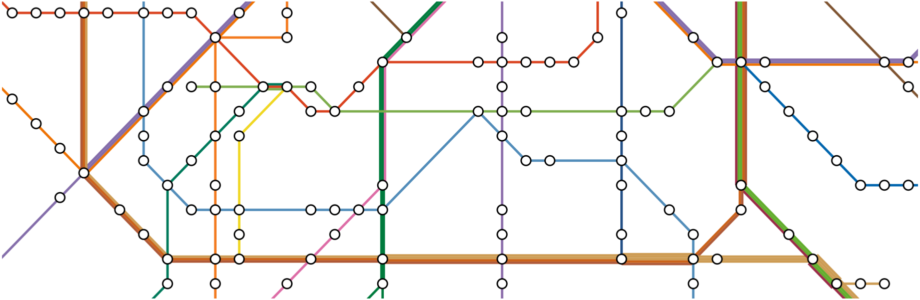
Mathematical Aspects of Public Transportation Networks

Niels Lindner



May 28, 2018

Chapter 3

Periodic Timetabling

§3.1 Overview

Periodic Event Scheduling Problem (PESP)

Input

- event-activity network $\mathcal{E} = (V, E)$,
- period time $T \in \mathbb{N}$,
- lower bound vector $\ell \in (\mathbb{R}_{\geq 0})^E$, $0 \leq \ell < T$,
- upper bound vector $u \in\in (\mathbb{R}_{\geq 0})^E$, $\ell \leq u < T - \ell$,
- weight vector $w \in (\mathbb{R}_{\geq 0})^E$

PESP

Find a *periodic timetable* $\pi \in [0, T)^V$ and *periodic tensions* $x \in \mathbb{R}^E$, $\ell \leq x \leq u$, such that

$$x_{ij} = [\pi_j - \pi_i - \ell_{ij}]_T + \ell_{ij} \quad \text{for all } ij \in E$$

and $\sum_{ij \in E} w_{ij} x_{ij}$ is minimal.

Complexity of PESP

Theorem

For fixed $T \geq 3$, the PESP feasibility problem is NP-complete.

Remark

This means that

- T is not regarded as part of the input data,
- finding a single feasible solution (π, x) is NP-hard.

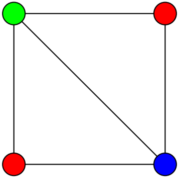
Strategy of the proof (Odijk, 1994)

We will reduce the *Vertex Coloring* problem to PESP feasibility.

Vertex Coloring

Definition

Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, the k-**Vertex Coloring** problem is to decide whether there is a map $f : V \to \{1, \dots, k\}$ such that for all edges $vw \in E$ holds $f(v) \neq f(w)$.



a 3-colorable graph
(not 2-colorable)

Complexity of k-Vertex Coloring

Theorem (Karp, 1972)

k-Vertex Coloring is NP-complete.

SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE
INPUT: Clauses $D_1, D_2, \ldots, D_r$, each consisting of at most 3 literals from the set $\{u_1, u_2, \ldots, u_m\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\}$
PROPERTY: The set $\{D_1, D_2, \ldots, D_r\}$ is satisfiable.

CHROMATIC NUMBER
INPUT: graph G, positive integer k
PROPERTY: There is a function $\phi: N \to Z_k$ such that, if u and v are adjacent, then $\phi(u) \neq \phi(v)$.

SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE
$\propto$ CHROMATIC NUMBER

Assume without loss of generality that $m \geq 4$.

$$N = \{u_1, u_2, \ldots, u_m\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\} \cup \{v_1, v_2, \ldots, v_m\} \cup \{D_1, D_2, \ldots, D_r\}$$

$$A = \{\{u_i, \bar{u}_i\} \mid i=1,2,\ldots,n\} \cup \{\{v_i, v_j\} \mid i \neq j\} \cup \{\{v_i, x_j\} \mid i \neq j\} \cup \{\{v_i, \bar{x}_j\} \mid i \neq j\} \cup \{\{u_i, D_f\} \mid u_i \notin D_f\} \cup \{\{\bar{u}_i, D_f\} \mid \bar{u}_i \in D_f\}$$

$$k = r + 1$$

3-SAT

Definition

Let $u_1, \ldots, u_m$ be *variables*.

