delta^+(w)) & = & 1 & \\ \sum_{j\in V\setminus\{w,i\}} [x_{wj} + x_{ij}] & \leq & 1 & \forall\, i \in V \setminus \{w\} \\ x_a & \geq & 0 & \forall\, a \in A. \end{array}$$

Moreover, the inequalities

$$\sum_{i\in V\setminus\{u,v\}} [x_{ui} + x_{vi}] \leq 1 \qquad \forall\, u,v \in V, u \neq v$$

define facets of $P^{\leq 3}_C(D_n)$.

(b) The polytope $P^3_C(D_n)|_{\delta^+(w)}$ is determined by the system

$$\begin{array}{rcll} x(\delta^+(i) \setminus \{(i,w)\}) - x_{wi} & = & 0 & \forall\, i \in V \setminus \{w\} \\ x(\delta^+(j) \setminus \{(w,j)\}) - x_{jw} & = & 0 & \forall\, j \in V \setminus \{w\} \\ x(A(V \setminus \{w\})) & = & 1 & \\ x_{ij} & \geq & 0 & \forall\, (i,j) \in A(V \setminus \{w\}). \end{array} \tag{3.74}$$

Proof. (a) The first statement is a direct consequence of Theorem 3.50 and the comments in Example 3.52. The second statement is so easy to show that we leave it as an exercise.

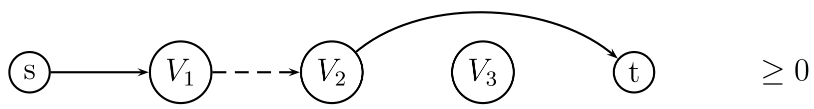
(b) Consider the above extended formulation. Set all binary variables to zero which are associated to (s,t)-paths of length at most two, i.e., $z_{st} = 0$, $z_j = 0$. Further define $x_{ij} := y_{ij}$ for all $i \neq s$, $j \neq t$, and $x_{wi} := y_{si}$, $x_{iw} := y_{it}$ for all $s \neq i \neq t$. Then Fourier-Motzkin elimination leads to the system (3.74). □

Thus one can suggest the following algorithm to solve the LRCPs

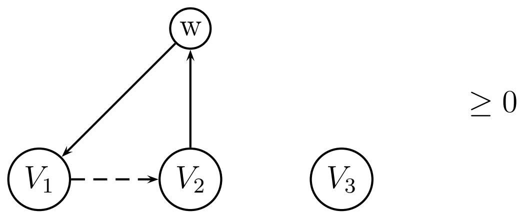
$$\begin{pmatrix} \min c^T\chi^C \\ C \in \mathcal{C}^{\le 3}(D_n) \end{pmatrix} \text{ and } \begin{pmatrix} \min c^T\chi^C \\ C \in \mathcal{C}^{3}(D_n) \end{pmatrix}:$$

For each $v \in V$ solve the LP $\min c^T x$, $x \in P_C^{\le 3}(D_n)|_{\delta^+(v)}$ ($x \in P_C^3(D_n)|_{\delta^+(v)}$), and choose among the optimal solutions $x^*(v)$ the solution with the minimum weight. Such an algorithm makes sense only when its running time is in general shorter than a simply enumeration algorithm.

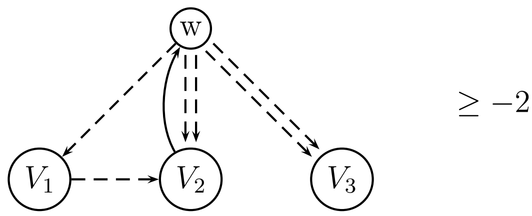
Example 3.54 (Inequality (3.73))**.** The support graph of an inequality (3.73), denoted by $d^T y \ge 0$, is sketched in Figure 3.6(a), where V_3 is maybe empty. The contraction of the nodeset $\{s, t\}$ to a single new node, say w, yields an inequality $d'^T x \ge 0$, whose support graph is sketched in Figure 3.6(b). Denote the digraph obtained by this contraction by D'. It is not hard to see that that $\min\{d'^T\chi^C \mid C \in P_C^{\le 4}(D'), w \notin C\} = -2$, and thus, $d'^T x - 2x(\delta^+(w)) \ge -2$, illustrated in Figure 3.6(c), is facet defining for $P_C^{\le 4}(D')$ if $d^T x \ge 0$ is facet defining for $P_{s-t \text{ path}}^{\le 4}(D)$.



(a)



(b)



(c)

Figure 3.6 Illustration of Example 3.55

By multiplying $d'^T x - 2x(\delta^+(w)) \ge -2$ with -1 we obtain

Corollary 3.55. *Let $D_n = (V, A)$ be the complete digraph on n nodes, $w \in V$, and $V_1, V_2 \subseteq V \setminus \{w\}$ be nonempty disjoint nodesets such that either*

(i) $V \setminus (\{w\} \cup V_1 \cup V_2)$ is nonempty or

(ii) $|V_1| \ge 2$ and $|V_2| \ge 2$.

Then the inequality

$$2x(\delta^+(w)) - x((w : V_1)) - x((V_2 : w)) + x((V_1 : V_2)) \le 2$$

is facet defining for $P_C^{\le 4}(D_n)$. □

A final comment is in order. Due to the fact that in the most cases $\dim P_C^L(D_n)|_{\delta^+(w)} = \dim P_C^L(D_n)$ we can deduce the dimension of the length restricted path polytope.

Corollary 3.56. *Let $D = (V, A)$ be a digraph on n nodes and let $s, t \in V$ such that D is obtained from the complete digraph on nodeset V by removing all arcs in $(\delta^-(s) \cup \delta^+(t))$. Moreover, let $\emptyset \neq L \subseteq \{1, \dots, n-1\}$ be a set of feasible lengths (with respect to $s-t$-paths) and let*

$$\alpha = \begin{cases} 1 & \text{if } 1 \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\dim P^L_{s-t \text{ path}}(D)$

$$= \begin{cases} & 0 & \text{if } L = \{1\}, \\ & n-3+\alpha & \text{if } L \setminus \{1\} = \{2\}, \\ & (n-2)(n-3)-1+\alpha & \\ & = n^2-5n+5+\alpha & \text{if } L \setminus \{1\} = \{3\}, \\ \dim P_C^k(D_{n-1}) - 1 + \alpha & = (n-2)^2 - 2 + \alpha & \text{if } L \setminus \{1\} = \{k\}, 4 \leq k \leq n-2, \\ \dim P_C^{n-1}(D_{n-1}) + \alpha & = n^2-5n+5+\alpha & \text{if } L \setminus \{1\} = \{n-1\}, \\ \dim P_C^(D_{n-1}) - 1 + \alpha & = (n-2)^2 - 1 + \alpha & \text{if } |L \setminus \{1\}| \geq 2. \end{cases}$$

□

Chapter 4

Separation

In this chapter we investigate the separation problem for the considered inequalities, which includes partly references to the literature for exact and heuristics algorithms as well as complexity results.

Let us start with some notes on the separation problem. It is usually defined as follows: Given a point $x^* \in \mathbb{R}^A$, $x^* \geq \mathbf{0}$, and a family $\mathcal{F}$ of inequalities, find a violated member of $\mathcal{F}$, i.e., an inequality $b^T x \leq b_0$ belonging to $\mathcal{F}$ with $b^T x^* > b_0$. But sometimes it is profitable to find a most violated member, i.e., an inequality $b^T x \leq b_0$ belonging to $\mathcal{F}$ and maximizing the **degree of violation** $b^T x^* - b_0$ (optimization version), because sometimes a maximally violated inequality exhibit a strong combinatorial structure, which can be exploited for separation (see Caprara et al. [10]).

