ON HYPOHAMILTONIAN GRAPHS

Carsten THOMASSEN*

Mathematisk Institut, Universitetsparken, Ny Munkegade, 8000 Aarhus C, Denmark

Received 22 January 1974

Abstract. Herz, Duby and Vigué [9] conjectured that every hypohamiltonian graph has girth $\geq 5$. In the present note hypohamiltonian graphs of girth 3 and 4 are described. Also two conjectures on hypohamiltonian graphs made by Bondy and Chvátal, respectively, are disproved.

1. Introduction and terminology

We adopt the notation and terminology of Harary [8] with the modifications that the terms vertices and edges are here used instead of the terms points and lines, respectively, in [8]. The set of vertices, respectively edges, of the graph $G$ is denoted by $V(G)$, respectively $E(G)$. The edge joining the vertices $x$ and $y$ is denoted by $(x, y)$ and $(y, x)$ and the degree of $x$ in $G$ is denoted by $d(x, G)$.

A graph $G$ is hypohamiltonian if and only if $G$ is not Hamiltonian but every vertex-deleted subgraph $G - v$ is Hamiltonian. Hypohamiltonian graphs were first studied by Sousselier (see [1, 2]) who among other things proved that the Petersen graph is the smallest one. Herz, Duby and Vigué [9] proved that there exists no hypohamiltonian graph with 11 or 12 vertices. Infinite families of hypohamiltonian graphs have been constructed by Sousselier (see [9]), Lindgren [11], Bondy [3], Chvátal [4], Doyen and Van Diest [7] and by the author [12]. In [12] it was shown that for every $p \geq 13$, except possibly for $p = 14, 17, 19$, there exists a hypohamiltonian graph with $p$ vertices. This improved on the result of Chvátal [4] for $p = 20, 25$. Doyen and Van Diest have constructed hypohamiltonian graphs with $3k + 1$ vertices for all $k \geq 3$ so the question of the existence of a hypohamiltonian graph with $p$ vertices is left open for $p = 14, 17$.

* Present address: Department of Combinatorics and Optimization, Faculty of Mathematics, University of Waterloo, Waterloo, Ont., Canada.
The following three conjectures have been made concerning the structure of hypohamiltonian graphs.

1. Every hypohamiltonian graph has girth $\geq 5$. (Herz, Duby and Vigué [9]. See also [4, 5, 6].)

2. If the deletion of an edge $e$ from a hypohamiltonian graph $G$ does not create a vertex of degree two, then $G - e$ is hypohamiltonian (Chvátal [4]).

3. If the addition of a new edge to a hypohamiltonian graph of girth $\geq 5$ does not create a cycle of length $< 5$, then it does not create a Hamiltonian cycle (Bondy, see [4]).

This note gives examples of hypohamiltonian graphs for which (1) and (2) are false and one for which (2) and (3) are false.

2. Construction of hypohamiltonian graphs

Let $G_1$, $G_2$ be disjoint graphs. Assume $G_1$, respectively $G_2$, contains a vertex $x_0$, respectively $y_0$, of degree $5$, and let $x_1$, $x_2$, $x_3$, respectively $y_1$, $y_2$, $y_3$, denote the vertices adjacent to $x_0$, respectively $y_0$. Assume that $G_2$ is hypohamiltonian. Bondy (see [4, p. 39]) pointed out that $G_2$ contains none of the edges $(y_1, y_2), (y_1, y_3), (y_2, y_3)$. We assume that the graph $G_1$ has at least six vertices. Let $G$ denote the graph obtained from $H_1 = G_1 - x_0$ and $H_2 = G_2 - y_0$ by identifying the vertices $x_1, y_1$ into a vertex $z_1$, the vertices $x_2, y_2$ into a vertex $z_2$ and the vertices $x_3, y_3$ into a vertex $z_3$. This construction is illustrated in [12, Fig. 1]. The special case in which $G_2$ is the Petersen graph is shown in Fig. 1. In this case we say that $x_0$ is replaced by a vertex-deleted subgraph of the Petersen graph. We consider $H_1$ and $H_2$ as subgraphs of $G$. In [12] it was shown that $G$ is hypohamiltonian provided $G_1$ is hypohamiltonian. By the same type of arguments we obtain the following stronger result.