- A *literal* is a symbol of the form u_i or $\overline{u_i}$ ("not u_i").
- A *clause* is a disjunction $D_j = \ell_{j_1} \vee \cdots \vee \ell_{j_k}$ of literals.
- A *formula in conjuctive normal form (CNF)* is a conjunction $F = D_1 \wedge \cdots \wedge D_r$ of clauses.
- A formula is in *3-CNF* if every clause contains at most three literals.

Given a formula F in 3-CNF, the **3-SAT** problem is to decide whether there is a map $a : i \to \{true, false\}$ (*truth assignment*) such that F evaluates to *true* when each variable u_i is set ot the truth value $a(i)$.

Theorem (Karp, 1972)

3-SAT is NP-complete.

Proof.

Transformation from SAT - the first known NP-complete problem. □

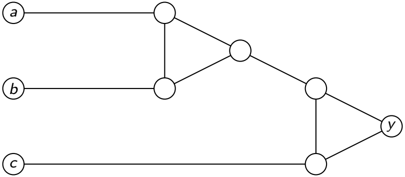
3-SAT $\leq$ 3-Vertex Coloring

Theorem (Garey/Johnson/Stockmeyer, 1976)

3-*Vertex Coloring is NP-complete.*

Proof.

Consider the following *clause gadget*:



- If at least one of a,b,c has color 1, then this extends to a coloring of the gadget where y is colored with 1.
- If a,b,c all have the same color i, then y must be colored with i.

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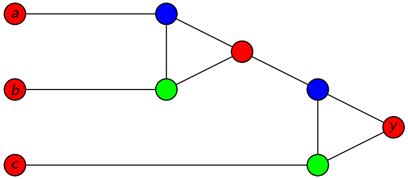

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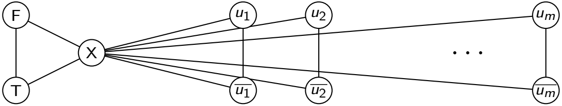
3-SAT $\leq$ 3-Vertex Coloring

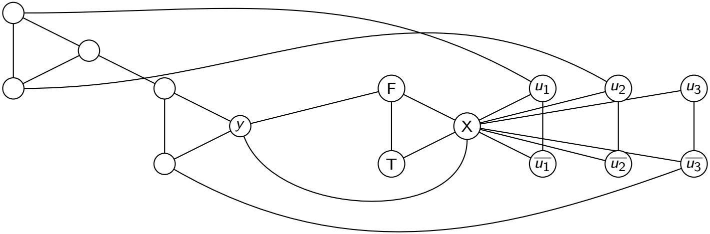
Given a formula F in 3-CNF, we construct a graph G as follows:

- Start with a *truth gadget* and a *variable gadget*:



- For each clause $a_i \vee b_i \vee c_i$ in F, insert the clause gadget, by replacing a, b, c with the corresponding vertex u_i or $\overline{u_i}$ of the variable gadget.
- Add edges $\{F, y_i\}$ and $\{X, y_i\}$ for each clause.

3-SAT $\leq$ 3-Vertex Coloring



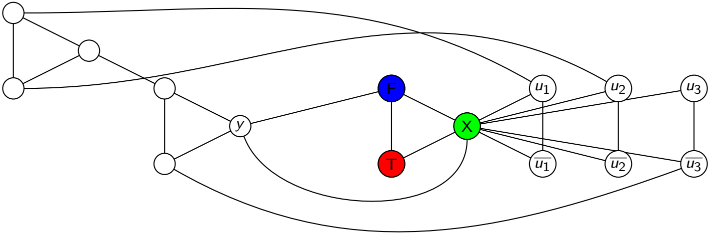
Graph for $F = u_1 \vee u_2 \vee \overline{u_3}$

Proof ($\Rightarrow$).

Suppose that F is satisfiable by some truth assignment.

- Color T with 1, F with 2 and X with 3.
- If u_i is *true*, then color the vertex u_i with 1 and $\overline{u_i}$ with 2. Otherwise, color u_i with 2 and $\overline{u_i}$ with 1.
- Since F is satisfied, for each clause, at least one of the literals a_i, b_i, c_i has color 1, so this extends to a coloring where all y_i have color 1.
- This coloring is compatible with the truth and variable gadget.