Further, note that the separation problem depends strongly on the exact definition of the family $\mathcal{F}$. For example, it can arise the case that the separation problem for $\mathcal{F}$ is solvable in polynomial time but for a subclass $\mathcal{F}' \subseteq \mathcal{F}$ it could be NP-hard (see for example [15]).

We come now to the results on the separation problem for $P_C^L(D_n)$. We suppose that a givem point $x^* \in \mathbb{R}^A$ satisfies the flow constraints (2.1).

4.1 Symmetric inequalities

The connection between symmetric inequalities $b^T x \leq b_0$ and the associated inequalities $\sum_{[i,j] \in E} b_{ij} y_{[i,j]} \leq b_0$ implies that every separation algorithm for $P_C^L(K_n)$ can be used, as a "black box", for $P_C^L(D_n)$ as well. To this end, given the point x^* and a family $\mathcal{F}$ of symmetric inequalities one first defines the undirected counterparts y^* and $\mathcal{F}^*$ of x^* and $\mathcal{F}$, respectively, by the transformation

$$\begin{array}{rcll} y^*_{[i,j]} & := & x^*_{ij} + x^*_{ji} & \forall\ [i,j] \in E, \\ \sum\limits_{(i,j) \in A} b_{ij} x_{ij} \leq b_0 & \rightarrow & \sum\limits_{[i,j] \in E} b_{ij} y_{[i,j]} \leq b_0, & \end{array}$$

and then applies the separation algorithm (for $P_C^L(K_n)$) to y^* and the class $\mathcal{F}^*$. On return, the detected violated inequality $b^T y \leq b_0 \in \mathcal{F}^*$ is transformed into its counterpart $\sum_{e \in E} b_e (x_{ij} + x_{ji}) \leq b_0 \in \mathcal{F}$.

The following inequalities are equivalent to symmetric inequalities.

4.1.1 Disjoint circuits elimination constraints

Class: $\{x(\delta^+(p)) + x(\delta^+(q)) - x((S:T)) \leq 1 \mid V = S \cup T, p \in S, q \in T\}$
Complexity: polynomial

Given $x^* \in \mathbb{R}^A$, there is a violated disjoint circuits elimination constraint if there are nodes $p, q \in V$ and a bipartition S, T of V with $p \in S$, $q \in T$ such that

$$x((S:T)) < x(\delta^+(p)) + x(\delta^+(q)) - 1.$$

Thus the separation problem for the dce inequalities can be solved in polynomial time, e.g., by applying $n(n-1)$ times a minimum (i,j)-cut algorithm.

Since x^* satisfies the flow constraints, there is a violated dce inequality if and only if there exists a bipartition $V = S \cup T$ with

$$x^*((S:T)) + x^*((T:S)) < x(\delta^+(p)) + x(\delta^-(p)) + x(\delta^+(q)) + x(\delta^-(q)) - 2.$$

Hence, we can transform the separation problem to the corresponding separation problem of finding a bipartition $V = S \cup T$ of V with

$$y^*((S:T)) < y^*(\delta(p)) + y^*(\delta(q)) - 2.$$

for the corresponding point $y^* \in \mathbb{R}^E$. This separation problem can be efficiently solved by applying the Gomory-Hu algorithm (see Bauer [7]).

4.1.2 Parity constraints

Class: $\{x(\delta^+(v) \setminus (v,w)) - x_{wv} \geq 0 \mid v, w \in V, v \neq w\}$
Complexity: polynomial

Given a point $x^* \in \mathbb{R}^A_+$, the separation problem for the parity constraints can be solved in computational time $\mathcal{O}(n^2)$ by checking all of them.

4.1.3 Cut inequalities

Class: $\{x((S:T) \setminus (i,j)) - x_{ji} \geq 0 \mid V = S \cup T, 1 \leq |S| \leq n-1\}$
Remark: not necessarily facet defining
Complexity: polynomial

Given a point $x^* \in \mathbb{R}^A_+$ satisfying the flow constraints, there is a violated cut inequality if there exists a bipartition $V = S \cup T$ of V with $x^*((S:T)) < x^*_{ij} + *x_{ji}$. Thus the separation problem for the cut inequalities can be solved in polynomial time, e.g., by applying $n(n-1)$ times a minimum (i,j)-cut algorithm. (See also 4.1.1.)

4.1.4 Bipartition inequalities

Class: $\{x(A(S))+x(A(T))+x((S:T)) \geq \frac{k+1}{2} \mid V = S \cup T, 1 \leq |S| \leq n-1\}$
Complexity: NP-hard

Given $x^* \in \mathbb{R}^A$, there is a violated bipartition inequality if there is a bipartition S, T of the nodeset V such that

$$x^*((T:S)) > x^*(A) - \frac{k+1}{2}.$$

Since x^* satisfies the flow constraints (2.1), the problem is equivalent to the problem of finding a bipartition S, T of V such that

$$x^*((S:T)) + x^*((T:S)) > 2x^*(A) - k - 1.$$

This inequality is symmetric and corresponds to the inequality

$$y^*((S:T)) > 2y^*(E) - k - 1$$

by setting $y^*_e := (x^*_{ij} + x^*_{ji})/2$ for all $e = ij \in E$ where $K_n = (V, E)$ is the associated graph.

But this is exactly the separation problem for the undirected bipartition inequalities, and Bauer [7] showed that it is NP-hard for $k = 3$. It is no problem to extend her proof to any odd k, $k \geq 5$. (See Appendix B). Thus the separation problem for the class of bipartition inequalities is NP-hard.

4.1.5 Cardinality-path inequalities

Class: $\{x(\mathrm{dir}(P)) - \sum_{v \in \dot{P}} x(\delta^-(v)) \leq 0 \mid P \text{ path in } K_n, |P| = k\}$
Complexity: NP-hard

Due to the flow constraints the cardinality-path inequality is equivalent to the symmetric inequality

$$2x(\mathrm{dir}(P)) - \sum_{v \in \dot{P}} [x(\delta^-(v)) + x(\delta^+(v))] \leq 0.$$

For the complexity result and a heuristic see Bauer, Linderoth, and Savelsbergh [8].

4.1.6 Cardinality-tree Inequalities

The complexity of the separation problem for the cardinality-tree inequalities is unknown, but there are some known polynomial cases for the symmetric counterpart (see Bauer, Linderoth, Savelsbergh [8].

4.2 Asymmetric inequalities

4.2.1 Linear ordering constraints

Class: $\{\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} x_{\pi(i),\pi(j)} \geq 1 \mid \pi \text{ permutation of } V\}$
Complexity: NP-hard

The separation problem for the class of linear ordering constraints is NP-hard, since it is a linear ordering problem (LOP). For a heuristic we refer to Chanas and Kobylanski [11]. They developed an algorithm which uses the fact that a permutation $(v_1, v_2, \ldots, v_n)$ is a solution of the LOP maximizing the objective function if and only if the permutation $v_n, v_{n-1}, \ldots, v_1$ is a solution of the LOP minimizing the objective function.

