Addendum: Symmetric vs. asymmetric timetables

- lines operate in both directions, frequency: 10 minutes
- waiting times: 2 minutes (A, D), 0 minutes (B, C)
- minimum transfer time: 2 minutes
- no turnarounds, no transfers to opposite direction of the same line
- weights: 1 (transfers), 0 (other activities)
Event-activity network $\mathcal{E} = (V, E)$

$T = 10$

[2, 11]
PESP MIP formulation

Timetable-based MIP formulation:

Minimize \[ \sum_{ij} x_{ij} - 64 \] (minimal slack)

s.t. \[ x_{ij} = \pi_j - \pi_i + 10p_{ij}, \]
\[ \ell_{ij} \leq x_{ij} \leq u_{ij}, \]
\[ p_{ij} \in \{0, 1, 2\}, \]
\[ 0 \leq \pi_i \leq 9, \]

Symmetry constraints (axis = 0):

\[ 0 = \pi_i + \pi_j - 10q_{ij}, \]
\[ (i, j) \in V \times V \text{ complementary} \]
\[ q_{ij} \in \{0, 1\}, \]
\[ (i, j) \in V \times V \text{ complementary} \]
Optimal asymmetric solution (computed by SCIP)

<table>
<thead>
<tr>
<th>station</th>
<th>slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>24</td>
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<tr>
<td>D</td>
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<tr>
<td>total</td>
<td>48</td>
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<table>
<thead>
<tr>
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<th>sym. axis</th>
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</thead>
<tbody>
<tr>
<td>red</td>
<td>4</td>
</tr>
<tr>
<td>yellow</td>
<td>4</td>
</tr>
<tr>
<td>blue</td>
<td>2</td>
</tr>
<tr>
<td>green</td>
<td>2</td>
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</table>
Optimal symmetric solution (computed by SCIP)

<table>
<thead>
<tr>
<th>station</th>
<th>slack</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>20</td>
</tr>
<tr>
<td>B</td>
<td>24</td>
</tr>
<tr>
<td>C</td>
<td>24</td>
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<tr>
<td>D</td>
<td>0</td>
</tr>
<tr>
<td>total</td>
<td>68</td>
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<table>
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<tbody>
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<td>0</td>
</tr>
<tr>
<td>yellow</td>
<td>0</td>
</tr>
<tr>
<td>blue</td>
<td>0</td>
</tr>
<tr>
<td>green</td>
<td>0</td>
</tr>
</tbody>
</table>
Chapter 5

Line Planning

§5.1 Overview
§5.1 Overview

Public transport planning cycle

Network Design → Line Planning → Timetabling → Vehicle Scheduling → Duty Scheduling → Crew Rostering

strategic planning

operational planning
5.1 Overview

Description

Let \( G \) be a graph modeling a public transportation network, e.g.,
- a road network (for buses)
- a railway track system (for railways, trams, underground trains, . . .)

Definition

A line plan is a set \( \mathcal{L} \) of paths (lines) in \( G \) together with frequencies \( f : \mathcal{L} \rightarrow \mathbb{N}_0 \).

Line Planning Problem

The line planning problem is to find a feasible line plan providing both convenient travel for passengers and small operational costs.

Feasible lines

Lines are either chosen from a line pool, or are computed on the fly subject to certain restrictions.
5.1 Overview

Optimization goals

Two oppositional goals

<table>
<thead>
<tr>
<th>passenger-oriented</th>
<th>cost-oriented</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimize travel time</td>
<td>Minimize operational costs</td>
</tr>
<tr>
<td>given an upper bound</td>
<td>given an upper bound</td>
</tr>
<tr>
<td>on operational costs</td>
<td>on travel time</td>
</tr>
</tbody>
</table>

Passenger quality

Minimize travel time (estimated: no timetable available), Maximize number of passengers having a direct connection, . . .

Operational costs

Minimize vehicle costs (estimated: no vehicle schedule), Minimize driver costs (estimated: no crew schedule), . . .
§5.1 Overview

Feasibility

Basic Line Planning Feasibility Problem (BLPFP)

Given a graph $G = (V, E)$, a line pool $L_0$, lower and upper frequency bounds $f_{\text{min}} \leq f_{\text{max}} : E \rightarrow \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$ with $\mathcal{L} \subseteq L_0$ such that

$$\forall e \in E : \quad f_{e, \text{min}} \leq \sum_{\ell \in \mathcal{L} : e \in \ell} f_{\ell} \leq f_{e, \text{max}}.$$ 

Example

Assume that there is an edge $e$ that has to be served at least 3 times per hour, i.e., $f_{e, \text{min}} = 3$. This might be satisfied by a line $\ell_1$ with $f_{\ell_1} = 2$ (riding twice per hour), together with a line $\ell_2$ with $f_{\ell_2} = 1$ (riding once per hour).

