Mathematical Aspects of Public Transportation Networks

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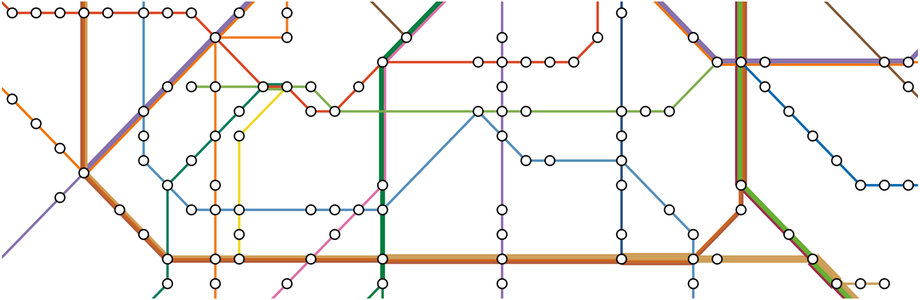


July 2, 2018

Excursion

Excursion to DB Fernverkehr on **Thursday, July 12**:

- talk on planning processes
  (Berlin Hauptbahnhof)
- guided tour through ICE maintenance building
  (Betriebsbahnhof Rummelsburg)

Please register soon!

There will be no tutorial on July 12.

Chapter 5

Line Planning

§5.2 Passenger-Oriented Models

Direct Travelers LPP

Input

- graph $G = (V, E)$
- OD matrix (d_{st}) with set of OD pairs $\mathcal{D} \subseteq V \times V$
- fixed passenger paths p_{st} for all $(s, t) \in \mathcal{D}$
- line pool $\mathcal{L}_0$
- frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$
- global capacity bound $C \geq 0$

Goal

Find a feasible line plan $(\mathcal{L}, f)$ maximizing the number of direct travelers over all OD pairs.

Remark

This is trivial to maximize if there are neither capacities nor upper bounds on line costs: Either p_{st} is covered by a line in $\mathcal{L}_0$ or not.

Direct Travelers LPP

MIP formulation

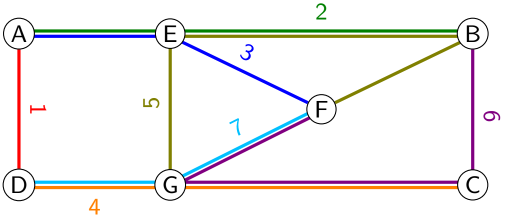
$$
\begin{aligned}
\text{Maximize} \quad & \sum_{\ell \in \mathcal{L}} \sum_{(s,t) \in \mathcal{D}\colon p_{st} \subseteq \ell} x_{st,\ell} & \\
\text{subject to} \quad & \sum_{\ell \in \mathcal{L}\colon p_{st} \subseteq \ell} x_{st,\ell} \leq d_{st}, & (s,t) \in \mathcal{D}, \\
& \sum_{(s,t) \in \mathcal{D}\colon e \in p_{st} \subseteq \ell} x_{st,\ell} \leq C \cdot f_\ell, & e \in E, \ell \in \mathcal{L}, \\
& f_e^{\min} \leq \sum_{\ell \in \mathcal{L}\colon e \in \ell} f_\ell \leq f_e^{\max}, & e \in E, \\
& \ell \in \mathcal{L}_0, & \ell \in \mathcal{L}, \\
& f_\ell \in \mathbb{N}_0, & \ell \in \mathcal{L}, \\
& x_{st,\ell} \geq 0, & (s,t) \in \mathcal{D}, \ell \in \mathcal{L}.
\end{aligned}
$$

Notation

$x_{st,\ell}$ is the number of direct travelers from s to t using line ℓ.

Direct Travelers LPP: Example

Graph G with line pool $\mathcal{L}_0$



OD matrix

		to		
		B	C	F
from	A	50	0	50
	D	0	80	20
	G	40	0	0

Further data: $f^{\min} = 0$, $f^{\max} = \infty$, $C = 50$

To serve all demands by direct connections, we need

OD pair	line	freq.	pass./cap.	OD pair	line	freq.	pass./cap.
A → B	2	1	50/50	D → F	7	1	20/50
A → F	3	1	50/50	G → B	5 or 6	1	40/50
D → C	4	2	80/100				

This clearly maximizes the number of direct travelers, which is 240.