4.2.2 Asymmetric maximal set inequalities

Class: $\{\sum_{i=1}^{s} x(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x((V_i : V_j)) \geq 1 \mid \text{Conditions see } (i), (ii)\}$

(i) $V = \bigcup_{i=1}^{s} V_i$ partition with $s \geq 1$

(ii) $M_i \subseteq E(V_i)$ maximal set not containing any undirected circuit of length less than or equal to k where $4 \leq k < n$, $i = 1, \ldots, s$

Complexity: unknown

Let $D = (V, A)$ be a complete digraph and $G = (V, E)$ be the associated graph. Given a point $x^* \in \mathbb{R}_+^A$, the separation problem for the asymmetric maximal set inequalities is to find a partition of the nodes $V = \bigcup_{i=1}^{s} V_i$ and in each V_i a maximal edge set M_i not containing any (undirected) circuit of length less than or equal to k such that

$$\sum_{i=1}^{s} x^*(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*((V_i : V_j)) < 1.$$

Since the polytope $P_C^{\{3,\ldots,k\}}$ contains amongst others symmetric points, i.e., such points x with $x_{ij} = x_{ji}$ for all $i, j \in V$, $i \neq j$, it is not absurd to assume that x^* is symmetric. Then

$$\begin{aligned}
&\exists\, V_i, M_i: && \textstyle\sum_{i=1}^{s} x^*(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*((V_i : V_j)) && < 1 \\
\Leftrightarrow\; &\exists\, V_i, M_i: && \textstyle\sum_{i=1}^{s} x^*(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*((V_j : V_i)) && < 1 \\
\Leftrightarrow\; &\exists\, V_i, M_i: && \textstyle 2\sum_{i=1}^{s} x^*(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*(\text{dir}((V_i : V_j))) && < 2.
\end{aligned}$$

Define $y^* \in \mathbb{R}^E$ by $y_e^* := \frac{1}{2}(x_{ij}^* + x_{ji}^*)$ for all $e = ij$. Then

$$\begin{aligned}
&\exists\, V_i, M_i: && \textstyle 2\sum_{i=1}^{s} x^*(\text{dir}(\bar{M}_i)) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*(\text{dir}((V_i : V_j))) && < 2. \\
\Leftrightarrow\; &\exists\, V_i, M_i: && \textstyle 2\sum_{i=1}^{s} x^*(\bar{M}_i) + \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} x^*((V_i : V_j)) && < 2,
\end{aligned}$$

i.e., if x^* is symmetric, then the separation problem for the asymmetric maximal set inequalities is as hard as the separation problem for the maximal set inequalities.

Bauer et al. [8] showed that the separation problem for a special subclass of maximal set inequalities is NP-hard. They restricted themselves to $s = 1$ and to a subset of maximal sets. The subset consists of all spanning trees whose fundamental circuits are of length at most k. Then the separation problem is to find a spanning tree T of G such that $y^*(T) > y^*(E) - 1$ and a longest path in T with at most $k - 1$ edges. Since this problem is NP-hard, the separation problem for the maximal set inequalities and thus for the asymmetric maximal set inequalities seems to be NP-hard. But this is not a proof for the class of facet defining maximal set inequalities, since in case $s = 1$ they are not facet defining for $P_C^{\leq k}(K_n)$. Moreover, it should be examined whether it is sufficient to consider only the set of spanning trees as subsets of maximal sets or not.

4.2.3 Generalized linear ordering constraints

Class: $\{\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} x((V_i : V_j)) \geq 1 | V = \bigcup_{i=1}^{m} V_i$ partition with $(i)-(iii)\}$
(i) $|V_i| \leq k-1$ for $1 \leq i \leq m$
(ii) $|V_i| + |V_j| \geq k$ for $1 \leq i < j \leq m$
(iii) $|V_1| + |V_i| \geq k$ for $i = 2, \ldots, m$
Complexity: unknown

Given a point $x^* \in \mathbb{R}_+^A$ and an integer k with $3 \leq k \leq |V| - 1$, the separation problem for the generalized linear ordering constraints is to find a partition $\bigcup_{i=1}^m V_i$ of the nodeset V satisfying (i)-(iii) such that

$$\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x((V_i : V_j)) < 1. \tag{4.1}$$

The class $\mathcal{F}$ of generalized linear ordering constraints is contained in the class $\mathcal{F}^*$ of the inequalities (4.1) whose associated partition satisfies only item (i). $\mathcal{F}^*$ is a family of valid inequalities for the polytopes $P_C^L(D_n)$ with $L \subseteq \{k, \ldots, n\}$. The next theorem indicates that the separation problem for $\mathcal{F}$ is hard.

Theorem 4.1. *The separation problem for the family $\mathcal{F}^*$ of inequalities is NP-hard.*

Proof. We will show that the corresponding decision problem is NP-complete.
PROBLEM: Generalized linear ordering separation problem (GLOSP1)
INSTANCE: Complete digraph $D = (V, A)$, a vector $x^* \in \mathbb{R}_+^A$ with $\mathbf{1}^T x^* \geq 1$ and $x_{ij}^* = x_{ji}^*$ for all $i, j \in V$, $i \neq j$, and an integer k with $3 \leq k < n$.
QUESTION: Is there a partition of $V = \bigcup_{i=1}^m V_i$ with $m \geq 2$ such that $|V_i| \leq k-1$ for $i = 1, \ldots, m$ and such that $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} x^*((V_i : V_j)) < 1$?

GLOSP1 is clearly in NP. Moreover, it is at least as hard as the following NP-complete graph partitioning problem (GPP) [22]:
PROBLEM: Graph partitioning problem (GPP)
INSTANCE: Graph $G = (V, E)$, a weight vector $c \in \mathbb{R}_+^E$, and positive integers K, J.
QUESTION: Is there a partition of $V = \bigcup_{i=1}^m V_i$ with $m \geq 2$ such that $|V_i| \leq K$ for $i = 1, \ldots, m$ and such that $\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} c((V_i : V_j)) \leq J$?

Given an instance of GPP, we construct an instance of GLOSP1 by letting $G' = (V, E')$ be the complete graph on nodeset V and setting $c'_e := c_e$ if $e \in E$ and $c'_e := 0$ otherwise, $D := \mathrm{dir}(G')$, $k := K + 1$, and

$$x_{ij}^* := x_{ji}^* := \frac{1}{2J} c'_e \qquad \forall\ e = ij \in E'.$$

With these definitions, there exists obviously a partition $V = \bigcup_{i=1}^{m} V_i$ with $m \geq 2$, $|V_i| \leq K$, and $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} c((V_i : V_j)) \leq J$ if and only if there exists a partition $V = \bigcup_{i=1}^{m} V_i$ with $m \geq 2$, $|V_i| \leq k-1$, and $\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} x^*((V_i : V_j)) \leq 1$. □

Chapter 5

Conclusions

At the end of this thesis we want to evaluate some results. The study of the facial structure of the length restricted circuit polytope $P_C^L(D_n)$ has shown that only the degree, disjoint circuits elimination, and nonnegativity constraints define facets of $P_C^L(D_n)$ independent of L, with some exceptions. All other considered inequalities depends on L.

For the study of the facial structure of $P_C^L(D_n)$ the polytopes $P_C^k(D_n)$ with $3 \leq k < n$ are the most important of all length restricted circuit polytopes, because facet defining inequalities for $P_C^k(D_n)$ can be lifted to facet defining inequalities for $P_C^L(D_n)$ with $L \subseteq \{2, \ldots, k\}$ or $L \subseteq \{k, \ldots, n\}$ by using standard sequential lifting.