Theorem (Bussieck, 1998)

BLPFP is NP-complete.
Complexity

Definition
The exact cover by 3-sets problem (X3C) is the following:
Given a set $X$ with $3q$ elements for some integer $q$, and a collection $C$ of 3-element subsets of $X$, is there a subcollection $S \subseteq C$ such that each $x \in X$ occurs in exactly one member of $S$?

Theorem (Karp, 1972)
$X3C$ is NP-complete.

Theorem (Bussieck, 1998)
$X3C \leq BLPFP.$
5.1 Overview

**X3C $\leq$ BLPFP**

Proof ($\iff$).

Let $(X, C)$ be an instance for X3C. We consider $C$ as set of triples $(x, y, z)$. Build a simple graph $G = (V, E)$ as follows:

- Add two vertices $x^+$ and $x^-$ for each $x \in X$.
- Add an edge $\{x^-, x^+\}$ for each $x \in X$.
- Add two edges $\{x^+, y^-, y^+, z^-, z^+\}$ for each $(x, y, z) \in C$.

Define the line pool $\mathcal{L}_0 := \{(x^-, x^+, y^-, y^+, z^-, z^+) \mid (x, y, z) \in C\}$ and the lower and upper frequency bounds

$$f_{e}^{\text{min}} := \begin{cases} 1 & \text{if } e = \{x^-, x^+\} \text{ for some } x \in X, \\ 0 & \text{otherwise,} \end{cases} \quad f_{e}^{\text{max}} := 1, \quad e \in E.$$

Let $(\mathcal{L}, f)$ be a feasible line plan. Then for each $x \in X$, the edge $\{x^-, x^+\}$ is covered by a unique line $\ell \in \mathcal{L}$ with $f_\ell = 1$, corresponding to a unique triple $(x, y, z) \in C$. 
Proof ($\Rightarrow$):

Conversely, let $S \subseteq C$ be a subcollection solving the X3C problem on $(X, C)$. Then taking all lines $(x^-, x^+, y^-, y^+, z^-, z^+)$ $\in L_0$ for triples $(x, y, z) \in S$ with frequency 1 yields a feasible line plan.

Example

$X := \{1, 2, 3, 4, 5, 6\}$, $C := \{(1, 3, 6), (1, 4, 6), (2, 3, 5), (4, 5, 6)\}$
§5.1 Overview

Cost-oriented LPP

Cost-oriented Line Planning Problem

Given a graph $G = (V, E)$, a line pool $\mathcal{L}_0$ with costs $c : \mathcal{L}_0 \to \mathbb{R}_{\geq 0}$, lower and upper frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$ minimizing

$$\sum_{\ell \in \mathcal{L}} c_\ell$$

subject to

$$f^{\min}_e \leq \sum_{\ell \in \mathcal{L} : e \in \ell} f_\ell \leq f^{\max}_e, \quad e \in E,$$

$$\ell \in \mathcal{L}_0, \quad \ell \in \mathcal{L}.$$

Lemma (Exercise)

The problem “Given $C$, is there a feasible line plan with cost $\leq C$” is NP-complete.

Remark

The quality for passengers is established by the minimum frequency requirement.
§5.1 Overview

Cost-oriented LPP: Example

Graph $G$ with line pool $\mathcal{L}_0$

Further data

$\begin{align*}
  c &\equiv 1 \\
  f_{\min} &\equiv 1 \\
  f_{\max} &\equiv 2
\end{align*}$

Since the edges AD, BC, EF, EG need to be served with frequency $\geq f_{\min} = 1$, the lines 1, 3, 5, 6 have to appear in every feasible line plan. This leaves the edge DG uncovered, which can be covered either by line 4 or line 7. In particular, the cost of an optimal line plan is at least 5.

Running each of the lines 1, 3, 4, 5, 6 with frequency 1 is a feasible line plan: Each edge is covered at least once, only CG is covered twice, and no edge is covered more than twice.
Chapter 5

Line Planning

§5.2 Passenger-Oriented Models
Passenger flow

Let $G = (V, E)$ be a graph.

**Definition**

- An **origin-destination matrix**, short **OD matrix**, is a $V \times V$-matrix $(d_{st})$ with non-negative entries.
- For $(s, t) \in V \times V$, the entry $d_{st}$ is called the **demand** from $s$ to $t$.
- An **OD pair** is a pair $(s, t) \in V \times V$ such that $d_{st} > 0$.

OD matrices are the standard tool to model demands in a public transportation network. However, without a timetable, it is hard to tell which routes passengers will take.

**Routing strategies**

- shortest paths without transfer times
- shortest paths with transfer penalty
- system split: divide into different transport modes
- ...
§5.2 Passenger-Oriented Models

Direct Travelers LPP

Input

- graph $G = (V, E)$
- OD matrix $(d_{st})$ with set of OD pairs $D \subseteq V \times V$
- fixed passenger paths $p_{st}$ for all $(s, t) \in D$
- line pool $\mathcal{L}_0$
- frequency bounds $f^{\min} \leq f^{\max} : E \rightarrow \mathbb{N}_0$
- global capacity bound $C \geq 0$

Goal

Find a feasible line plan $(\mathcal{L}, f)$ maximizing the number of direct travelers over all OD pairs.