Travel Time LPP

Remark

The direct travelers approach ignores all passengers that cannot travel directly.

Idea

Allow transfers:

- The transfer times are unknown (no timetable). However, it makes sense to penalize the number of transfers.
- Consider paths with weight $w_1 \cdot \text{travel time} + w_2 \cdot \text{no. of transfers}$.

Model (Schöbel/Scholl, 2003)

Model the line planning problem minimizing passenger travel time on the *change & go* graph.

Change & go

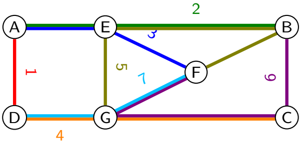
Let $G = (V, E)$ be a public transportation network with a line pool $\mathcal{L}_0$. Assume w.l.o.g $0 \notin \mathcal{L}_0$. Define

- $V_{\mathrm{CG}} := \{(v, \ell) \in V \times \mathcal{L}_0 \mid \ell \text{ stops at } v\}$ (change&go vertices)
- $V_{\mathrm{OD}} := \{(s, 0) \in V \mid \exists\, t \in V : d_{st} > 0 \text{ or } d_{ts} > 0\}$ (OD vertices)
- $E_{\mathrm{change}} := \{((v, \ell_1), (v, \ell_2)) \in V_{\mathrm{CG}} \times V_{\mathrm{CG}}\}$ (changing edges)
- $E_{\mathrm{go}}(\ell) := \{((v, \ell), (w, \ell)) \in V_{\mathrm{CG}} \times V_{\mathrm{CG}} \mid vw \in E\}$
  (driving edges for $\ell \in \mathcal{L}_0$)
- $E_{\mathrm{go}} := \bigcup_{\ell \in \mathcal{L}_0} E_{\mathrm{go}}(\ell)$ (driving edges)
- $E_{\mathrm{OD}} := \{((s, 0), (s, \ell)) \in V_{\mathrm{OD}} \times V_{\mathrm{CG}} \mid \exists\, t \in V : d_{st} > 0\}$
  $\cup \{((t, \ell), (t, 0)) \in V_{\mathrm{CG}} \times V_{\mathrm{OD}} \mid \exists\, s \in V : d_{st} > 0\}$ (OD edges)

Definition

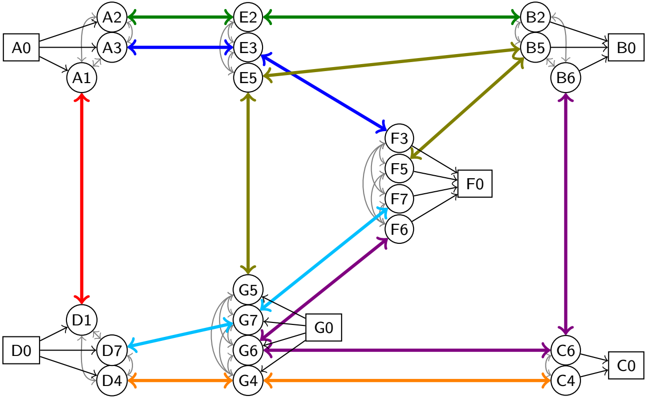
The **change & go graph** is the digraph with vertex set $V_{\mathrm{CG}} \cup V_{\mathrm{OD}}$ and edge set $E_{\mathrm{change}} \cup E_{\mathrm{go}} \cup E_{\mathrm{OD}}$.

Change & go: Example



	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

↓



Travel Time LPP via change & go

Input

- graph $G = (V, E)$
- OD matrix (d_{st}) with set of OD pairs $\mathcal{D} \subseteq V \times V$
- line pool $\mathcal{L}_0$
- frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$
- line capacities $C : \mathcal{L}_0 \to \mathbb{R}_{\geq 0}$
- change & go graph $G' = (V', E')$ obtained from G
- incidence matrix A of G'
- for $v \in V'$, balances $b_{st,v} = \begin{cases} d_{st} & \text{if } v = (s,0), \\ -d_{st} & \text{if } v = (t,0), \\ 0 & \text{otherwise.} \end{cases}$
- for $e \in E'$, edge costs c_e,
  e.g., $w_1 \cdot$ travel time for $e \in E_{\text{go}}$ and w_2 for $e \in E_{\text{change}}$

