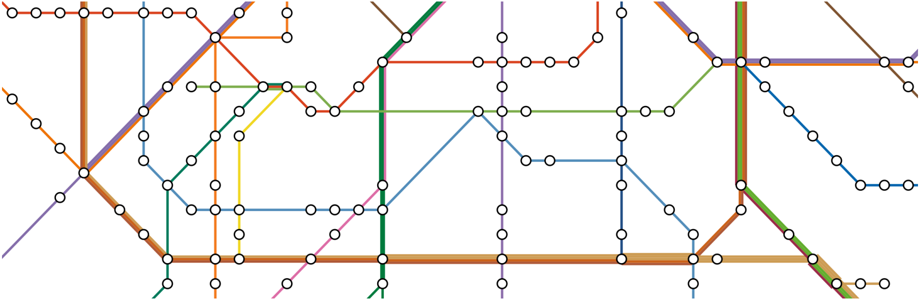
Mathematical Aspects of Public Transportation Networks

Niels Lindner



April 23, 2018

Chapter 1

S-Bahn Challenge

§1.3 The Traveling Salesman Problem

Interlude: P vs. NP

Informal definitions

- A **decision problem** is a problem whose solution is either *yes* or *no*.
- The **complexity class P** consists of all decision problems that can be *solved* in polynomial time.
- The **complexity class NP** consists of all decision problems that can be *verified* in polynomial time

P vs. NP

The question whether $\mathrm{P} = \mathrm{NP}$ is a millenium problem.

Notation

For a decision problem Π with an input x, we write $x \in \Pi$ iff x is a "yes"-instance for Π.

Interlude: P vs. NP

How to show membership to P or NP

Let Π be a decision problem.

- $\Pi \in \mathrm{P} \Leftrightarrow \exists$ polynomial p and an algorithm A that decides for each input x if $x \in \Pi$, and the running time of A is $\leq p(\mathrm{size}(x))$.
- $\Pi \in \mathrm{NP} \Leftrightarrow \exists$ polynomial p and a problem $\Lambda \in \mathrm{P}$ such that each input x has a certificate $c(x)$ satisfying $x \in \Pi \Leftrightarrow (x, c(x)) \in \Lambda$, and $\mathrm{size}(c(x)) \leq p(\mathrm{size}(x))$.

Examples

- "Does a graph G admit an Euler tour?" is in P.
- "Is a graph G Hamiltonian?" is in NP.
  (certificate: a Hamiltonian circuit C)
- "Is a graph G *not* Hamiltonian?" is not known to be in NP.
  (certificate: all circuits in G – too large!)

Polynomial-time reduction

Definition

Let Π and Λ be decision problems. Π **reduces polynomially** to Λ (short: $\Pi \leq \Lambda$) if there is a function f on the inputs for Π such that

$$x \in \Pi \Leftrightarrow f(x) \in \Lambda,$$

and f can be computed by a polynomial-time algorithm.

Remarks

- This is a partial order.
- Intuitively, $\Pi \leq \Lambda$ if and only if Π is at most as hard to solve as Λ.
- If $\Pi \leq \Lambda$ and $\Lambda \leq \Pi$, then Π and Λ are **polynomially equivalent**.

Lemma

- $\Pi \in \mathrm{P} \Leftrightarrow \Pi \leq \Lambda$ *for some* $\Lambda \in \mathrm{P}$.
- $\Pi \in \mathrm{NP} \Leftrightarrow \Pi \leq \Lambda$ *for some* $\Lambda \in \mathrm{NP}$.

NP-completeness

Definition

Let Π be a decision problem.

- Π is **NP-hard** if $\Lambda \leq \Pi$ for each $\Lambda \in \mathrm{NP}$.
- Π is **NP-complete** if Π is NP-hard and $\Pi \in \mathrm{NP}$.

Lemma (How to show NP-hardness)

Suppose there is an NP-hard problem Λ with $\Lambda \leq \Pi$. Then Π is NP-hard.

Optimization problems

We also call a minimization problem $\min_{x \in X} f(x)$ **NP-hard/-complete** if the decision problem

$$\text{“Given } q \in \mathbb{Q}\text{, is there an } x \in X \text{ with } f(x) \leq q\text{?”}$$

is NP-hard/-complete. (Similar: maximization with "$\geq$".)

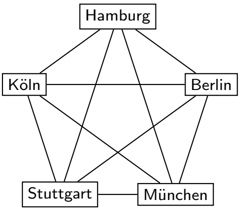
Complete Graphs

Let $n \in \mathbb{N}$. The **complete graph** K_n is the graph with

- vertex set $V(K_n) = \{1, \ldots, n\}$,
- edge set $E(K_n) = \{\{i, j\} \mid 1 \leq i < j \leq n\}$.

Definition

The **Traveling Salesman Problem (TSP)** on a complete graph K_n is to find a minimum-cost Hamilton circuit in K_n w.r.t. a cost function $c : E(K_n) \to \mathbb{R}_{\geq 0}$.



distance/km	HH	K	S	M	B
Hamburg	0	366	534	613	256
Köln	366	0	288	456	478
Stuttgart	534	288	0	191	512
München	613	456	191	0	505
Berlin	256	478	512	505	0

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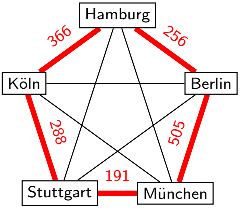

optimal cost: 1606

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Hardness

Theorem

TSP is NP-hard.