Expectedly, the undirected circuit polytopes and directed circuit polytopes are closely related. We have some inequalities which are facet defining for $P_C(K_n)$ or $P_C^{\leq k}(K_n)$ transformed to facet defining inequalities for $P_C^k(D_n)$ or $P_C^{\{3,\ldots,k\}}(D_n)$ and generalized partly. For the considered inequalities the transformation is very easy (for example, undirected cuts $(V_i : V_j) \cup (V_j : V_i)$ will be substituted by directed cuts $(V_i : V_j)$). Conversely we have shown that the undirected versions of symmetric inequalities which are facet defining for $P_C^L(D_n)$, $2 \notin L$, are facet defining for $P_C^L(K_n)$.

Moreover, we have seen that in generally the length restricted path polytope can be transformed into a facet of an appropriate length restricted circuit polytope. Hence it would be an important subject to study the facial structure of the length restricted path polytope. Some good reasons argue for solving the LRCP by means of branch and cut algorithms for associated path polytopes. For example, the results so far seem not to lead to a tractable description of the dominant of $P_C(D_n)$, although the corresponding optimization problem can be solved in polynomial time. In contrast, the dominant $\text{dmt}(P_{s-t \text{ path}}(D))$ of the path polytope $P_{s-t \text{ path}}(D)$ is determined by nonnegativity constraints and a class of cut inequalities which can be separated in polynomial time (see Schrijver [26]). It should be possible to give a linear description of the upper path polyhedron $U_{s-t \text{ path}}(D) := C_{s-t \text{ path}}(D) + P_{s-t \text{ path}}(D)$, if there is an analogous connection between $\text{dmt}(P_{s-t \text{ path}}(D))$ and $U_{s-t \text{ path}}(D)$ as between $\text{dmt}(P_C(D))$ and $U_C(D)$. Another reason is that we have complete linear descriptions of the polytopes $P^3_{s-t \text{ path}}(D)$ and $P^{\leq 3}_{s-t \text{ path}}(D)$ (see Dahl und Gouveia [12] and 3.7), but for the corresponding polytopes $P_C^3(D_n)$ and $P_C^{\leq 3}(D_n)$ we have not found them. Hence, I would plead to focus the study to length restricted path polytopes.

Appendix A

Generalized bipartition inequalities

Theorem A.1. *Let $2 \notin L$, $3, 4 \in L$, $D_n = (V, A)$, $n \geq 6$, be the complete digraph on n nodes, and S, T be a bipartition of V with $|S| \geq 3$. Further, let $K_n = (V, E)$ be the associated complete graph on nodeset V, that is, $\mathrm{dir}(K_n) = D_n$, let $M \subseteq E(S)$ be a matching, and let $H = \cup_{e \in M} H_e$ be a M-cover of T. Then a facet of $P_C^L(D_n)$ is given by the* ***generalized bipartition inequality***

$$c^T x \geq 4, a \in \mathbb{R}^A, \text{ with } c_a = \begin{cases} 0, & \text{if } a \in \mathrm{dir}M, \\ 3, & \text{if } a \in \mathrm{dir}H, \\ 1, & \text{if } a \in \mathrm{dir}((S:T)) \setminus \mathrm{dir}(H), \\ 2, & \text{otherwise, i.e. if } a \in A(S) \setminus \mathrm{dir}(M) \\ & \text{or } a \in A(T), \end{cases} \tag{A.1}$$

as long as we do not have one of the following two cases:

(i) $M = \{(p, q), (q, p)\}$ *and all arcs of H are either incident with p or with q;*

(ii) $n = 6$, $|S| = 4$ *and* $|M| = 2$, *i.e.,* $|\mathrm{dir}(M)| = 4$.

Sketch of proof. The validity of the generalized bipartition inequality is easily checked.

We show the facet defining property by assuming that we have a valid inequality $b^T x \geq b_0$, $b \in \mathbb{R}^A$, $b \neq \mathbf{0}$, in 1-rooted form, such that $\{x \in P_C^L(D_n) \mid c^T x = 4\} \subseteq \{x \in P_C^L(D_n) \mid b^T x = b_0\}$. Let w.l.o.g. $1 \in S$, and define $S' := S \setminus \{1, 2\}$. Moreover, let w.l.o.g. $M \neq \emptyset$, $(1, 2), (2, 1) \in \mathrm{dir}(M)$, and define

$$B := \{a = (u, v) \in A(T) \mid (1, u) \in \mathrm{dir}(H) \text{ and } (2, v) \in \mathrm{dir}(H), \text{ or conversly}\}.$$

Case 1: $|M| = 1$, i.e., $|\mathrm{dir}(M) = 2|$.

Since $n \geq 6$, we have $|S| \geq 4$ or $|T| \geq 3$. Due to (i), to each arc $a \in A$ exists at least one weighted subdigraph D^c of D_n containing a as depicted in Figures A.1(a), A.2(a), and A.3(a).

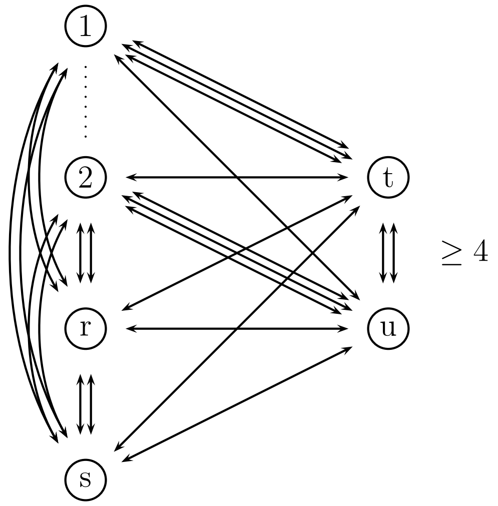


Figure A.1(a)

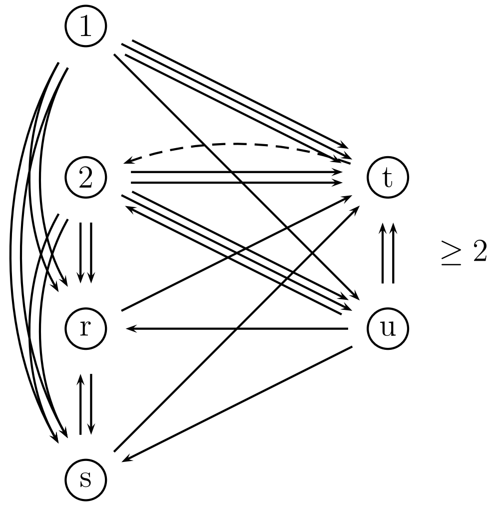


Figure A.1(b) 1-rooted form

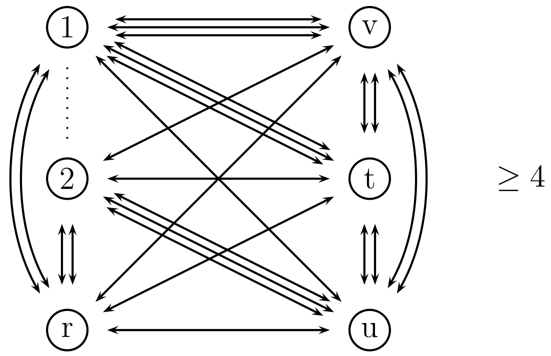


Figure A.2(a)