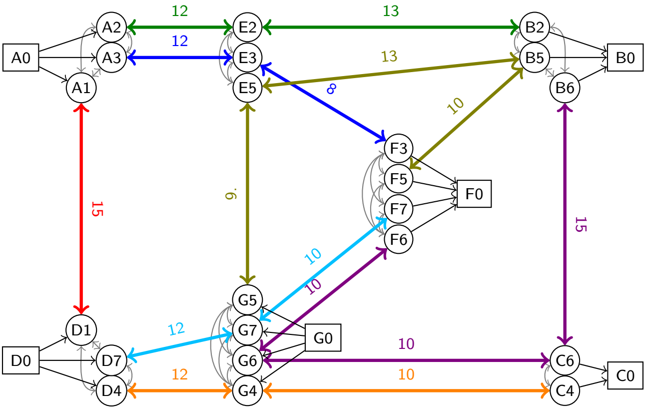
Travel Time LPP via change & go

MIP formulation

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{(s,t)\in\mathcal{D}} \sum_{e\in E'} c_e x_{st,e} & \\
\text{subject to} \quad & \sum_{(s,t)\in\mathcal{D}} x_{st,e} \le C_\ell \cdot f_\ell, & \ell \in \mathcal{L}_0, e \in E_{\text{go}}(\ell), \\
& A x_{st} = b_{st}, & (s,t) \in \mathcal{D}, \\
& x_{st,e} \ge 0, & (s,t) \in \mathcal{D}, e \in E', \\
& f_e^{\min} \le \sum_{\ell\in\mathcal{L}_0:\, e\in\ell} f_\ell \le f_e^{\max}, & e \in E, \\
& f_\ell \in \mathbb{N}_0, & \ell \in \mathcal{L}_0.
\end{aligned}
$$

This is $|\mathcal{D}|$-commodity flow + basic line planning feasibility.

Travel Time LPP via change & go: Example



	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$f_{\min} \equiv 0$

$f_{\max} \equiv \infty$

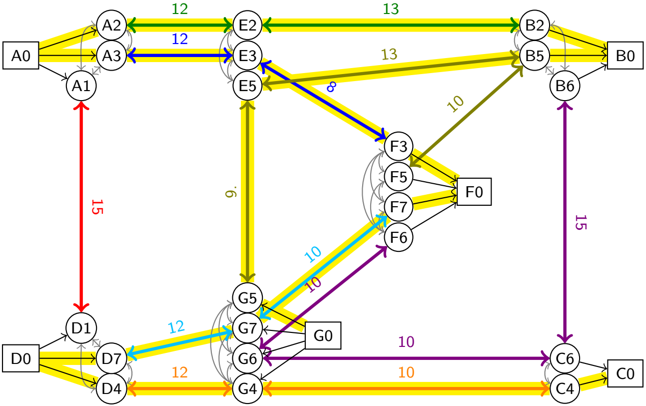
$C \equiv 50$

$c_e = 0, e \in E_{\text{OD}}$

$c_e = 2, e \in E_{\text{change}}$

Since the frequencies are not bounded from above, this problem decomposes into the 5 shortest path problems for the 5 OD pairs.

Travel Time LPP via change & go: Example



	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$f_{\min} \equiv 0$

$f_{\max} \equiv \infty$

$C \equiv 50$

$c_e = 0, e \in E_{\text{OD}}$

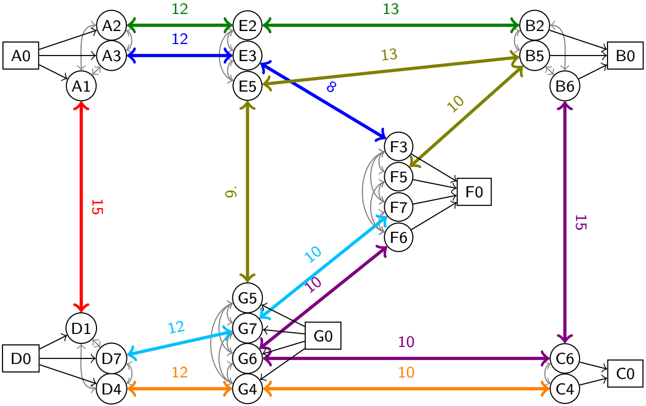
$c_e = 2, e \in E_{\text{change}}$

Since the frequencies are not bounded from above, this problem decomposes into the 5 shortest path problems for the 5 OD pairs.