Proof.

Let G be a graph on n vertices with edge set $E(G)$. Define a cost function on $E(K_n)$ via

$$c(\{i,j\}) := \begin{cases} 1 & \text{if } \{i,j\} \in E(G), \\ 2 & \text{otherwise,} \end{cases} \qquad 1 \leq i < j \leq n.$$

Then G contains a Hamiltonian circuit if and only if K_n has a Hamiltonian circuit with cost $\leq n$. □

Combinatorial Explosion

K_n contains $(n-1)!/2$ Hamilton circuits.

Approximation hardness

Definition

Let P be an optimization problem with non-negative cost and $k \geq 1$. A k-**factor approximation algorithm** for P is a polynomial-time algorithm A for P such that

$$\frac{1}{k} \cdot \mathrm{OPT}(I) \leq A(I) \leq k \cdot \mathrm{OPT}(I)$$

for all instances I of P. Here, $\mathrm{OPT}(I)$ denotes the cost of an optimal solution, and $A(I)$ is the cost of the solution computed by A.

A k-factor approximation algorithm is a polynomial-time heuristic with a worst-case estimate on the solution quality (the lower k, the better).

Theorem

Let A be a k-factor approximation algorithm for TSP for some $k \geq 1$. Then $P = NP$.

Approximation hardness

Proof.

- Let A be such an algorithm, i.e., for every TSP instance $I = (K_n, c)$ with optimal solution $\mathrm{OPT}(I)$, A computes a Hamiltonian circuit of cost $A(I) \leq k \cdot \mathrm{OPT}(I)$.
- Let G be a graph with edge set $E(G)$ and n vertices. Define a cost function on $E(K_n)$ via

$$c(\{i,j\}) := \begin{cases} 1 & \text{if } \{i,j\} \in E(G), \\ 2 + (k-1)n & \text{otherwise,} \end{cases} \qquad 1 \leq i < j \leq n.$$

- If $A(I) \leq n$, then G admits a Hamiltonian circuit.
- Otherwise $k \cdot \mathrm{OPT}(I) \geq A(I) \geq n - 1 + 2 + (k-1)n = kn + 1$, thus $\mathrm{OPT}(I) > n$ and G cannot have a Hamiltonian circuit.
- A is a polynomial-time algorithm deciding the NP-complete Hamilton circuit problem on an arbitrary graph. This implies $\mathrm{P} = \mathrm{NP}$.

Metric TSP

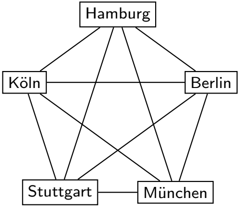
Definition

A TSP instance (K_n, c) is called **metric** if the triangle inequality $c(\{i,j\}) \leq c(\{i,k\}) + c(\{k,j\})$ holds for all $1 \leq i,j,k \leq n$.

Theorem (Christofides, 1976)

There is a $\frac{3}{2}$-factor approximation algorithm for metric TSP.

Christofides' algorithm



1. Compute a minimum spanning tree T in K_n w.r.t. c.
2. Find a min-weight perfect matching M of the odd-degree vertices of T w.r.t. c.
3. Take the Hamiltonian circuit by sorting the vertices by order of appearance in an Euler tour in $(V(K_n), E(T) \cup M)$.

Metric TSP

Definition

A TSP instance (K_n, c) is called **metric** if the triangle inequality $c(\{i,j\}) \leq c(\{i,k\}) + c(\{k,j\})$ holds for all $1 \leq i,j,k \leq n$.

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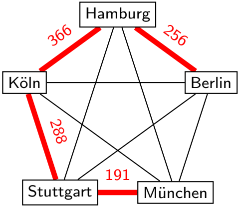

MST: 1101

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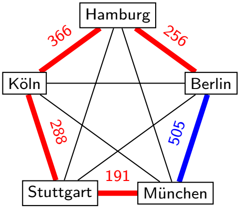


MST: 1101, Matching: 505
TSP: 1606

1. Compute a minimum spanning tree T in K_n w.r.t. c.
2. Find a min-weight perfect matching M of the odd-degree vertices of T w.r.t. c.
3. Take the Hamiltonian circuit by sorting the vertices by order of appearance in an Euler tour in $(V(K_n), E(T) \cup M)$.

Christofides' Algorithm

Proof.

- Let $I = (K_n, c)$ be a TSP instance. Removing a single edge from any Hamilton circuit gives a spanning tree. Hence for a minimum spanning tree T of K_n w.r.t. c, we have $\mathrm{OPT}(I) \geq c(T) := \sum_{e \in E(T)} c(e)$.
- A shortest path from i to j is simply given by the edge $\{i, j\}$ because of the triangle inequality.
- Denote by $c(M)$ the weight of the min-weight perfect matching M. Each Hamiltonian circuit decomposes into two matchings of the odd-degree nodes of T. Hence $\mathrm{OPT}(I) \geq 2c(M)$ (triangle inequality).
- The graph $(V(K_n), E(T) \cup M)$ is clearly Eulerian.
- Computing a Hamiltonian circuit from an Euler tour does not increase the cost (again triangle inequality).
- Thus $A(I) \leq c(T) + c(M) \leq \mathrm{OPT}(I) + \frac{1}{2}\mathrm{OPT}(I) = \frac{3}{2}\mathrm{OPT}(I)$.
- The algorithm runs in polynomial time.