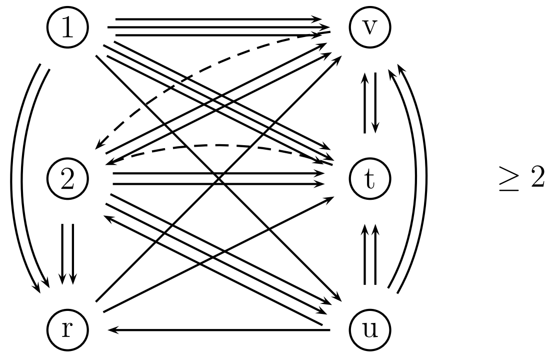


Figure A.2(b) 1-rooted form

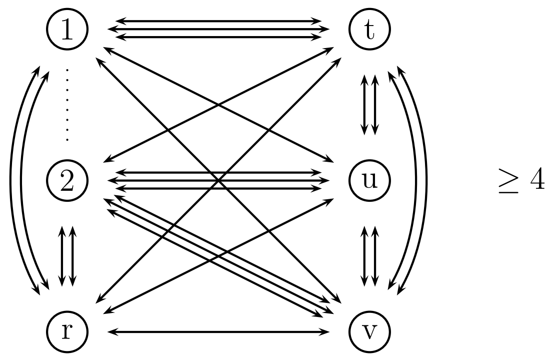


Figure A.3(a)

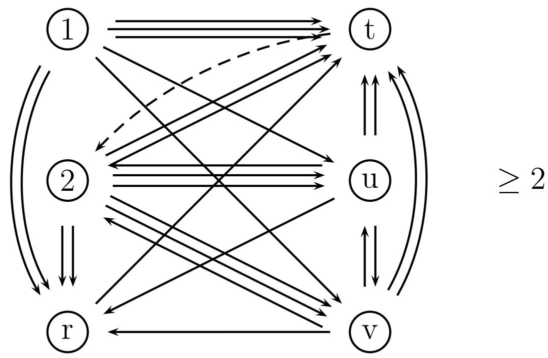


Figure A.3(b) 1-rooted form

The subdigraphs have a common subdigraph, namely

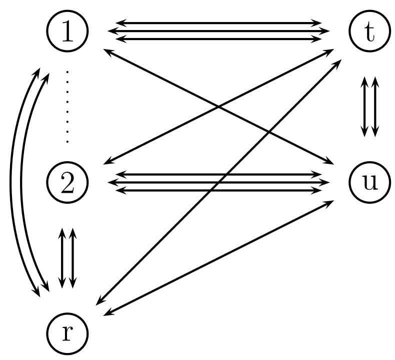


Figure A.4

Let us first consider some coefficients b_a, $a \in A$, on the common subdigraph.

1. We have $b_{i1} = 0$ for $i = 2, r, t, u$, since $b^T x \geq b_0$ is 1-rooted.

2. $b_{r2} = 0$ and $b_{1r} = b_0$.

   From the circuit $(1, u, r, 1)$, we derive $b_{1u} + b_{ur} = b_0$, and hence the circuit $(1, u, r, 2, 1)$ yields $b_{r2} = 0$. Further, we have $b_{1r} + b_{r2} + b_{21} = b_0$, and thus $b_{1r} = b_0$.

3. $b_{ru} = 0$.

   Consider the circuit $(1, r, u, 1)$.

4. $b_{2r} = b_0$, $b_{12} = 0$, $b_{2t} = b_0$ and $b_{2u} = b_0$.

   From the circuit $(1, r, t, 2, 1)$, we derive $b_{rt} + b_{t2} = 0$, and hence follows from the circuit $(r, t, 2, r)$ $b_{2r} = b_0$. Moreover, we can conclude $b_{12} = 0$ and b_{2t} by considering the circuits $(1, 2, r, 1)$ and $(1, 2, t, 1)$ Finally, we derive $b_{2u} = b_0$ from the circuit $(1, 2, u, 1)$.

5. $b_{tr} = 0$.

   Since $b_{r2} = 0$ and $b_{2t} = b_0$, the circuit $(t, r, 2, t)$ yields the desired results.

6. $b_{ut} = b_0$ and $b_{tu} = 0$.

   Consider the circuits $(1, 2, t, u, 1)$ and (r, u, t, r).

We investigate now the remaining coefficients b_a of the subdigraph as depicted in Figure A.1(a). Since r and s are clones it follows that $b_{s1} = b_{s2} = b_{su} = b_{ts} = 0$ and $b_{1s} = b_{2s} = b_0$. Further, we have

$$\begin{array}{rlcl} & b_{ur} + b_{rt} + b_{tu} & = & b_0 \\ \wedge & b_{ur} + b_{rs} + b_{su} & = & b_0 \end{array} \quad \overset{b_{tu}=b_{su}=0}{\Longrightarrow} \quad b_{rs} = b_{rt},$$

$$\begin{array}{rlcl} & b_{t2}+b_{2s}+b_{st} & = & b_0 \\ \wedge & b_{t2} \qquad +b_{21}+b_{1u}+b_{ut} & = & b_0 \end{array} \overset{b_{2s}=b_{ut}=b_0,}{\underset{b_{21}=0}{\Longrightarrow}} \quad b_{st} = b_{1u},$$

and as can easily be seen that $b_{rt} = b_{st}$, and thus $b_{1u} = b_{st} = b_{rt} = b_{rs}$. From the circuit (r, s, t, r) we obtain finally $b_{1u} = b_{st} = b_{rt} = b_{rs} = \frac{b_0}{2}$. Now it is easy to see that $b_{u2} = \frac{b_0}{2}$, $b_{t2} = -\frac{b_0}{2}$, $b_{1t} = \frac{3}{2}b_0$ and $b_{ur} = b_{us} = b_{sr} = \frac{b_0}{2}$.

In Figure A.2(a) the nodes t and v are clones, and hence $b_{2v} = b_{uv} = b_0$ and $b_{vu} = b_{vr} = 0$. Further,

$$\begin{array}{rlcl} & b_{t2}+b_{2r}+b_{rt} & = & b_0 \\ \wedge & b_{t2} \qquad +b_{2v}+b_{vt} & = & b_0 \end{array} \overset{b_{2r}=b_{2v}=b_0}{\Longrightarrow} \quad b_{rt} = b_{vt},$$

$$\begin{array}{rlcl} & b_{rv}+b_{vt}+b_{tr} & = & b_0 \\ \wedge & b_{rv} \qquad +b_{vu}+b_{ur} & = & b_0 \end{array} \overset{b_{tr}=b_{vu}=0}{\Longrightarrow} \quad b_{vt} = b_{ur},$$

and thus $b_{ur} = b_{rt}$. The circuit (u, r, t, u) yields now $b_{ur} = b_{rt} = \frac{b_0}{2}$. The rest is an easy task.

Finally, we consider Figure A.3(a). Here are u and v clones. Thus $b_{2v} = b_{vt} = b_0$, $b_{rv} = b_{tv} = 0$. Further, we have

$$\begin{array}{rlcl} & b_{uv}+b_{vr}+b_{ru} & = & b_0 \\ \wedge & b_{uv} \qquad +b_{v1}+b_{1u} & = & b_0 \end{array} \overset{b_{ru}=b_{u1}=0}{\Longrightarrow} \quad b_{vr} = b_{1u}, \text{ and}$$

$$\begin{array}{rlcl} & b_{ur}+b_{rt}+b_{tu} & = & b_0 \\ \wedge & b_{ur} \qquad +b_{r1}+b_{1u} & = & b_0 \end{array} \overset{b_{tu}=b_{r1}=0}{\Longrightarrow} \quad b_{rt} = b_{1u}.$$

Together with the equations $b_{rt} + b_{tv} + b_{vr}$ and $b_{tv} = 0$ we conclude $b_{rt} = b_{vr} = \frac{b_0}{2}$. The remaining coefficients can be determined as an exercise.
Case 2: $|M| \geq 2$, i.e., $|\text{dir}(M)| \geq 4$.

