Mathematical Aspects of Public Transportation Networks

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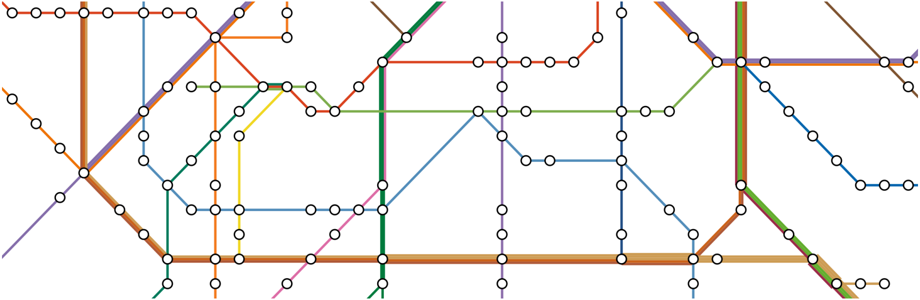

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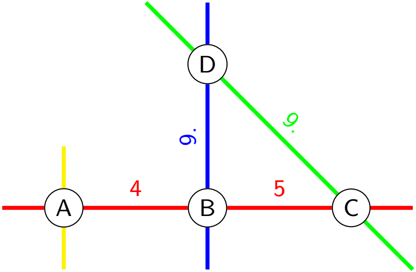
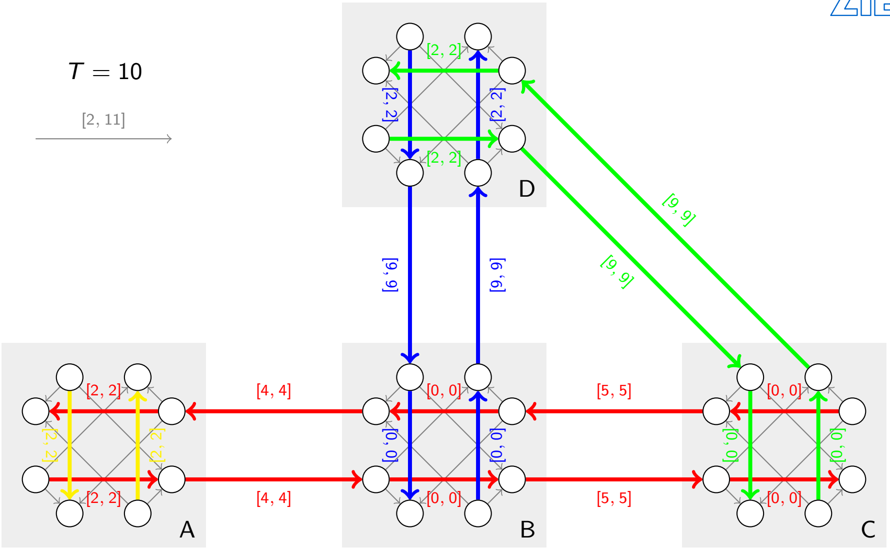
Addendum: Symmetric vs. asymmetric timetables



- lines operate in both directions, frequency: 10 minutes
- waiting times: 2 minutes (A, D), 0 minutes (B, C)
- minimum transfer time: 2 minutes
- no turnarounds, no transfers to opposite direction of the same line
- weights: 1 (transfers), 0 (other activities)

Event-activity network $\mathcal{E} = (V, E)$

$T = 10$



PESP MIP formulation

Timetable-based MIP formulation:

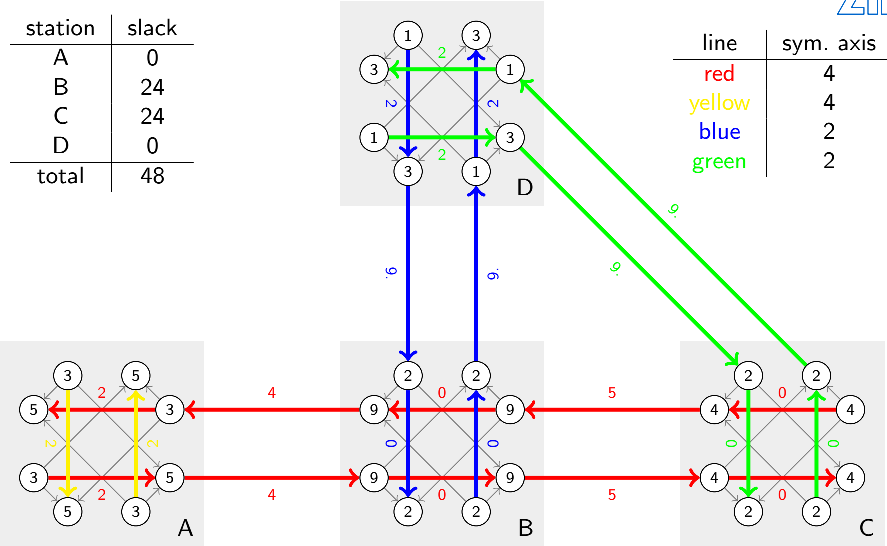
$$
\begin{array}{lll}
\text{Minimize} & \displaystyle\sum_{ij \text{ transfer activity}} x_{ij} - 64 & \text{(minimal slack)} \\
\text{s.t.} & x_{ij} = \pi_j - \pi_i + 10 p_{ij}, & ij \in E \\
& \ell_{ij} \leq x_{ij} \leq u_{ij}, & ij \in E \\
& p_{ij} \in \{0, 1, 2\}, & ij \in E \\
& 0 \leq \pi_i \leq 9, & i \in V
\end{array}
$$

Symmetry constraints (axis $= 0$):

$$
\begin{array}{ll}
0 = \pi_i + \pi_j - 10 q_{ij}, & (i, j) \in V \times V \text{ complementary} \\
q_{ij} \in \{0, 1\}, & (i, j) \in V \times V \text{ complementary}
\end{array}
$$

Optimal asymmetric solution (computed by SCIP)

station	slack
A	0
B	24
C	24
D	0
total	48

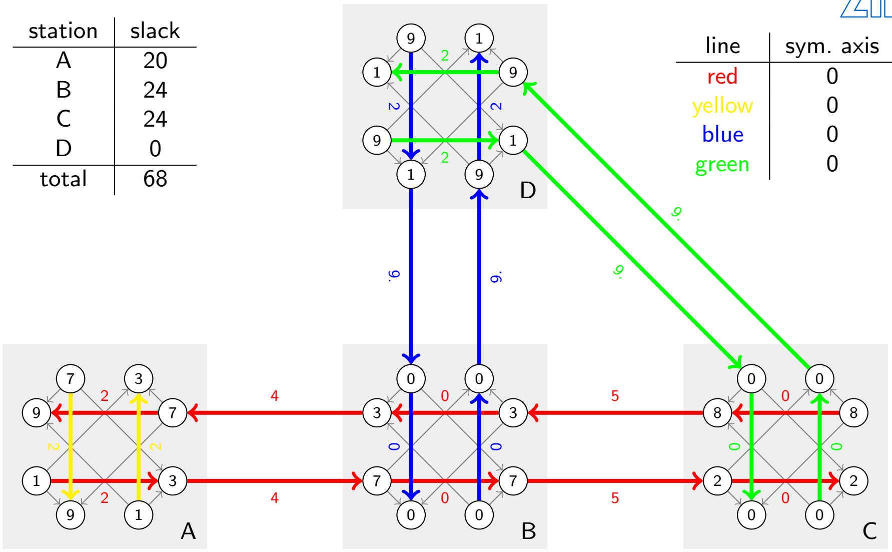


line	sym. axis
red	4
yellow	4
blue	2
green	2

Optimal symmetric solution (computed by SCIP)

station	slack
A	20
B	24
C	24
D	0
total	68

line	sym. axis
red	0
yellow	0
blue	0
green	0

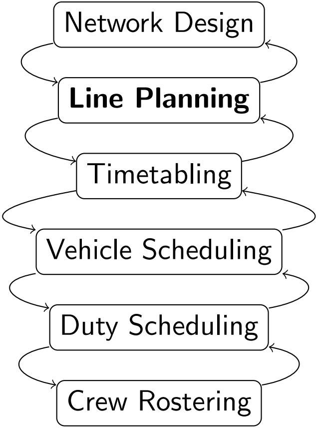


Chapter 5

Line Planning

§5.1 Overview

Public transport planning cycle



strategic planning

operational planning

Description

Let G be a graph modeling a public transportation network, e.g.,

- a road network (for buses)
- a railway track system (for railways, trams, underground trains, ...)