The k-opt heuristic

For non-metric TSP instances $I = (K_n, c)$, there is a family of heuristics based on local search:

k-opt heuristic

Fix an integer $k \geq 2$.

1. Let C be any Hamiltonian circuit.
2. Let $\mathcal{S}$ be the collection of all k-element subsets of $E(C)$.
3. Let $C' := \arg\min\{c(C') \mid C' \text{ Ham. circuit}, E(C) \setminus S \subseteq E(C'), S \in \mathcal{S}\}$.
4. If $c(C') < c(C)$, set $C := C'$ and go to 2. Otherwise return C'.

Remarks

- For all $k \geq 2$, the worst-case running time is exponential in n.
- n-opt would be exact, but enumerates all possibilites.
- In Step 3, 2-opt simply replaces two edges $(i, j), (k, \ell)$ by $(i, k), (j, \ell)$.

Integer Programming

The TSP on (K_n, c) has the following classical formulation as an IP:

$$\begin{aligned} \text{Minimize} \quad & \sum_{e \in E(K_n)} c(e) x_e \\ \text{s. t.} \quad & \sum_{e \in E(K_n): v \in e} x_e = 2, && v \in V(K_n), \\ & \sum_{e \in E(K_n): e \in S \times S} x_e \leq |S| - 1, && \emptyset \subsetneq S \subsetneq V(K_n), \\ & x_e \in \{0, 1\}, && e \in E(K_n). \end{aligned}$$

The second constraint is called *subtour elimination* constraint. It excludes solutions that are unions of disjoint circuits. Unfortunately, there are exponentially many of those.

Separating Subtour Constraints

Theorem

Let $x \in [0,1]^{E(K_n)}$ satisfy $\sum_{e: v \in e} x_e = 2$. Then there is a polynomial-time algorithm that decides if there is a subset $\emptyset \subsetneq S \subsetneq V(K_n)$ such that x violates the subtour elimination constraint w.r.t. S.

Proof.

Tutorial. □

This yields the following IP-based solution method:

1. Let $\mathcal{S} := \emptyset$.
2. Solve the IP with subtour elimination constraints only for $S \in \mathcal{S}$.
3. If the optimal solution violates the constraint for some S, add it to $\mathcal{S}$. Otherwise, an optimal solution is found.

There are also IP formulations for the TSP with a polynomial number of constraints, but they have weaker LP relaxations and are hence harder for IP solvers.

Solving TSP: Summary

Heuristics

- Metric TSP: Christofides' $\frac{3}{2}$-factor approximation algorithm
- Local search: 2-opt, 3-opt, Lin-Kernighan (combines both, implementation: LKH)

Exact algorithms

- Integer programming: Branch-and-cut (implementation: concorde)
- Dynamic programming: Held-Karp $\mathcal{O}(2^n n^2)$ algorithm

TSP Record

In 2006, concorde computed a solution for a TSP instance on 85 900 vertices, and proved optimality. LKH can solve this instance as well nowadays, but cannot provide lower bounds.

Directed graphs

Let $n \in \mathbb{N}$. The **complete directed graph** K_n^* is the digraph with

- vertex set $V(K_n^*) = \{1, \ldots, n\}$,
- edge set $E(K_n^*) = \{(i,j) \mid 1 \leq i \neq j \leq n\}$.

Definition

The **Asymmetric Traveling Salesman Problem (ATSP)** on K_n^* is to find a minimum-cost directed Hamiltonian circuit w.r.t. a cost function $c : E(K_n^*) \to \mathbb{R}_{\geq 0}$.

Remarks

- If $c(i,j) = c(j,i)$ for all $1 \leq i \neq j \leq n$, then the problem is called *symmetric* and is equivalent to the TSP on the undirected complete graph K_n with cost function $c(\{i,j\}) := c(i,j)$.
- ATSP is NP-complete.

Asymmetric TSP

Theorem (Jonker-Volgenant, 1983)

TSP is polynomially equivalent to ATSP.

Proof.

- Clearly, any TSP instance can be transformed into an ATSP instance by replacing each undirected edge $\{i,j\}$ with cost $c(\{i,j\})$ by the two anti-parallel edges (i,j) and (j,i), and setting $c(i,j) := c(j,i) := c(\{i,j\})$.
- Conversely, let $I = (K_n^*, c)$ be an ATSP instance. Create a TSP instance $I' = (K_{2n}, c')$ as follows: For $i \in V(K_n^*)$, let $i^+ := 2i$ and $i^- := 2i - 1$. Set

$$\begin{aligned} c'_{\{i^+,j^-\}} &:= c(i,j) + M, && (i,j) \in E(K_n^*), \\ c'_{\{i^-,i^+\}} &:= 0, && i \in V(K_n^*), \end{aligned}$$

  and let c' have value $(n+1)M + 1$ on all other edges.

Asymmetric TSP

Proof.