First we will determine all coefficients b_a, $a \in A(S)$.

Claim 1. We have

- $b_a = 0$ for all $a \in (S' : \{1, 2\})$,
- $b_a = b_0$ for all $a \in (\{1, 2\} : S')$,
- $b_a = 0$ for all $a \in \text{dir}(M)$, and
- $b_a = \frac{b_0}{2}$ for all $a \in A(S') \setminus \text{dir}(M)$.

Clearly, $b_{i1} = 0$ for $i = 2$ and for all $i \in S'$, since $b^T x \geq 0$ is 1-rooted. Now let $a \in \text{dir}(M) \setminus \{(1, 2), (2, 1)\}$. Then there is a subdigraph of D_n^c containing a as given in Figure A.5(a), i.e., $a = (p, q)$ or $a = (q, p)$.

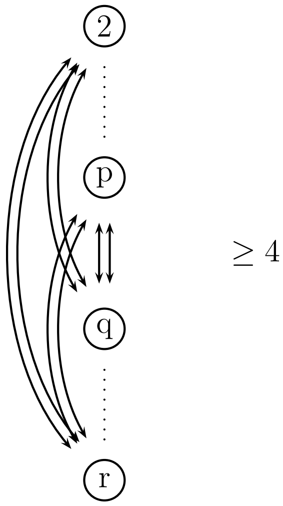


Figure A.5(a)

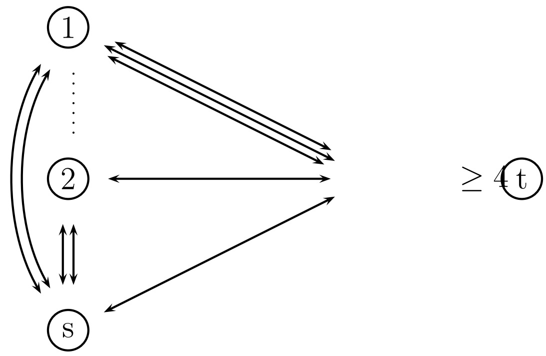


Figure A.5(b)

From the circuits $(1, 2, p, 1)$ and $(1, 2, p, q, 1)$ we derive $b_{pq} = 0$, and from the circuits $(1, 2, q, 1)$ and $(1, 2, q, p, 1)$ we get $b_{qp} = 0$.

Since $|\text{dir}(M)| \geq 4$, such a subdigraph always exists. Hence we can show also $b_{12} = 0$ as follows.

$$\text{The circuits} \left\{ \begin{array}{c} (1, q, p, 1) \\ (1, q, 2, 1) \\ (2, p, q, 2) \\ (1, 2, p, 1) \end{array} \right\} \text{yield} \left\{ \begin{array}{ccc} b_{1q} & = & b_0 \\ b_{q2} & = & 0 \\ b_{2,p} & = & b_0 \\ b_{12} & = & 0. \end{array} \right.$$

Next consider the coefficients b_a, $a \in (\{1, 2\} : S') \cup (S' : \{1, 2\})$. If a is adjacent with an arc $a^* \in M$, then the statement is clear by the preceding arguments. Otherwise there is a directed subgraph of D_n^c containing a as given in Figure A.5(b), i.e., $a = (1, s)$, $a = (s, 1)$, $a = (2, s)$ or $a = (s, 2)$. Clearly, $b_{s1} = 0$, and thus the circuit $(1, 2, s)$ yields $b_{2s} = b_0$. In order to show $b_{s2} = 0$ consider the circuits $(1, 2, t, s, 1)$ and $(2, t, s, 2)$. Since $b_{12} = b_{s1} = 0$, we get $b_{2t} + b_{ts} = b_0$. Thus we can derive from the second circuit $b_{s2} = 0$, and finally, we obtain $b_{1s} = b_0$ by considering the circuit $(1, s, 2, 1)$.

We have to show yet $b_a = \frac{b_0}{2}$ for the remaining arcs in $A(S)$. Let a be adjacent with a matching arc $a^* \in M \setminus \{1, 2\}$. Then it is contained in the subdigraph of D_n^c as given in Figure A.5(c).

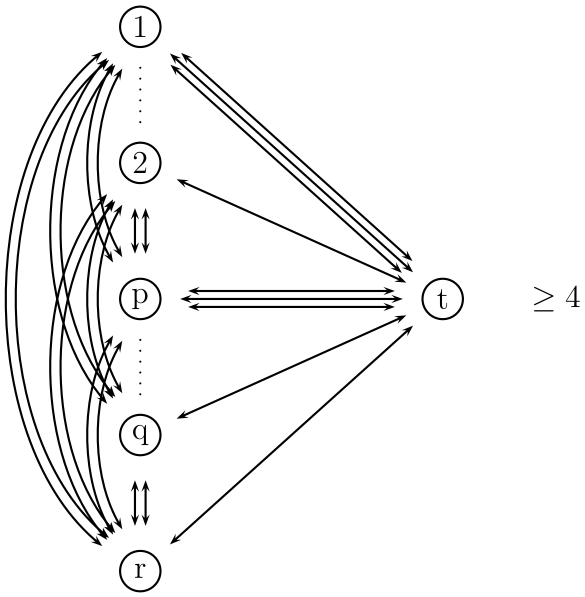


Figure A.5(c)

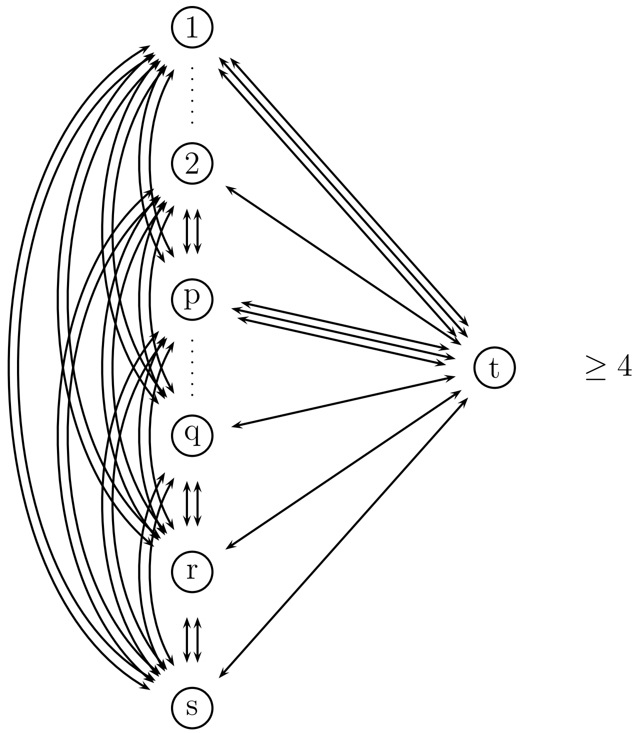


Figure A.5(d)

It is easy to see that $b_{2t} = b_0$, and further $b_{tq} = b_{tr} = 0$. Moreover, we have the equations

$$\begin{array}{llll} b_{rq}+b_{qt}+b_{tr} & & = & b_0 \\ b_{rq} & +b_{qp}+b_{pr} & = & b_0, \\ & & & \\ b_{t2}+b_{2q}+b_{qt} & & = & b_0 \\ b_{t2} & +b_{2r}+b_{rt} & = & b_0, \\ & & & \\ \text{and} \qquad b_{rt} & +b_{tq}+b_{qp}+b_{pr} & = & b_0 \end{array}$$

which lead to $b_{qt} = b_{pr} = b_{rt} = \frac{b_0}{2}$. In particular, $b_{pr} = \frac{b_0}{2}$, and hence the circuit (p, r, q) yields $b_{rq} = \frac{b_0}{2}$. Analogously, we get $b_{rp} = b_{qr} = \frac{b_0}{2}$.