**Lemma 1.** (a) $G$ is Hamiltonian if and only if $G_1$ is Hamiltonian.

(b) For every $z \in V(H_1)$, $G - z$ is Hamiltonian if and only if $G_1 - z$ is Hamiltonian.

(c) If $G_1 - x_i$ is Hamiltonian for $i = 1, 2, 3$, then for every $z \in V(H_2)$, $G - z$ is Hamiltonian.
Proof. Suppose first that $G_1$ is Hamiltonian. Let $C$ be a Hamiltonian cycle of $G_1$. Then $P_1 = C - x_0$ is a Hamiltonian path of $H_1$ connecting two of the vertices $x_1, x_2, x_3$ ($x_1$ and $x_2$, say). Since $G_2 - y_3$ is Hamiltonian, $G_2 - y_3 - y_0 = H_2 - y_3$ contains a Hamiltonian path $P_2$ connecting $y_1$ and $y_2$. Then $P_1 \cup P_2$ is a Hamiltonian cycle of $G$. Suppose next that $G$ is Hamiltonian and let $C$ be a Hamiltonian cycle of $G$. $C = P_1 \cup P_2 \cup P_3$, where $P_1$ is a $z_1 - z_2$ path, $P_2$ is a $z_2 - z_3$ path and $P_3$ is a $z_3 - z_1$ path. Each of the paths $P_i$ is a path of either $H_1$ or $H_2$ for $i = 1, 2, 3$. Two of these paths are contained in $H_j$, where $j = 1$ or $j = 2$. Then $H_j$ has a Hamiltonian path connecting two of the vertices $z_1, z_2, z_3$ and clearly $G_j$ is Hamiltonian. Since $G_2$ is assumed to be non-Hamiltonian, we have proved that $G_1$ is Hamiltonian and we have proved (a). If $z \in V(H_1) - \{z_1, z_2, z_3\}$, then, by (a), $G - z$ is Hamiltonian if and only if $G_1 - z$ is Hamiltonian since $G - z$ is obtained from $G_1 - z$ and $G_2$ in the same way as $G$ is obtained from $G_1$ and $G_2$. Since $H_2 - y_i$ ($i = 1, 2, 3$) has a Hamiltonian path connecting the two vertices of $\{y_1, y_2, y_3\} - \{y_i\}$, clearly $G - z_i$ is Hamiltonian if and only if $G_1 - x_i$ is Hamiltonian. This proves (b). If $z \in V(H_2)$, then $H_2 - z$ has a Hamiltonian path $P_2$ connecting two of the vertices $y_1, y_2, y_3$ ($y_1$ and $y_2$, say). If $G_1 - x_3$ is Hamiltonian, then $H_1 - x_3$ contains a Hamiltonian path $P_1$ connecting $x_1$ and $x_2$ and $P_1 \cup P_2$ is a Hamiltonian cycle of $G - z$, so (c) holds.