- Then any directed Hamiltonian circuit $(i_1, \ldots, i_n, i_1)$ in K_n^* yields a Hamiltonian circuit $(i_1^+, i_2^-, i_2^+, \ldots, i_n^-, i_n^+, i_1^-)$ in K_{2n}, the cost increases by $n \cdot M$. This shows $\mathrm{OPT}(I) + n \cdot M \geq \mathrm{OPT}(I')$.
- Let C' be the optimal solution to I'. Suppose $M > \mathrm{OPT}(I)$. Then C' contains all n edges $i^- \to i^+$, as otherwise

$$\mathrm{OPT}(I') \geq (n+1)M = n \cdot M + M > n \cdot M + \mathrm{OPT}(I).$$

- Moreover, C' contains none of the $(n+1)M + 1$ cost edges, because otherwise also

$$\mathrm{OPT}(I') \geq (n+1)M + 1 > n \cdot M + \mathrm{OPT}(I).$$

- Hence C' can be transformed to a Hamiltonian circuit in K_n^*, the cost decreasing by $n \cdot M$. Thus $\mathrm{OPT}(I) + n \cdot M = \mathrm{OPT}(I')$.
- Take e.g. $M := 1 +$ sum of n heaviest edges of K_n^* w.r.t. c.

Asymmetric TSP

Summary

- A TSP instance on n nodes can be transformed into an ATSP instance on n nodes, with the same optimal cost.
- An ATSP instance on n nodes can be transformed into a TSP instance on $2n$ nodes, the cost increasing by $n \cdot M$ for a large M.

General undirected graphs

Let $G = (V, E)$ be a not necessarily complete undirected graph with a cost function $c : E \to \mathbb{R}_{\geq 0}$.

Definition

- A **Traveling Salesman tour** is a closed walk $(e_1, \ldots, e_k)$ in G such that every vertex in G is visited *at least* once.
- The **Traveling Salesman Problem (TSP)** is to find a Traveling Salesman tour $(e_1, \ldots, e_k)$ of minimum cost $\sum_{i=1}^{k} c(e_i)$.

Lemma

If G is Hamiltonian and c satisfies the triangle inequality, then the optimal TSP solution is one of the Hamiltonian circuits of G.

In particular, it is important to know if TSP refers to the "exactly once" or "at least once" version.

Reducing ≥ 1 to $= 1$

Let G be an undirected graph on n nodes, $c : E(G) \to \mathbb{R}_{\geq 0}$ a cost function.

Theorem

The "at least once" TSP on G w.r.t. c can be polynomially transformed to an "exactly once" metric TSP instance on K_n with the same optimal cost.

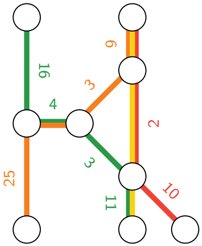
Proof.

- Let $v_1, \ldots, v_n$ denote the vertices of G. For $1 \leq i < j \leq n$, set

  $$c'(\{i, j\}) := \text{length of shortest path from } v_i \text{ to } v_j \text{ in } G \text{ w.r.t. } c.$$

- The optimal Hamiltonian circuit on (K_n, c') produces a closed walk in G by transforming $i \to j$ to the shortest path from $v_i \to v_j$. The cost does not change, hence $\mathrm{OPT}(G, c) \leq \mathrm{OPT}(K_n, c')$.
- The optimal TSP tour in G w.r.t. c. gives a Hamiltonian circuit in K_n by sorting the vertices in their order of appearance. Since c' consists of the shortest distances, we have $\mathrm{OPT}(G, c) \geq \mathrm{OPT}(K_n, c')$.

Example: Amsterdam metro



c	Amsterdam Zuid	Centraal Station	Gaasperplas	Gein	Isolatorweg	Overamstel	Spaklerweg	van der Madeweg	Westwijk
Amsterdam Zuid									
Centraal Station									
Gaasperplas									
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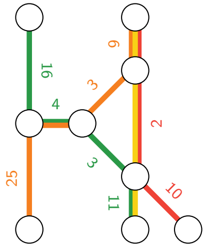
van der Madeweg									
Westwijk									

1. Compute all shortest paths (e.g., using Floyd-Warshall).

Example: Amsterdam metro



c	Amsterdam Zuid	Centraal Station	Gaasperplas	Gein	Isolatorweg	Overamstel	Spaklerweg	van der Madeweg	Westwijk
Amsterdam Zuid	0				16	4			25
Centraal Station		0					9		
Gaasperplas			0					10	
Gein				0				11	
Isolatorweg					0				
Overamstel						0	3	3	
Spaklerweg							0	2	
van der Madeweg								0	
Westwijk									0

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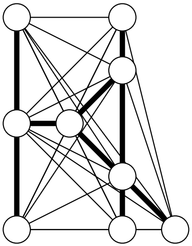
Example: Amsterdam metro



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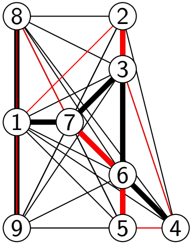
Amsterdam Zuid	0	16	17	18	16	4	7	7	25
Centraal Station		0	21	22	32	12	9	11	41
Gaasperplas			0	21	33	13	12	10	42
Gein				0	34	14	13	11	43
Isolatorweg					0	20	23	23	41
Overamstel						0	3	3	29
Spaklerweg							0	2	32
van der Madeweg								0	32
Westwijk									0

1. Compute all shortest paths (e.g., using Floyd-Warshall).
2. Solve the TSP on the complete graph (20 160 Hamiltonian circuits).