If there is an arc $a \in A(S')$ not adjacent with a matching arc, we have the situation illustrated in Figure A.5(d), i.e., $a = (r, s)$ or $a = (s, r)$. The nodes r and s are clones, and with the above results, we have $b_{rt} = b_{st} = \frac{b_0}{2}$ and $b_{tr} = b_{ts} = 0$. Hence, from the circuits (r, t, s, r) and (s, t, r) we derive $b_{rs} = b_{sr} = \frac{b_0}{2}$.

Finally, let $a \in A(S')$ adjacent with two matching arcs not belonging to $\{(1, 2), (2, 1)\}$. Then there is a driected subgraph of D_n^c as given in Figure A.5(e).

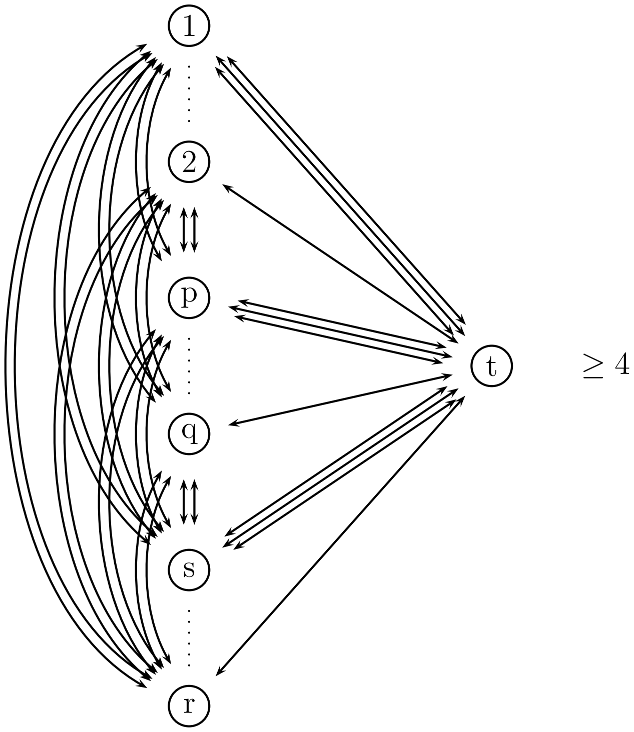


Figure A.5(e)

The subgraph in Figure A.5(e) contains the subgraph as given in A.5(c). Hence, $b_{qr} = b_{rq} = b_{rp} = b_{pr} = \frac{b_0}{2}$. Now it easy to see that also $b_{qs} = b_{sq} = b_{sp} = b_{ps} = \frac{b_0}{2}$.

We will now determine the remaining coefficients. Due to (ii) D_n^c contains one of the directed subgraphs as given in Figure A.5(c)-(m).

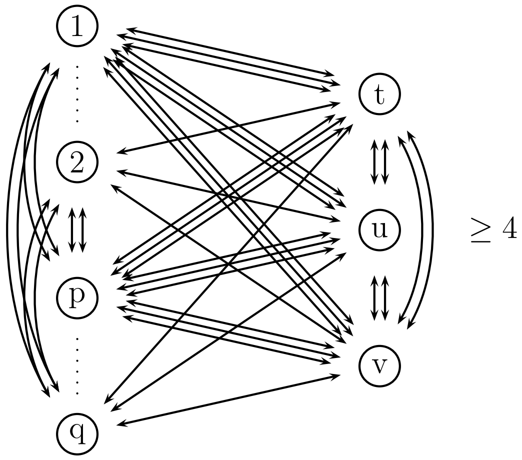


Figure A.5(f)

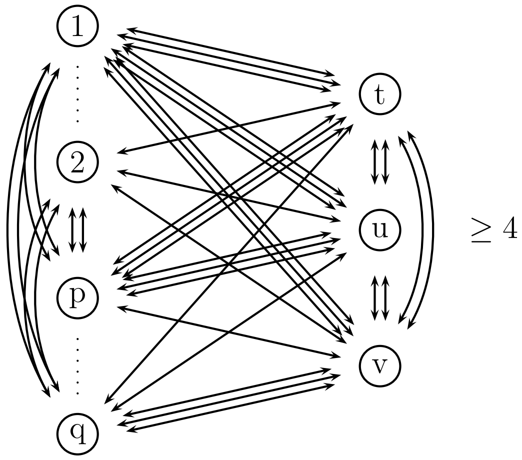


Figure A.5(g)

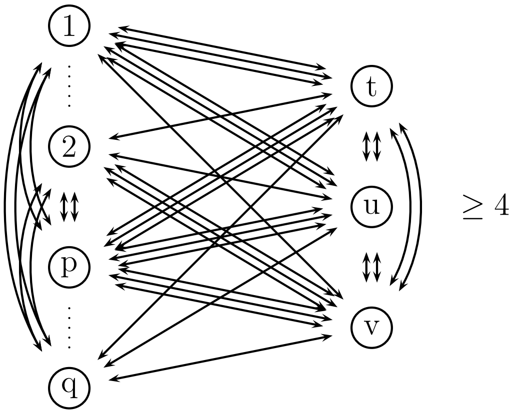


Figure A.5(h)

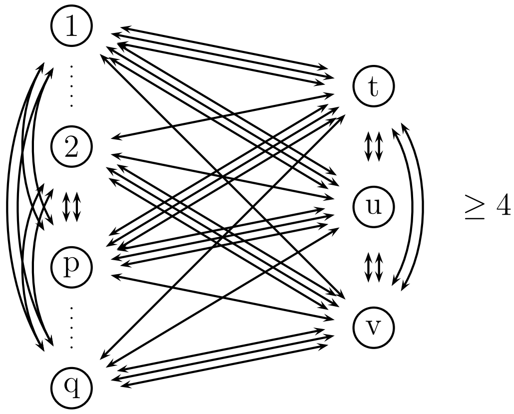


Figure A.5(i)

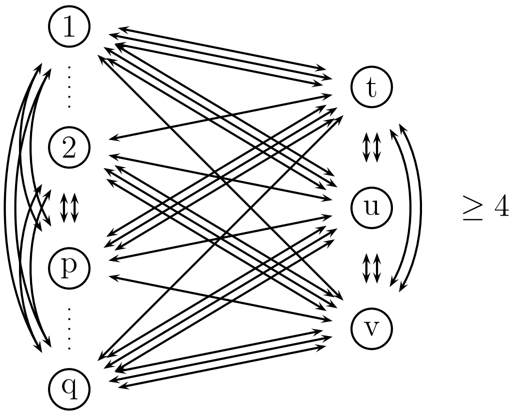


Figure A.5(j)