3-SAT $\leq$ 3-Vertex Coloring



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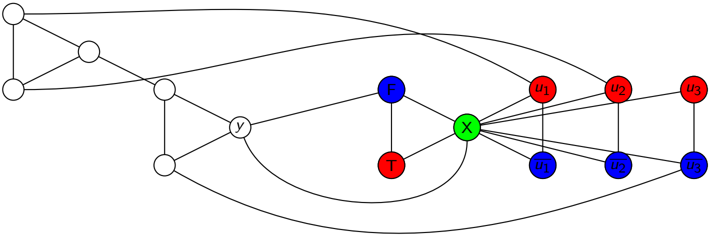

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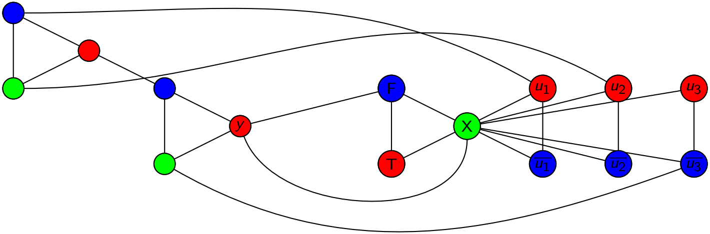

Graph for $F = u_1 \vee u_2 \vee \overline{u_3}$

Proof ($\Rightarrow$).

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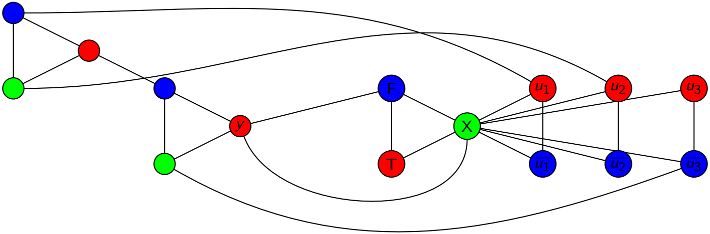
- Color T with 1, F with 2 and X with 3.
- If u_i is *true*, then color the vertex u_i with 1 and $\overline{u_i}$ with 2. Otherwise, color u_i with 2 and $\overline{u_i}$ with 1.
- Since F is satisfied, for each clause, at least one of the literals a_i, b_i, c_i has color 1, so this extends to a coloring where all y_i have color 1.
- This coloring is compatible with the truth and variable gadget.

3-SAT $\leq$ 3-Vertex Coloring



Graph for $F = u_1 \vee u_2 \vee \overline{u_3}$

Proof ($\Leftarrow$).

Conversely, suppose that G has a 3-coloring.

- W.l.o.g. T has color 1, F has color 2 and X has color 3.
- This yields a truth assignment on the variables (1: true, 2: false).
- Moreover, y_i is colored with 1 for all clauses.
- For a clause, not all of a_i, b_i, c_i can have color 2, because this would imply that y_i has color 2.
- In particular, F is satisfiable.

k-Vertex Coloring

Corollary

For fixed $k \geq 3$, k-Vertex Coloring is NP-complete.

Proof.

Probably an exercise. □

T-Vertex Coloring $\leq$ T-PESP

Theorem (Odijk, 1994)

Fix an integer T. Then T-Vertex Coloring can be reduced to PESP feasibility with period time T.

Proof.

Let $G = (V, E)$ be an arbitrary graph (w.l.o.g. directed). Define a PESP instance on G as follows (weights are unimportant for feasibility):

$$\ell_e := 1, \quad u_e := T - 1, \quad e \in E.$$

Suppose that G has a T-coloring $f : V \to \{1, 2, \ldots, T\}$. Define $\pi_v := f(v) - 1$ for all $v \in V$. Then π takes values in $\{0, 1, \ldots, T - 1\}$. Set

$$x_{ij} := \begin{cases} \pi_j - \pi_i & \text{if } \pi_j \geq \pi_i, \\ \pi_j - \pi_i + T & \text{otherwise,} \end{cases} \quad ij \in E.$$

Clearly $x_{ij} \geq 0$. Since f is a coloring, also $x_{ij} \geq 1 = \ell_{ij}$. Moreover $x_{ij} \leq T - 1 = u_{ij}$. Hence (π, x) is feasible for PESP.

T-Vertex Coloring $\leq$ T-PESP

Proof.