Theorem 1. Let $G$ be a non-Hamiltonian graph and let $A \subseteq V(G)$. Suppose that the vertices of $A$ are mutually non-adjacent and that they all have degree 3. If for every vertex $z \in V(G) - A$, $G - z$ is Hamiltonian, then there exists a hypo-Hamiltonian graph $G'$ containing $G - A$ as a subgraph. If furthermore for every edge $e \in E(G - A)$ there is a vertex $z_e \in V(G) - A$ such that $G - e - z_e$ is non-Hamiltonian, then we can construct $G'$ such that for every edge $e \in E(G')$, $G' - e$ is not hypo-Hamiltonian.
Proof. Let \( x_0 \) be any vertex of \( A \). Replace \( x_0 \) by a vertex-deleted subgraph of the Petersen graph. Denote the resulting graph by \( G_1 \) and put \( A_1 = A - \{x_0\} \). Then for every vertex \( z \in V(G_1) - A_1 \), \( G_1 - z \) isHamiltonian by Lemma 1. The vertices of \( A_1 \) are mutually non-adjacent and they all have degree 3 in \( G_1 \). If \( e \in E(G - A) \), \( z_e \in V(G) - A \) and \( G - e - z_e \) is non-Hamiltonian, then also \( G_1 - e - z_e \) is non-Hamiltonian by Lemma 1. If \( e \) is any edge of \( G_1 \) not contained in \( G \), then \( G_1 - e \) contains a vertex of degree 2. So it is easy to see that \( G_1 \) contains a vertex \( z_e \in V(G_1) - A_1 \) such that \( G_1 - e - z_e \) is non-Hamiltonian. If \( A_1 = \emptyset \), \( G_1 \) has the desired properties. If \( A_1 \neq \emptyset \), we replace any vertex \( x_1 \) of \( A \) by a vertex-deleted subgraph of the Petersen graph and we put \( A_2 = A_1 - \{x_1\} \), etc. Since \( |A_1| > |A_1| > A_2 | > \ldots \), we obtain in a finite number of steps a graph \( G' \) which satisfies the assertion of the theorem.

3. Disproof of the conjectures (1), (2), (3);

Using Theorem 1, it is easy to see that there exists a hypohamiltonian graph containing a cycle of length 4. Let for \( k \geq 2 \), \( R_k \) denote the graph consisting of the vertices
\[
\{x_1, x_2, \ldots, x_{2k+1}, y_1, y_2, \ldots, x_{2k+1}, z_1, z_2\}
\]
and the edges
\[
\{(x_1, x_2), (x_2, x_3), \ldots, (x_{2k}, x_{2k+1}), (x_{2k+1}, x_1), (y_1, y_2), (y_2, y_3), \ldots,
(y_{2k+1}, y_1), (x_1, z_1), (z_1, y_1), (z_2, y_2), (x_1, z_2), (z_2, y_3), (x_2, y_4),
(x_3, y_5), \ldots, (x_{2k+1}, y_{2k+1})\}.
\]

Fig. 2. The graph \( R_2 \).
$R_2$ is obtained from a pentagonal prism by subdividing two edges through the insertion of two new vertices of degree 2 (see Fig. 2). Put $A_k = \{x_1, y_1, x_3, y_3\} \subseteq V(R_k)$. Tutte [13] pointed out that $R_2$ is non-Hamiltonian. More generally, it is easy to see that $R_k$ is non-Hamiltonian for all $k \geq 2$ and that $R_k - z$ is Hamiltonian whenever $z \in V(R_k) - A_k$.

Also it is easy to see that for every edge $e \in E(R_k - A_k)$ there is a $z_e \in V(R_k) - A_k$ such that $R_k - e - z_e$ is non-Hamiltonian. By Theorem 1, there exists a hypohamiltonian graph $R'_k \supseteq R_k - A_k$ such that for any edge $e \in E(R'_k), R'_k - e$ is not hypohamiltonian. $R'_2$ is shown in Fig. 3.
Clearly, $F'_k$ contains a cycle of length 4 and it contains edges whose removal does not create vertices of degree 2. So for every $k \geq 2$, $F'_k$ is a counterexample to (1) and (2).