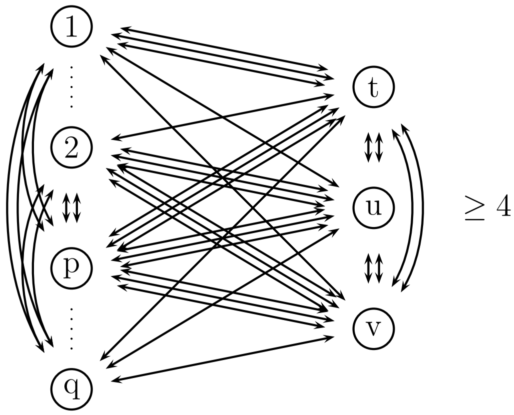


Figure A.5(k)

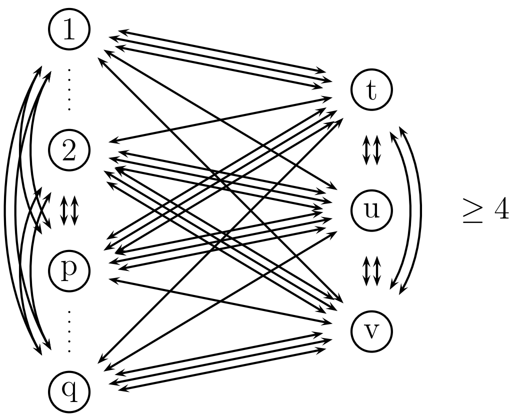


Figure A.5(l)

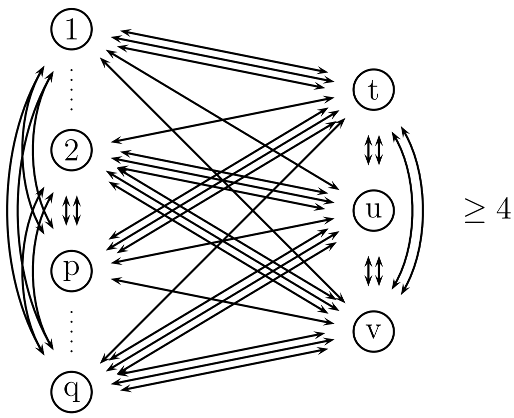


Figure A.5(m)

One can verify that

- $b_a = \frac{3}{2}b_0$ for all $a \in (1:T) \cap \mathrm{dir}(H)$,
- $b_a = 0$ for all $a \in (T:1) \cap \mathrm{dir}(H)$,
- $b_a = b_0$ for all $a \in (2:T) \cap \mathrm{dir}(H)$,
- $b_a = \frac{b_0}{2}$ for all $a \in (T:2) \cap \mathrm{dir}(H)$,
- $b_a = \frac{b_0}{2}$ for all $a \in (1:T) \setminus \mathrm{dir}(H)$,
- $b_a = 0$ for all $a \in (T:1) \setminus \mathrm{dir}(H)$,
- $b_a = b_0$ for all $a \in (2:T) \setminus \mathrm{dir}(H)$,
- $b_a = -\frac{b_0}{2}$ for all $a \in (T:2)$,
- $b_{st} = \frac{b_0}{2}$ for all $(s,t) \in (S':T) \setminus \mathrm{dir}(H)$, with $(1,t) \in \mathrm{dir}(H)$,
- $b_{st} = 0$ for all $(s,t) \in (S':T) \setminus \mathrm{dir}(H)$, with $(1,t) \notin \mathrm{dir}(H)$,
- $b_{ts} = 0$ for all $(t,s) \in (T:S') \setminus \mathrm{dir}(H)$, with $(1,t) \in \mathrm{dir}(H)$,
- $b_{ts} = \frac{b_0}{2}$ for all $(t,s) \in (T:S') \setminus \mathrm{dir}(H)$, with $(1,t) \notin \mathrm{dir}(H)$,
- $b_{st} = \frac{b_0}{2}$ for all $(s,t) \in (S':T) \cap \mathrm{dir}(H)$, with $(1,t) \notin \mathrm{dir}(H)$,
- $b_{st} = b_0$ for all $(s,t) \in (S':T) \cap \mathrm{dir}(H)$, with $(1,t) \in \mathrm{dir}(H)$,
- $b_{ts} = b_0$ for all $(t,s) \in (T:S') \cap \mathrm{dir}(H)$, with $(t,1) \notin \mathrm{dir}(H)$,
- $b_{ts} = \frac{b_0}{2}$ for all $(t,s) \in (T:S') \cap \mathrm{dir}(H)$, with $(t,1) \in \mathrm{dir}(H)$,
- $b_{uv} = 0$ for all $(u,v) \in A(T)$, with $(1,u) \in \mathrm{dir}(H)$ and $(1,v) \notin \mathrm{dir}(H)$,
- $b_{uv} = b_0$ for all $(u,v) \in A(T)$, with $(1,u) \notin \mathrm{dir}(H)$ and $(1,v) \in \mathrm{dir}(H)$,
- $b_{uv} = \frac{b_0}{2}$ for all $(u,v) \in A(T)$, with $(1,u) \notin \mathrm{dir}(H)$ and $(1,v) \notin \mathrm{dir}(H)$,
- $b_{uv} = \frac{b_0}{2}$ for all $(u,v) \in A(T)$, with $(1,u) \in \mathrm{dir}(H)$ and $(1,v) \in \mathrm{dir}(H)$.

$\square$

Appendix B

MAXCUT Problem

The separation problem for the bipartition inequalities (3.36) can be reduced to a separation problem for undirected bipartition inequalities (see 4.1.4):

PROBLEM:	Bipartition separation problem (BSP)
INSTANCE:	Complete graph $G = (V, E)$, a vector $x^* \in \mathbb{R}^E$.
QUESTION:	Is there a cut K in K_n with $w^T\chi^K > 2w(E) - (k+1)$?

Theorem B.1. *BSP is NP-hard.*

Proof. BSP is obviously in NP. We show it is NP-hard by a reduction from the maximum cut problem (MAXCUT) [14]:

PROBLEM:	Maximum cut problem (MAXCUT)
INSTANCE:	Complete graph $G = (V, E)$, a vector $w \in \mathbb{R}^E$, $w \geq \mathbf{0}$, of edge weigths and a positive number $p \in \mathbb{R}$.
QUESTION:	Is there a cut K in K_n with $w^T\chi^K > p$?

Let $\lambda \in \mathbb{R}$, $\lambda > 0$ and set $\tilde{w} = \lambda w$. Clearly, MAXCUT is equivalent to the problem: Is there a cut K in K_n with $\tilde{w}\chi^K > \lambda p$?

Choosing $\lambda = \frac{k+1}{2w(E)-p}$, we get

$$\begin{aligned} 2\tilde{w}(E) - (k+1) &= 2\lambda w(E) - (k+1) \\ &= 2\frac{k+1}{2w(E)-p} - (k+1) \\ &= \frac{2(k+1)w(E) - (k+1)(2(w(E)-p))}{2w(E)-p} \\ &= (k+1)\frac{2w(E) - 2w(E) + p}{2w(E)-p} \\ &= \lambda p \end{aligned}$$

which proves the statement. □

Since the problem MAXCUT is NP-complete, the separation problem for the bipartition inequalities (3.36) is NP-hard.

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Die selbständige und eigenhändige Anfertigung dieser Arbeit versichere ich an Eides statt.

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