Definition

A **line plan** is a set $\mathcal{L}$ of paths (*lines*) in G together with *frequencies* $f : \mathcal{L} \to \mathbb{N}_0$.

Line Planning Problem

The **line planning problem** is to find a feasible line plan providing both convenient travel for passengers and small operational costs.

Feasible lines

Lines are either chosen from a *line pool*, or are computed on the fly subject to certain restrictions.

Optimization goals

Two oppositional goals

passenger-oriented	*cost-oriented*
Minimize travel time given an upper bound on operational costs	Minimize operational costs given an upper bound on travel time

Passenger quality

Minimize travel time (estimated: no timetable available), Maximize number of passengers having a direct connection, ...

Operational costs

Minimize vehicle costs (estimated: no vehicle schedule), Minimize driver costs (estimated: no crew schedule), ...

Feasibility

Basic Line Planning Feasibility Problem (BLPFP)

Given a graph $G = (V, E)$, a line pool $\mathcal{L}_0$, lower and upper frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$ with $\mathcal{L} \subseteq \mathcal{L}_0$ such that

$$\forall e \in E : \quad f_e^{\min} \leq \sum_{\ell \in \mathcal{L} : e \in \ell} f_\ell \leq f_e^{\max}.$$

Example

Assume that there is an edge e that has to be served at least 3 times per hour, i.e., $f_e^{\min} = 3$. This might be satisfied by a line ℓ_1 with $f_{\ell_1} = 2$ (riding twice per hour), together with a line ℓ_2 with $f_{\ell_2} = 1$ (riding once per hour).

Theorem (Bussieck, 1998)

BLPFP is NP-complete.

Complexity

Definition

The **exact cover by 3-sets problem (X3C)** is the following:

Given a set X with $3q$ elements for some integer q, and a collection C of 3-element subsets of X, is there a subcollection $S \subseteq C$ such that each $x \in X$ occurs in exactly one member of S?

Theorem (Karp, 1972)

X3C is NP-complete.

Theorem (Bussieck, 1998)

X3C $\leq$ BLPFP.

X3C $\leq$ BLPFP

Proof ($\Leftarrow$).

Let (X, C) be an instance for X3C. We consider C as set of triples (x, y, z). Build a simple graph $G = (V, E)$ as follows:

- Add two vertices x^+ and x^- for each $x \in X$.
- Add an edge $\{x^-, x^+\}$ for each $x \in X$.
- Add two edges $\{x^+, y^-\}$, $\{y^+, z^-\}$ for each $(x, y, z) \in C$.

Define the line pool $\mathcal{L}_0 := \{(x^-, x^+, y^-, y^+, z^-, z^+) \mid (x, y, z) \in C\}$ and the lower and upper frequency bounds

$$f_e^{\min} := \begin{cases} 1 & \text{if } e = \{x^-, x^+\} \text{ for some } x \in X, \\ 0 & \text{otherwise,} \end{cases} \qquad f_e^{\max} := 1, \quad e \in E.$$

Let $(\mathcal{L}, f)$ be a feasible line plan. Then for each $x \in X$, the edge $\{x^-, x^+\}$ is covered by a unique line $\ell \in \mathcal{L}$ with $f_\ell = 1$, corresponding to a unique triple $(x, y, z) \in C$.