Example: Amsterdam metro



TSP: 158

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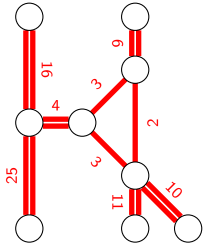
1. Compute all shortest paths (e.g., using Floyd-Warshall).
2. Solve the TSP on the complete graph (20 160 Hamiltonian circuits).
3. Trace back the shortest paths.

Example: Amsterdam metro



TSP: 158

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Westwijk									0

1. Compute all shortest paths (e.g., using Floyd-Warshall).
2. Solve the TSP on the complete graph (20 160 Hamiltonian circuits).
3. Trace back the shortest paths.

Result: The optimal TSP tour is identical to the optimal CPP tour.

Chapter 1

S-Bahn Challenge

§1.4 Generalized Routing Problems

GATSP and GDRPP

Let $G = (V, E)$ be a directed graph with a cost function $c : E \to \mathbb{R}_{\geq 0}$.

Definition

Let $V_1, \ldots, V_k$ be disjoint subsets of V (*clusters*). The **Generalized Asymmetric Traveling Salesman Problem (GATSP)** is to find a directed closed walk $C = (e_1, \ldots, e_k)$ in G such that

- C visits at least one vertex from each cluster at least once,
- C has minimal cost w.r.t. c.

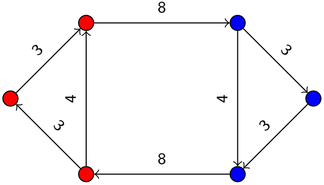
Definition

Let $E_1, \ldots, E_k$ be disjoint subsets of E (*clusters*). The **Generalized Directed Rural Postman Problem (GDRPP)** is to find a directed closed walk $C = (e_1, \ldots, e_k)$ in G such that

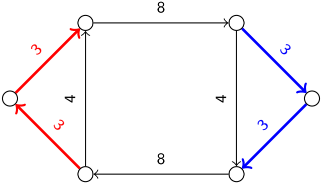
- C visits at least one edge from each cluster at least once,
- C has minimal cost w.r.t. c.

We will model the S-Bahn Challenge problem as GDRPP.

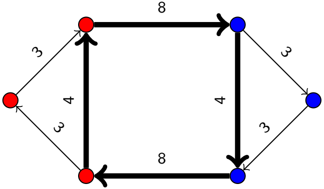
GATSP and GDRPP: Example



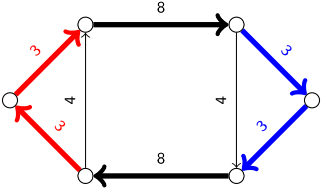
GATSP instance with 2 clusters



GDRPP instance with 2 clusters



an optimal GATSP tour of cost 24



an optimal GDRPP tour of cost 28

GATSP and GDRPP: Equivalence

Theorem (Drexl, 2007)

GATSP and GDRPP are polynomially equivalent.

Proof (GATSP $\leq$ GDRPP).

- Let $I = (G, c, \{V_1, \ldots, V_k\})$ be a GATSP instance. Set

$$E_i := \{(v, w) \mid v \in V_i, w \notin V_i\}, \quad i = 1, \ldots, k,$$

  and define a GDRPP instance $I' := (G, c, \{E_1, \ldots, E_k\})$.
- For each i, any solution to the GDRPP on I' visits at least one edge of E_i, and hence at least one vertex of V_i. We conclude $\text{OPT}(I') \geq \text{OPT}(I)$.
- Conversely, any solution to the GATSP on I visits at least one edge of E_i, because E_i comprises all outgoing edges from V_i. Hence $\text{OPT}(I') \leq \text{OPT}(I)$.

GATSP and GDRPP: Equivalence

Proof (GDRPP $\leq$ GATSP).

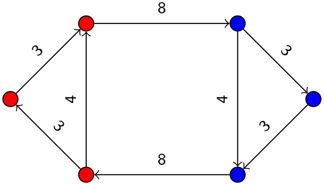
- Let $I' = (G, c, \{E_1, \ldots, E_k\})$ be a GDRPP instance. Split each edge $e = (v, w) \in \bigcup_{i=1}^{k} E_i$ by a new vertex z_e. That is, remove e, and add the edges (v, z_e) and (z_e, w) with cost $c(e)$ and 0, respectively. Set

$$V_i := \{z_e \mid e \in E_i\}, \quad i = 1, \ldots, k.$$

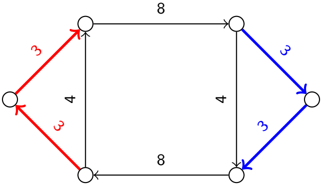
  and define a GATSP instance $I := (G, c, \{V_1, \ldots, V_k\})$.
- Any solution to the GDRPP on I' visits at least one edge $e_i \in E_i$ for all i, and hence gives rise to a GATSP solution visiting at least one vertex z_{e_i} for all i. The cost does not change, thus $\mathrm{OPT}(I') \geq \mathrm{OPT}(I)$.
- Conversely, any solution to the GATSP on I visits at least one vertex $z_{e_i} \in V_i$ for all i, and yields a GDRPP solution visiting at least one edge e_i for all i. Therefore $\mathrm{OPT}(I') \leq \mathrm{OPT}(I)$.