Cost of line plan: $50 \cdot 25 + 50 \cdot 20 + 80 \cdot 22 + 20 \cdot 22 + 40 \cdot 22 = 5330$

Frequencies: 0, 1, 1, 2, 1, 0, 1

Travel Time LPP via change & go: Example



	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$f_{\text{min}} \equiv 0$

$f_{\text{max}} \equiv 2$

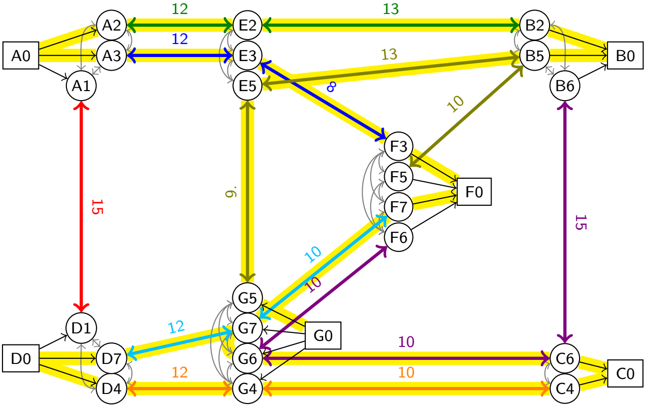
$C \equiv 50$

$c_e = 0, e \in E_{\text{OD}}$

$c_e = 2, e \in E_{\text{change}}$

Now each edge can be used at most by either one line with frequency 2 or two lines with frequency 1 ($\rightsquigarrow$ true 5-commodity flow). 30 passengers from D to C change at G

Travel Time LPP via change & go: Example



	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$f_{\min} \equiv 0$

$f_{\max} \equiv 2$

$C \equiv 50$

$c_e = 0, e \in E_{\text{OD}}$

$c_e = 2, e \in E_{\text{change}}$

Now each edge can be used at most by either one line with frequency 2 or two lines with frequency 1 ($\rightsquigarrow$ true 5-commodity flow).

Cost of plan: $50 \cdot 25 + 50 \cdot 20 + (50 \cdot 22 + 30 \cdot 24) + 20 \cdot 22 + 40 \cdot 22 = 5390$

Frequencies: 0, 1, 1, 1, 1, 1, 1 30 passengers from D to C change at G

Travel Time LPP: Solution methods

Solution approaches to the Travel Time LPP using the change&go graph include:

- Dantzig-Wolfe decomposition on an edge-based MIP formulation
- column generation on a path-based MIP formulation

Both methods have simple shortest path problems as subproblems.

Lines ($\|\mathcal{L}_0\|$)	OD pairs ($\|\mathcal{D}\|$)	running time (s)
100	2602	1
200	10126	78
300	17507	1171
400	22191	4789

Schöbel/Scholl, 2006

In contrast, the Xpress LP solver could not solve LP relaxations for instances of even smaller size due to memory overflow (Scholl, 2005).

Variable Lines LPP

Main Ideas (Borndörfer/Grötschel/Pfetsch, 2007)

- extends change & go, but ignores transfers
- path-based model
- variable routing of passengers (as in change & go)
- variable choice of lines (no line pool)

Solution method

- exponentially many path variables
  $\rightsquigarrow$ column generation is reasonable
- pricing of path variables: shortest path problem
- pricing of line variables: longest path problem

The pricing of line variables was not an issue in the change & go approach, because the line pool was fixed. Here, the pricing determines which lines are considered at all.

Variable Lines LPP: MIP formulation

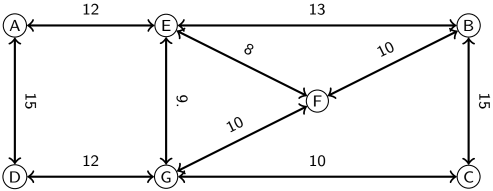
Input

- graph $G = (V, E)$
- OD matrix (d_{st}) with set of OD pairs $\mathcal{D} \subseteq V \times V$
- edge frequency bounds $f^{\min} \leq f^{\max} : E \to \mathbb{N}_0$
- set $\mathcal{L}_0$ of all (simple) paths in G
- line frequency bounds $F : \mathcal{L}_0 \to \mathbb{N}_0$
- line capacities $C : \mathcal{L}_0 \to \mathbb{R}_{\geq 0}$
- routing digraph $G' = (V', E')$ obtained from G by replacing each edge $\{v, w\}$ by two antiparallel edges (v, w), (w, v)
- for $(v, w) \in E'$, set $\mathcal{L}_{(v,w)} := \{\ell \in \mathcal{L}_0 \mid \{v, w\} \in \ell\}$
- edge costs c_e for $e \in E'$ (travel time)
- set $\mathcal{P}_{st}$ of all directed s-t-paths in G' for all $(s, t) \in \mathcal{D}$
- $\mathcal{P} := \bigcup_{(s,t)\in\mathcal{D}} \mathcal{P}_{st}$