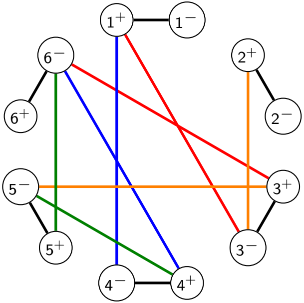
X3C $\leq$ BLPFP

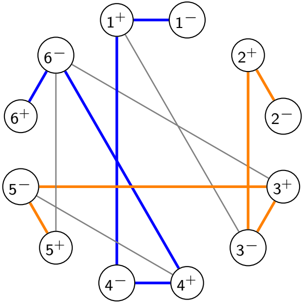
Proof ($\Rightarrow$):

Conversely, let $S \subseteq C$ be a subcollection solving the X3C problem on (X, C). Then taking all lines $(x^-, x^+, y^-, y^+, z^-, z^+) \in \mathcal{L}_0$ for triples $(x, y, z) \in S$ with frequency 1 yields a feasible line plan. $\square$

Example

$X := \{1, 2, 3, 4, 5, 6\}$, $C := \{(1, 3, 6), (1, 4, 6), (2, 3, 5), (4, 5, 6)\}$





Cost-oriented LPP

Cost-oriented Line Planning Problem

Given a graph $G = (V, E)$, a line pool $\mathcal{L}_0$ with costs $c : \mathcal{L}_0 \to \mathbb{R}_{\geq 0}$, lower and upper frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$, find a line plan $(\mathcal{L}, f)$

$$
\begin{aligned}
&\text{minimizing} && \sum_{\ell \in \mathcal{L}} c_\ell && \\
&\text{subject to} && f_e^{\min} \leq \sum_{\ell \in \mathcal{L} : e \in \ell} f_\ell \leq f_e^{\max}, && e \in E, \\
& && \ell \in \mathcal{L}_0, && \ell \in \mathcal{L}.
\end{aligned}
$$

Lemma (Exercise)

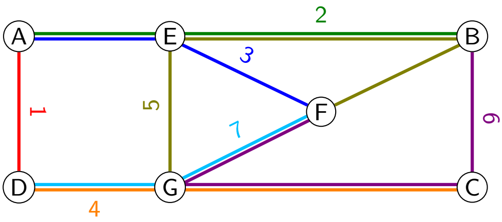
The problem "Given C, is there a feasible line plan with cost $\leq C$" is NP-complete.

Remark

The quality for passengers is established by the minimum frequency requirement.

Cost-oriented LPP: Example

Graph G with line pool $\mathcal{L}_0$



Further data

$$c \equiv 1$$
$$f^{\min} \equiv 1$$
$$f^{\max} \equiv 2$$

Since the edges AD, BC, EF, EG need to be served with frequency $\geq f^{\min} = 1$, the lines 1, 3, 5, 6 have to appear in every feasible line plan. This leaves the edge DG uncovered, which can be covered either by line 4 or line 7. In particular, the cost of an optimal line plan is at least 5.

Running each of the lines 1, 3, 4, 5, 6 with frequency 1 is a feasible line plan: Each edge is covered at least once, only CG is covered twice, and no edge is covered more than twice.

Chapter 5

Line Planning

§5.2 Passenger-Oriented Models

Passenger flow

Let $G = (V, E)$ be a graph.

Definition

- An **origin-destination matrix**, short **OD matrix**, is a $V \times V$-matrix (d_{st}) with non-negative entries.
- For $(s, t) \in V \times V$, the entry d_{st} is called the **demand** from s to t.
- An **OD pair** is a pair $(s, t) \in V \times V$ such that $d_{st} > 0$.

OD matrices are the standard tool to model demands in a public transportation network. However, without a timetable, it is hard to tell which routes passengers will take.

Routing strategies

- shortest paths without transfer times
- shortest paths with transfer penalty
- system split: divide into different transport modes
- ...

Direct Travelers LPP

Input

- graph $G = (V, E)$
- OD matrix (d_{st}) with set of OD pairs $\mathcal{D} \subseteq V \times V$
- fixed passenger paths p_{st} for all $(s, t) \in \mathcal{D}$
- line pool $\mathcal{L}_0$
- frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$
- global capacity bound $C \geq 0$

Goal

Find a feasible line plan $(\mathcal{L}, f)$ maximizing the number of direct travelers over all OD pairs.

Remark

This is trivial to maximize if there are neither capacities nor upper bounds on line costs: Either p_{st} is covered by a line in $\mathcal{L}_0$ or not.