□

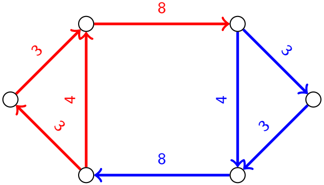
GATSP and GDRPP: Equivalence Example



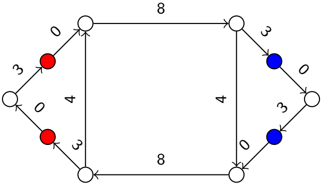
GATSP instance



GDRPP instance



equivalent GDRPP instance



equivalent GATSP instance

GATSP and GDRPP: NP-hardness

Theorem

GATSP and GDRPP are NP-hard.

Proof.

- It suffices to show NP-hardness for GDRPP. We already know that the Rural Postman Problem (RPP) is NP-hard.
- Let $G = (V, E)$ be an undirected graph with a cost function $c : E \to \mathbb{R}_{\geq 0}$, and let $S \subseteq E$ be a subset of edges. The RPP is to find a closed walk $(e_1, \ldots, e_k)$ in G covering S of minimal cost w.r.t. c.
- Let D be the digraph obtained from G where each undirected edge $\{v, w\}$ is replaced by two anti-parallel edges $(v, w), (w, v)$. Extend the cost function c to D by defining $c(v, w) := c(w, v) := c(\{v, w\})$. For each edge $e = \{v, w\} \in S$, add a cluster $E_e = \{(v, w), (w, v)\}$.
- The GDRPP on $(D, c, \{E_e \mid e \in S\})$ is equivalent to the RPP on (G, c, S).

GATSP and ATSP

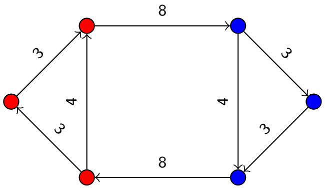
Theorem (Noon/Bean, 1991)

GATSP $\leq$ *ATSP.*

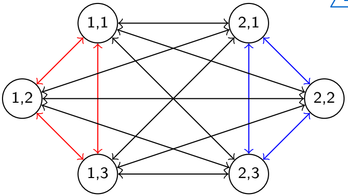
Proof.

- Let $I = (G, c, \{V_1, \ldots, V_k\})$ be an arbitrary GATSP instance, let $n = \sum_{i=1}^{k} |V_i|$. We will define an ATSP instance $I' = (K_n^*, c')$.
- For each $i = 1, \ldots, k$, choose any ordering $(v_{i,1}, v_{i,2}, \ldots, v_{i,r_i})$ of V_i.
- Set $M := 1 +$ sum of lengths of the k longest shortest paths in G and
  $c'(v_{i,1}, v_{i,2}) := c'(v_{i,2}, v_{i,3}) := \cdots := c'(v_{i,r_i}, v_{i,1}) := 0,$
  $c'(v_{i,j}, v_{p,q}) := M +$ shortest path length from $v_{i,(j+1) \bmod r_i}$ to $v_{p,q}$ in G
  for all i resp. all $(i,j), (p,q)$ with $i \neq p$.
- All other edges receive cost M.

GATSP and ATSP: Example



GATSP instance

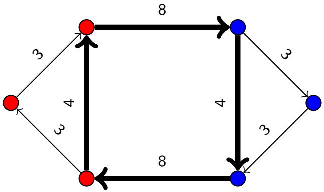


equivalent ATSP instance

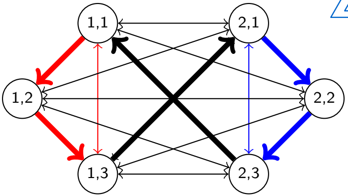
c'	1,1	1,2	1,3	2,1	2,2	2,3
1,1		0	M	$M+11$	$M+14$	$M+15$
1,2	M		0	$M+12$	$M+15$	$M+16$
1,3	0	M		$M+8$	$M+11$	$M+12$
2,1	$M+15$	$M+14$	$M+11$		0	M
2,2	$M+12$	$M+11$	$M+8$	M		0
2,3	$M+16$	$M+15$	$M+12$	0	M	

$$M = 33$$

GATSP and ATSP: Example



GATSP instance
optimal tour length: 24



equivalent ATSP instance
optimal tour length: $2M + 24$

c'	1,1	1,2	1,3	2,1	2,2	2,3
1,1		0	M	$M+11$	$M+14$	$M+15$
1,2	M		0	$M+12$	$M+15$	$M+16$
1,3	0	M		$M+8$	$M+11$	$M+12$
2,1	$M+15$	$M+14$	$M+11$		0	M
2,2	$M+12$	$M+11$	$M+8$	M		0
2,3	$M+16$	$M+15$	$M+12$	0	M	

$$M = 33$$

GATSP and ATSP

Proof (cont.)

- Let C' be a Hamiltonian circuit in $I' = (K_n^*, c')$ that visits all vertices of a cluster V_i in ascending order before moving to another cluster. This way, C contains precisely k edges of weight $\geq M$, and we have $c(C') < kM + M$.
- We claim that the optimal solution to I' is such a circuit. Otherwise, $\text{OPT}(I') \geq (k+1)M = kM + M > c(C')$.
- In particular, if the optimal solution enters V_i at $v_{i,j}$, it leaves V_i at $v_{i,(j-1) \bmod r_i}$. The cost of the edge to $v_{i,j}$ is $(M +)$ the shortest path length to $v_{i,j}$, and the cost of the edge from $v_{i,(j-1) \bmod r_i}$ is $(M +)$ the shortest path length from $v_{i,j}$ (note the shift!).
- Hence we find a GATSP tour by tracing the shortest paths back. For the cost we find $\text{OPT}(I) \leq \text{OPT}(I') - kM$.