Conversely, let (π, x) be feasible for PESP on the graph G. As lower und upper bounds are integer, we can assume that this holds for π and x as well (total unimodularity). Then

$$f(v) := \pi_v + 1 \quad \in \{1, 2, \ldots, T\}, \quad v \in V,$$

is a T-vertex coloring for G. □

Chapter 3

Periodic Timetabling

§3.2 Cycle Spaces

Motivation: PESP MIP formulation

So far, we considered the following MIP formulation of PESP:

$$
\begin{array}{llll}
\text{Minimize} & \displaystyle\sum_{ij \in E} w_{ij} x_{ij} & & \\
\text{s.t.} & x_{ij} = \pi_j - \pi_i + p_{ij} T, & ij \in E, & \\
& \ell_{ij} \leq x_{ij} \leq u_{ij}, & ij \in E, & \text{(periodic tension)} \\
& 0 \leq \pi_i \leq T - 1, & i \in V, & \text{(periodic timetable)} \\
& p_{ij} \in \mathbb{Z}, & ij \in E. & \text{(periodic offset)}
\end{array}
$$

If the event-activity network has n events and m activities, then this formulation uses m constraints, $m + n$ continuous variables, and m integer variables.

We will now construct a formulation with $m - n + 1$ constraints, m continuous variables, and $m - n + 1$ integer variables. This variant behaves much better in practice.

Cycles

Let $G = (V, E)$ be an undirected graph.

Definition

A **cycle** in G is an Eulerian subgraph of G.

Remarks

- In other words, a cycle is a subgraph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ such that $\deg_{G'}(v)$ is even for all $v \in V'$.
- Any cycle decomposes as an edge-disjoint union of circuits.
- We will sometimes identify a cycle with its sequence of edges or vertices.

Symmetric difference of cycles

Lemma

Let C_1, C_2 be two cycles in G. Then the symmetric difference

$$C_1 \Delta C_2 := (C_1 \cup C_2) \setminus (C_1 \cap C_2)$$

is a cycle in G.

Proof.

Let $v \in V(C_1 \Delta C_2)$. Then

$$\deg_{C_1 \Delta C_2}(v) = \underbrace{\deg_{C_1}(v)}_{\text{even}} + \underbrace{\deg C_2(v)}_{\text{even}} - 2 \deg_{C_1 \cap C_2}(v)$$

is even. □

Incidence vectors and cycle space

Let G be an undirected graph.

Definition

For a cycle C define its **incidence vector** $\gamma_C \in \{0,1\}^E$ as

$$\gamma_e := \begin{cases} 1 & \text{if } e \in E(C), \\ 0 & \text{if } e \notin E(C), \end{cases} \quad e \in E(G).$$

The **cycle space** of G is the set

$$\mathcal{C}(G) := \{\gamma_C \mid C \text{ is a cycle in } G\} \subseteq \{0,1\}^E.$$

Lemma

$\mathcal{C}(G)$ is an $\mathbb{F}_2$-vector space.

Proof.

Addition $\leftrightarrow$ symmetric difference. □

Cyclomatic number

Let G be an undirected graph.

Definition

The **cyclomatic number** of G is defined as

$$\mu(G) := \dim_{\mathbb{F}_2} \mathcal{C}(G).$$

In other words, the cyclomatic number is the length of any **cycle basis**.

Lemma

Suppose that G has n vertices, m edges, and c connected components. Then $\mu(G) = m - n + c$.

Proof.

Suppose first that G is connected. Let T be a *spanning tree* of G, i.e., a maximal cycle-free subgraph containing all n vertices.

Cyclomatic number

Proof (cont.)

We call an edge $e \in E(G)$ a *co-tree edge* if $e \notin E(T)$. Since T contains $n-1$ edges, there are precisely $m-n+1$ co-tree edges.

Since T is a spanning tree, adding a single co-tree edge e to T produces a cycle containing e. This way, we obtain $m-n+1$ cycles in G, one for each co-tree edge.

The incidence vectors of these cycles are $\mathbb{F}_2$-linearly independent, since each co-tree edge is contained in precisely one cycle. In formulae, if $\gamma_e \in \{0,1\}^E$ denotes the incidence vector of the cycle produced by the co-tree edge e, then

$$\gamma_{e,e'} = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e' \text{ for all co-tree edges } e' \notin E(T). \end{cases}$$

This shows $\dim_{\mathbb{F}_2} \mathcal{C}(G) \geq m-n+1$ for connected G.

Cycolmatic number

Proof (cont.)