Remark

This is trivial to maximize if there are neither capacities nor upper bounds on line costs: Either $p_{st}$ is covered by a line in $\mathcal{L}_0$ or not.
Direct Travelers LPP

MIP formulation

Maximize

\[ \sum_{\ell \in \mathcal{L}} \sum_{(s, t) \in \mathcal{D}: p_{st} \subseteq \ell} x_{st, \ell} \]

subject to

\[ \sum_{\ell \in \mathcal{L}: p_{st} \subseteq \ell} x_{st, \ell} \leq d_{st}, \quad (s, t) \in \mathcal{D}, \]

\[ \sum_{(s, t) \in \mathcal{D}: e \in p_{st} \subseteq \ell} x_{st, \ell} \leq C \cdot f_e, \quad e \in \mathcal{E}, \ell \in \mathcal{L}, \]

\[ f_{e, \text{min}} \leq \sum_{\ell \in \mathcal{L}: e \in \ell} f_{\ell} \leq f_{e, \text{max}}, \quad e \in \mathcal{E}, \]

\[ \ell \in \mathcal{L}_0, \quad \ell \in \mathcal{L}, \]

\[ f_{\ell} \in \mathbb{N}_0, \quad \ell \in \mathcal{L}, \]

\[ x_{st, \ell} \geq 0, \quad (s, t) \in \mathcal{D}, \ell \in \mathcal{L}. \]

Notation

\( x_{st, \ell} \) is the number of direct travelers from \( s \) to \( t \) using line \( \ell \).
5.2 Passenger-Oriented Models

Direct Travelers LPP

Remarks

- There is no point in taking $x_{st,\ell}$ integral: Capacities are in general only rough estimates, and the number of direct travelers is usually large.
- The capacity $C$ may be replaced by capacities for each pair of edge and line.
- One may also integrate budget constraints in terms of upper bounds on the frequencies.
- We have $\mathcal{L} = \{\ell \in \mathcal{L}_0 \mid f_\ell > 0\}$. We can therefore replace $\mathcal{L}$ by $\mathcal{L}_0$ in the MIP formulation. In other words, $f_\ell$ also takes the role of a decision variable if line $\ell$ should by included into $\mathcal{L}$ or not.
- In particular, this is a mixed integer linear program.
- This model is due to Bussieck/Kreuzer/Zimmermann, 1995.
- Disadvantage: Hard to solve exactly.
Set $x_{st} := \sum_{\ell \in \mathcal{L}_0: p_{st} \subseteq \ell}^{} x_{st, \ell}$, i.e., count all directly traveling passengers from $s$ to $t$ using any line.

### Aggregated MIP formulation

Maximize

$$\sum_{(s, t) \in \mathcal{D}: p_{st} \subseteq \ell}^{} x_{st}$$

subject to

$$x_{st} \leq d_{st}, \quad (s, t) \in \mathcal{D},$$

$$x_{st} \leq C \cdot \sum_{\ell \in \mathcal{L}_0: p_{st} \subseteq \ell}^{} f_{\ell}, \quad (s, t) \in \mathcal{D},$$

$$f_{e}^{\min} \leq \sum_{\ell \in \mathcal{L}: e \in \ell}^{} f_{\ell} \leq f_{e}^{\max}, \quad e \in \mathcal{E},$$

$$\ell \in \mathcal{L}_0,$$

$$f_{\ell} \in \mathbb{N}_0,$$

$$x_{st} \geq 0, \quad (s, t) \in \mathcal{D}.$$
Remarks

- Any feasible solution for the bigger model is feasible for the aggregated model.
- The converse is in general not true.
- However, this gives a heuristic for solving the bigger model.
5.2 Passenger-Oriented Models

Direct Travelers LPP: Example

Graph $G$ with line pool $\mathcal{L}_0$

OD matrix

<table>
<thead>
<tr>
<th></th>
<th>B</th>
<th>C</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>from</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>50</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>80</td>
<td>20</td>
</tr>
<tr>
<td>G</td>
<td>40</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Further data: $f^{\text{min}} = 0$, $f^{\text{max}} = \infty$, $C = 50$

To serve all demands by direct connections, we need

<table>
<thead>
<tr>
<th>OD pair</th>
<th>line</th>
<th>freq.</th>
<th>pass./cap.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A → B</td>
<td>2</td>
<td>1</td>
<td>50/50</td>
</tr>
<tr>
<td>A → F</td>
<td>3</td>
<td>1</td>
<td>50/50</td>
</tr>
<tr>
<td>D → C</td>
<td>4</td>
<td>2</td>
<td>80/100</td>
</tr>
<tr>
<td>D → F</td>
<td>7</td>
<td>1</td>
<td>20/50</td>
</tr>
<tr>
<td>G → B</td>
<td>5 or 6</td>
<td>1</td>
<td>40/50</td>
</tr>
</tbody>
</table>

This clearly maximizes the number of direct travelers, which is 240.