GATSP and ATSP

Proof (cont.)

- Consider an optimal GATSP tour C. Create a Hamiltonian circuit C' in K_n^* by sorting the clusters by their order of appearance in C and traversing the whole cluster before proceeding. The cost of C' increases at most by kM. Thus

$$\mathrm{OPT}(I') \leq c(C') \leq c(C) + kM = \mathrm{OPT}(I) + kM.$$

Corollary

GDRPP $\leq$ *GATSP* $\leq$ *ATSP* $\leq$ *TSP.*

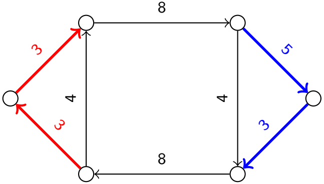
Corollary

A GDRPP with clusters $E_1, \ldots, E_k$ *can be polynomially transformed into a TSP on* $2\sum_{i=1}^{k} |E_i|$ *vertices (with a large increase in cost).*

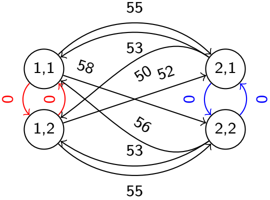
Lemma (Exercise)

Metric TSP $\leq$ *Metric ATSP* $\leq$ *GATSP.*

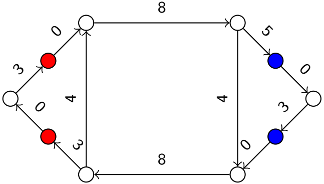
GDRPP to TSP: Example



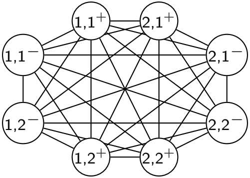
GDRPP instance
optimal tour: 30



ATSP instance ($M = 39$)
optimal tour: $108 = 2 \cdot 39 + 30$



GATSP instance
optimal tour: 30



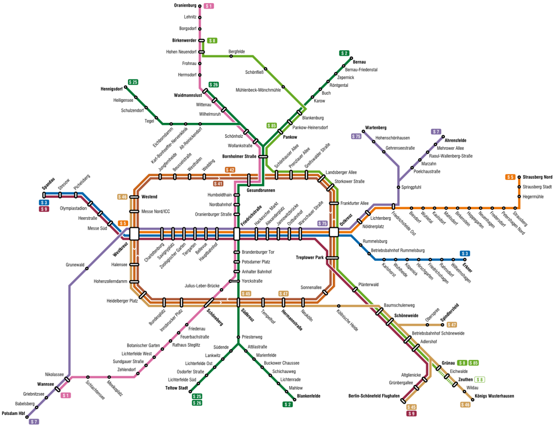
TSP instance ($M = 225$)
optimal tour: $1008 = 4 \cdot 225 + 108$

Chapter 1

S-Bahn Challenge

§1.5 Public Transportation Networks

Line Networks



Definition

A **line network** is a graph G together with a **line cover** $\mathcal{L}$, i.e., $\mathcal{L}$ is a set of walks in G such that $E(G) = \bigcup_{L \in \mathcal{L}} E(L)$.

Line Networks and Event-Activity Networks

Remarks

- Depending on the application, line networks may be undirected or directed.
- The vertices of a line network are *stations* or *stops*.
- The elements of $\mathcal{L}$ are *lines* or *routes*.
- The two directions of a classical path-shaped line can be modeled by two separated walks or by a closed walk.

Definition

An **event-activity network (EAN)** is a directed graph $\mathcal{E}$ whose vertices are called *events* and whose edges are called *activities*.

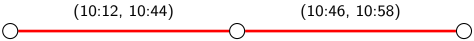
Timetables for Line Networks

Definition

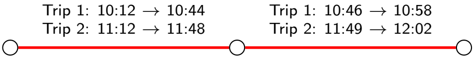
Let $\mathcal{N} = (G, \mathcal{L})$ be a line network.

- A **trip** of a line $L = (e_1, \dots, e_k) \in \mathcal{L}$ is a pair $(\tau_{\text{dep}}, \tau_{\text{arr}})$ of maps $\tau_{\text{dep}}, \tau_{\text{arr}} : \{1, \dots, k\} \to \mathbb{R}$ such that

$$\begin{aligned} \tau_{\text{dep}}(i) &\leq \tau_{\text{arr}}(i), && i = 1, \dots, k \\ \tau_{\text{arr}}(i) &\leq \tau_{\text{dep}}(i+1), && i = 1, \dots, k-1. \end{aligned}$$



- A **schedule** for L is a collection of trips of L.



- A **timetable** for $\mathcal{N}$ assigns a schedule to each line.