We shall go a step further and show that a hypohamiltonian graph may contain a cycle of length 3. Let $M$ denote the graph in Fig. 4. $V(M) = \{1, 2, ..., 30\}$. Put $A = \{3, 5, 17, 19, 24, 26\}$. We shall show by reductio ad absurdum that $M$ is non-Hamiltonian. Suppose $C$ is a Hamiltonian cycle of $M$. Then $C$ contains the edges $(3, 4), (4, 5), (17, 18), (18, 19), (24, 25), (25, 26)$. Suppose first that $C$ contains the edges $(1, 5), (3, 7)$. Then $C$ contains the edges $(1, 2), (2, 23), (7, 6), (6, 9), (9, 8), (8, 11)$. Also $C$ contains $(21, 20), (20, 19), (17, 16), (16, 15), (15, 14), (14, 13), (13, 12)$. The two edges of $C$ which are incident with 10 are then $(10, 12)$ and $(10, 11)$. $C$ must contain the edge $(21, 22)$ and if $C$ contains $(22, 23)$ also then $C$ contains a cycle as a proper subgraph. So $C$ does not contain $(22, 23)$. But then $C$ contains $(22, 26)$ and $(23, 24)$ and again we see that $C$ contains a cycle as a proper subgraph, which is a contradiction. By symmetry, $C$ cannot contain the edges $(2, 3), (5, 6)$. So $C$ contains either none of both of the edges $(1, 5), (2, 3)$, or, in other words, $C$ either contains the path 20, 1, 2, 23 or the path 20, 1, 5, 4, 3, 2, 23. Because of the symmetry, $C$ contains either none of both of the edges $(19, 20), (17, 21)$ and either none of both of the edges $(22, 26), (23, 24)$. It is, however, easy to see that this leads to a contradiction and we have proved that $M$ is non-Hamiltonian.

Next we show that $M - z$ is Hamiltonian whenever $z \in V(M) - A$. Because of the symmetry, it is sufficient to consider the cases $z = 1, 4, 6, 8, 10$. In the case $z = 4, M - z$ has the following Hamiltonian cycle: 1, 5, 6, 9, 8, 7, 3, 2, 23, 22, 26, 25, 24, 28, 27, 30, 29, 11, 12, 10, 14, 13, 16, 15, 19, 18, 17, 21, 20, 1. Let $P$ denote the path 11, 29, 28, 24, 25, 26, 27, 30, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 2 and let $P_1, P_6, P_8, P_{10}$ be the paths defined as follows:

$P_1: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11,$

$P_6: 2, 1, 5, 4, 3, 7, 8, 9, 10, 11,$

$P_8: 2, 1, 5, 4, 3, 7, 6, 9, 10, 11,$

$P_{10}: 2, 1, 5, 4, 3, 7, 6, 9, 8, 11.$

Then $P \cup P_2$ is a Hamiltonian cycle of $M - z$ for $z = 1, 6, 8, 10$. Furthermore we can show that for every edge $e \in E(M - A)$ there exists a $z_{e} \in V(M) - A$ such that $M - e - z_{e}$ is non-Hamiltonian. If $e$ is incident with a vertex of degree 3, then this vertex in $M - e$ is adjacent to a vertex $z_{e} \in A$. Clearly, $M - e - z_{e}$ is non-Hamiltonian. If, on the other hand, $e$ joins two vertices of degree $\geq 4$, then $e$ is one of the edges $(10, 11)$.
(11, 12), (12, 10). If \( e = (10, 11) \), we put \( z_e = 1 \) and it is easy to prove that \( M - e - z_e \) is non-Hamiltonian (we leave this to the reader). By Theorem 1 there exists a hypohamiltonian graph \( M' \) (Fig. 5) such that \( M' \) contains \( M - A \) as a subgraph and for any edge \( e \) of \( M' \), \( M' - e \) is not hypohamiltonian. Clearly, \( M' \) is another counterexample to the conjectures (1) and (2).

We shall finally give a counterexample to the conjectures (2) and (3). Let \( G \) denote the Petersen graph and let \( A \) be a set consisting of two non-adjacent vertices of \( G \). For every \( z \in V(G) - A \), \( G - z \) is Hamiltonian and for every \( e \in E(G - A) \) there exists a \( z_e \in V(G) - A \) such that \( G - e - z_e \) is non-Hamiltonian. Let \( G' \) denote the graph obtained from \( G \) by replacing each vertex of \( A \) by a vertex-deleted subgraph of the Petersen graph (Fig. 6). Then \( G' \) is hypohamiltonian and the deletion of
any edge of \( G' \) results in a graph which is not hypohamiltonian. So \( G' \) is clearly a counterexample to (2). A Hamiltonian path of \( G' \) is drawn with thick lines in Fig. 6. If we add the edge joining the endvertices of this path we create a Hamiltonian cycle of \( G' \) but we do not create a cycle of length < 5. So \( G' \) is a counterexample to conjecture (3).

References