Let ζ be the incidence vector of an arbitrary cycle of G. Let

$$\zeta' := \zeta - \sum_{e \notin E(T)} \zeta_e \gamma_e \quad \in \mathcal{C}(G).$$

Then for any co-tree edge $e' \notin E(T)$, the corresponding entry of ζ' is

$$\zeta'_{e'} = \zeta_{e'} - \sum_{e \notin E(T)} \zeta_e \gamma_{e,e'} = \zeta_{e'} - \zeta_{e'} = 0,$$

so that ζ' corresponds to a cycle inside the tree T. Since trees cannot have cycles, $\zeta' = 0$ and ζ is therefore in the $\mathbb{F}_2$-span of $\{\gamma_e \mid e \notin E(T)\}$.

This finishes the proof for $c = 1$. If G has several connected components, then add the cyclomatic numbers of all components. □

Fundamental cycles

Definition

Let T be a spanning tree of an undirected graph G.

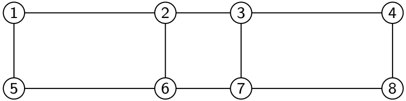
- A cycle created by adding a co-tree edge to T is called **fundamental cycle**.
- A **fundamental cycle basis** is a cycle basis consisting of fundamental cycles.

Remark

The following is an algorithm to construct a fundamental cycle basis: Compute first a spanning tree (Prim, Kruskal, . . .) and then take the fundamental cycle for each co-tree edge.

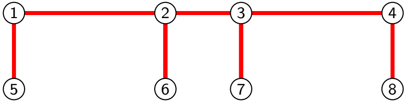
Example: Undirected cycle basis

Consider the following graph G:



G has 8 vertices, 10 edges, 1 connected component and hence $\mu(G) = 10 - 8 + 1 = 3$.

Fundamental cycle basis:



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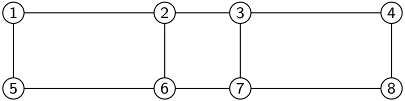

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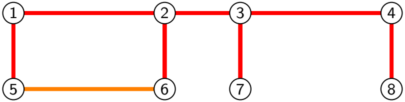

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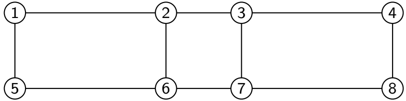

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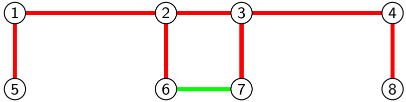

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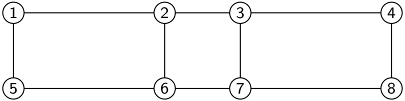

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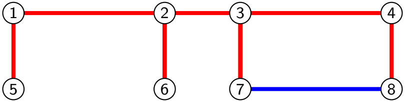

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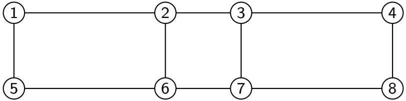

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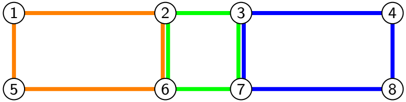
Fundamental cycle basis:



Directed cycles

Let G be a *directed* graph.

Definition

A **directed cycle** in G is an Eulerian subgraph of G.
An **oriented cycle** in G is a cycle of the undirected graph $|G|$ underlying G.

Remarks

- Any directed cycle is an oriented cycle.
- An oriented cycle uses edges either in *forward* or in *backward* direction.
- Any directed/oriented cycle decomposes as an edge-disjoint union of directed/oriented circuits.

Incidence vectors and cycle space

Definition

Let C be an oriented cycle in G. Then its **incidence vector** $\gamma_C \in \{-1, 0, 1\}^E$ is defined as

$$\gamma_e := \begin{cases} 1 & \text{if } C \text{ uses } e \text{ as forward edge,} \\ -1 & \text{if } C \text{ uses } e \text{ as backward edge,} \\ 0 & \text{otherwise} \end{cases} \quad e \in E(G).$$

The $\mathbb{Q}$-**cycle space** of G is the $\mathbb{Q}$-vector space

$$\mathcal{C}_{\mathbb{Q}}(G) := \operatorname{span}_{\mathbb{Q}} \{\gamma_C \mid C \text{ oriented cycle of } G\}.$$

A basis consisting of incidence vectors of true oriented cycles is called a **cycle basis** of G.

The **cyclomatic number** of G is defined as $\mu(G) := \dim_{\mathbb{Q}} \mathcal{C}_{\mathbb{Q}}(G)$.