Time Expansion

Definition

Consider a timetable $\mathcal{T}$ for a line network $\mathcal{N}$. The **time expansion** of $\mathcal{N}$ w.r.t. $\mathcal{T}$ is the event-activity network $\mathcal{E}$, together with the length function $\ell : E(\mathcal{E}) \to \mathbb{R}_{\geq 0}$, constructed as follows:

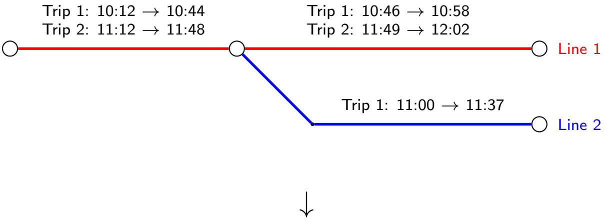
1. For each trip $\tau = (\tau_{\text{dep}}, \tau_{\text{arr}})$ of a line $L = (e_1, \ldots, e_k)$ in $\mathcal{N}$:
   - Add *departure events* (L, τ, i, dep) for $i = 1, \ldots, k$.
   - Add *arrival events* (L, τ, i, arr) for $i = 1, \ldots, k$.
   - Add *driving activities* $(L, \tau, i, \text{dep}) \to (L, \tau, i, \text{arr})$ with length $\tau_{\text{arr}}(i) - \tau_{\text{dep}}(i)$, $i = 1, \ldots, k$.
   - Add *waiting activities* $(L, \tau, i, \text{arr}) \to (L, \tau, i+1, \text{dep})$ with length $\tau_{\text{dep}}(i+1) - \tau_{\text{arr}}(i)$, $i = 1, \ldots, k-1$.
2. Add a *transfer activity* $(L, \tau, i, \text{arr}) \to (L', \tau', i', \text{dep})$ with length $\tau'_{\text{dep}}(i') - \tau_{\text{arr}}(i)$ for each pair of trips (τ, τ') associated to a pair of lines (L, L') whenever:
   - $\tau'_{\text{dep}}(i') - \tau_{\text{arr}}(i) \geq 0$, and
   - the $(i+1)$-st vertex of L and the i'-th vertex of L' coincide in $\mathcal{N}$,
   - (L, τ, i, arr) and $(L', \tau', i', \text{dep})$ are not connected by a waiting activity.

Time Expansion

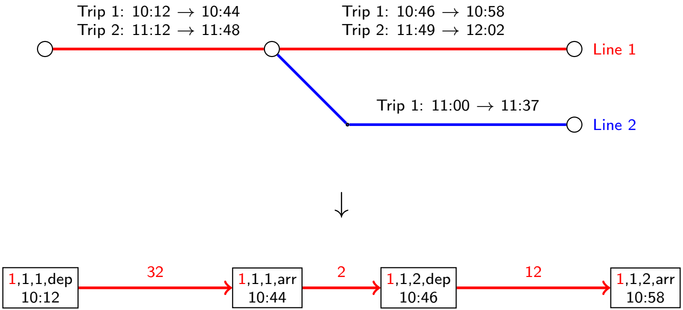
Remarks

- Trips correspond to certain disjoint directed paths in the EAN.
- The EAN is bipartite, as there are no departure-departure and no arrival-arrival activities.
- No activity goes "backward in time": Circuits can only have length 0.
- The number of driving and waiting activities is linear in the number of trips, whereas the number of transfer activities is quadratic.
- A transfer activity between two trips of a line at one of its endpoints is called a *turnaround activity*.
- Often there is no point in a transfer between trips of parallel lines, and the corresponding transfer activities can be removed.
- Sometimes we want to establish a minimum transfer time, and hence only add transfer activities where $\tau'_{\mathrm{dep}}(i') - \tau_{\mathrm{arr}}(i)$ is large enough.
- Footpath information can also be included using transfer activities.

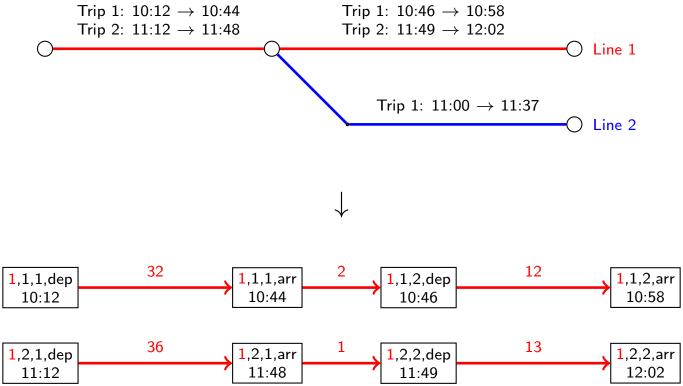
Time Expansion: Example



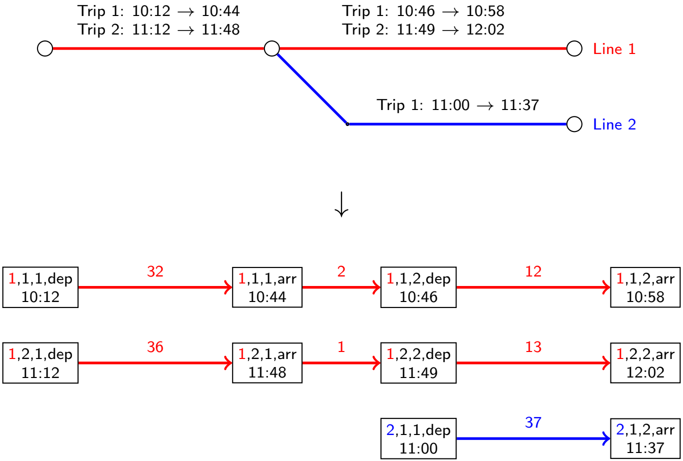
Time Expansion: Example



Time Expansion: Example



Time Expansion: Example



Time Expansion: Example