Variable Lines LPP: MIP formulation

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{p\in\mathcal{P}} y_p \sum_{e\in p} c_e \\
\text{subject to} \quad & \sum_{p\in\mathcal{P}:\, e\in p} y_p \le \sum_{\ell\in\mathcal{L}_e} C_\ell \cdot f_\ell, && e \in E', \\
& \sum_{p\in\mathcal{P}_{st}} y_p = d_{st}, && (s,t) \in \mathcal{D}, \\
& y_p \ge 0, && p \in \mathcal{P}, \\
& f_e^{\min} \le \sum_{\ell\in\mathcal{L}_e} f_\ell \le f_e^{\max}, && e \in E, \\
& f_\ell \le F_\ell \cdot x_\ell, && \ell \in \mathcal{L}_0, \\
& f_\ell \ge 0, && \ell \in \mathcal{L}_0, \\
& x_\ell \in \{0,1\}, && \ell \in \mathcal{L}_0.
\end{aligned}
$$

Variable Lines LPP: Example

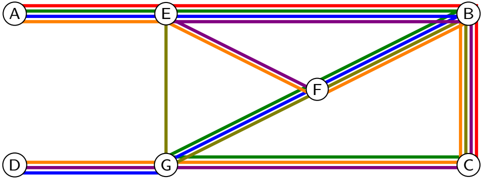


$$|\mathcal{L}_0| = 164, \quad |\mathcal{D}| = 5, \quad |\mathcal{P}| = 48$$

	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$$f_{\text{min}} \equiv 0$$
$$f_{\text{max}} \equiv 2$$
$$F \equiv 2$$
$$C \equiv 50$$

Variable Lines LPP: Example



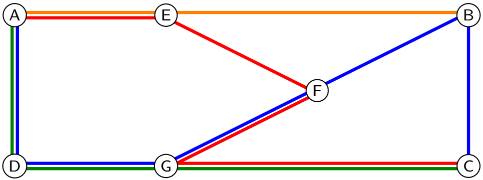
$$|\mathcal{L}_0| = 164, \quad |\mathcal{D}| = 5, \quad |\mathcal{P}| = 48$$

	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$$f_{\min} \equiv 0$$
$$f_{\max} \equiv 2$$
$$F \equiv 2$$
$$C \equiv 50$$

Optimal fractional line plan: lines: 6, cost: 5250, frequencies: 0.4, 0.267, 0.667, 0.667, 0.667, 0.267

Variable Lines LPP: Example



$|\mathcal{L}_0| = 164, \quad |\mathcal{D}| = 5, \quad |\mathcal{P}| = 48$

	B	C	F
A	50	0	50
D	0	80	20
G	40	0	0

$$f_{\min} \equiv 0$$
$$f_{\max} \equiv 2$$
$$F \equiv 2$$
$$C \equiv 50$$

Optimal integral line plan: lines: 4, cost: 5250, frequencies: 1, 1, 1, 1.
Since lines can be extended with no cost, e.g., AD carries no passengers.

Variable Lines LPP: More aspects

Operational and fixed costs

Minimize

$$\sum_{p \in \mathcal{P}} y_p \sum_{e \in p} c_e + \sum_{\ell \in \mathcal{L}_0} c'_\ell x_\ell + \sum_{\ell \in \mathcal{L}_0} c''_\ell f_\ell$$

The costs c'_ℓ are the fixed costs for adding line ℓ, and c''_ℓ model operational costs for line ℓ depending on the frequency.

Length restriction

If paths or lines are restricted in length, e.g., by

$$\sum_{e \in p} c_e \leq L, \quad p \in \mathcal{P},$$

this leads to NP-hard pricing problems (constrained shortest path).

Chapter 5

Line Planning

§5.3 Steiner Connectivity