Direct Travelers LPP

MIP formulation

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{\ell \in \mathcal{L}} \sum_{(s,t) \in \mathcal{D}\colon p_{st} \subseteq \ell} x_{st,\ell} \\
\text{subject to} \quad & \sum_{\ell \in \mathcal{L}\colon p_{st} \subseteq \ell} x_{st,\ell} \leq d_{st}, && (s,t) \in \mathcal{D}, \\
& \sum_{(s,t) \in \mathcal{D}\colon e \in p_{st} \subseteq \ell} x_{st,\ell} \leq C \cdot f_\ell, && e \in E, \ell \in \mathcal{L}, \\
& f_e^{\min} \leq \sum_{\ell \in \mathcal{L}\colon e \in \ell} f_\ell \leq f_e^{\max}, && e \in E, \\
& \ell \in \mathcal{L}_0, && \ell \in \mathcal{L}, \\
& f_\ell \in \mathbb{N}_0, && \ell \in \mathcal{L}, \\
& x_{st,\ell} \geq 0, && (s,t) \in \mathcal{D}, \ell \in \mathcal{L}.
\end{aligned}
$$

Notation

$x_{st,\ell}$ is the number of direct travelers from s to t using line ℓ.

Direct Travelers LPP

Remarks

- There is no point in taking $x_{st,\ell}$ integral: Capacities are in general only rough estimates, and the number of direct travelers is usually large.
- The capacity C may be replaced by capacities for each pair of edge and line.
- One may also integrate budget constraints in terms of upper bounds on the frequencies.
- We have $\mathcal{L} = \{\ell \in \mathcal{L}_0 \mid f_\ell > 0\}$. We can therefore replace $\mathcal{L}$ by $\mathcal{L}_0$ in the MIP formulation. In other words, f_ℓ also takes the role of a decision variable if line ℓ should by included into $\mathcal{L}$ or not.
- In particular, this is a mixed integer *linear* program.
- This model is due to Bussieck/Kreuzer/Zimmermann, 1995.
- Disadvantage: Hard to solve exactly.

Direct Travelers LPP: Aggregation

Aggregation

Set $x_{st} := \sum_{\ell \in \mathcal{L}_0 \colon p_{st} \subseteq \ell} x_{st,\ell}$, i.e., count all directly traveling passengers from s to t using any line.

Aggregated MIP formulation

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{(s,t) \in \mathcal{D} \colon p_{st} \subseteq \ell} x_{st} & \\
\text{subject to} \quad & x_{st} \leq d_{st}, & (s,t) \in \mathcal{D}, \\
& x_{st} \leq C \cdot \sum_{\ell \in \mathcal{L}_0 \colon p_{st} \subseteq \ell} f_\ell, & (s,t) \in \mathcal{D}, \\
& f_e^{\min} \leq \sum_{\ell \in \mathcal{L} \colon e \in \ell} f_\ell \leq f_e^{\max}, & e \in E, \\
& \ell \in \mathcal{L}_0, & \ell \in \mathcal{L}, \\
& f_\ell \in \mathbb{N}_0, & \ell \in \mathcal{L}, \\
& x_{st} \geq 0, & (s,t) \in \mathcal{D}.
\end{aligned}
$$

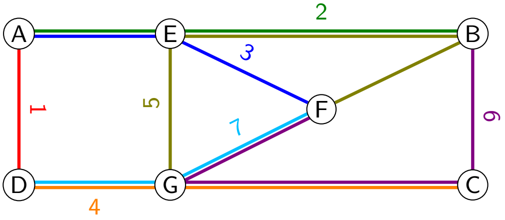
Direct Travelers LPP: Aggregation

Remarks

- Any feasible solution for the bigger model is feasible for the aggregated model.
- The converse is in general not true.
- However, this gives a heuristic for solving the bigger model.

Direct Travelers LPP: Example

Graph G with line pool $\mathcal{L}_0$



OD matrix

		to		
		B	C	F
from	A	50	0	50
	D	0	80	20
	G	40	0	0

Further data: $f^{\min} = 0,\ f^{\max} = \infty,\ C = 50$

To serve all demands by direct connections, we need

OD pair	line	freq.	pass./cap.	OD pair	line	freq.	pass./cap.
A → B	2	1	50/50	D → F	7	1	20/50
A → F	3	1	50/50	G → B	5 or 6	1	40/50
D → C	4	2	80/100				

This clearly maximizes the number of direct travelers, which is 240.