Undirected cycle bases

Lemma

Let $\mathcal{B}$ be a cycle basis for $|G|$. Then lifting each cycle in $\mathcal{B}$ to an oriented cycle in G yields a $\mathbb{Q}$-basis of $\mathcal{C}_{\mathbb{Q}}(G)$.

Proof.

Let $\mathcal{B} = \{\gamma_1, \dots, \gamma_\mu\}$ and let γ_i' be the incidence vector of an oriented cycle in G projecting to γ_i in $|G|$, $i = 1, \dots, \mu := \mu(|G|)$.

Linear independence: Suppose $\sum_{i=1}^{\mu} \lambda_i \gamma_i' = 0$ for some $\lambda_1, \dots, \lambda_\mu \in \mathbb{Q}$. Clearing denominators, we can assume that $\lambda_1, \dots, \lambda_\mu \in \mathbb{Z}$ and $\gcd(\lambda_1, \dots, \lambda_\mu) = 1$. Reducing modulo 2, we have $\sum_{i=1}^{\mu} [\lambda_i]_2 \gamma_i \equiv_2 0$, which implies that $\lambda_i \equiv_2 0$ for all i, as $\mathcal{B}$ is an $\mathbb{F}_2$-basis. Since all λ_i were coprime, this means that $\lambda_1 = \dots = \lambda_\mu = 0$.

Undirected cycle bases

Proof (cont.)

It remains to show that $\dim_{\mathbb{Q}} \mathcal{C}_{\mathbb{Q}}(G) = \mu(|G|)$. Consider a spanning tree T of $|G|$ with its fundamental cycle basis $\mathcal{B} = \{\gamma_e \mid e \notin E(T)\}$.
Let $\zeta \in \{-1, 0, 1\}^{E(G)}$ be the incidence vector of an arbitary oriented cycle in G and suppose that ζ does not lie in the span of the lifts $\{\gamma'_e \mid e \notin E(T)\}$ of the vectors in $\mathcal{B}$ to G. Then also

$$\zeta' := \zeta - \sum_{e \notin E(T)} \zeta_e \cdot \gamma'_{e,e} \quad \notin \operatorname{span}\{\gamma'_e \mid e \notin E(T)\}$$

Then $\zeta'_e = 0$ for any edge in $E(G)$ corresponding to a co-tree edge of T, so that ζ has support only in the tree edges. But T is a tree and hence cannot contain a cycle, so $\zeta' = 0$ (contradiction). □

Remark

In particular, fundamental cycle bases work as in the undirected case.

Cycle basis names

Let G be a directed graph.

Corollary

If G has n vertices, m edges and c weakly connected components, then $\mu(G) = \mu(|G|) = m - n + c$.

Definition

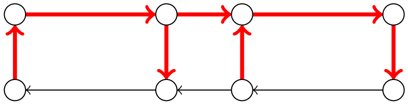
- A cycle basis in G coming from a cycle basis in $|G|$ is called an **undirected cycle basis**.
- A cycle basis in G coming from a spanning tree is called a **strictly fundamental basis**.

Definition

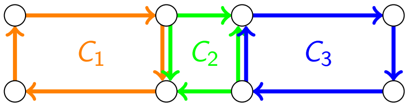
Let $\mathcal{B} = (\gamma_1, \ldots, \gamma_{\mu(G)})$ be a cycle basis. The $(\mu(G) \times m)$-matrix Γ whose rows are given by γ_i, $i = 1, \ldots, \mu(G)$, is called the **cycle matrix** of $\mathcal{B}$.

Cycle basis example

Consider the following digraph G with red spanning tree T:



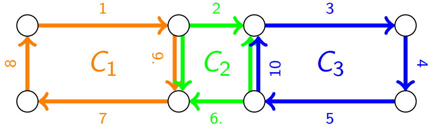
We produce a strictly fundamental cycle basis by taking the oriented cycle for each co-tree edge of T:



The cycles C_1 and C_3 use only forward edges, whereas C_2 uses two backward edges.

Cycle basis example

Label the edges by $1, \ldots, 10$:



Collecting the incidence vectors of C_1, C_2, C_3 yields the 3×10-cycle matrix:

	1	2	3	4	5	6	7	8	9	10
γ_1	1	0	0	0	0	0	1	1	1	0
γ_2	0	1	0	0	0	1	0	0	-1	-1
γ_3	0	0	1	1	1	0	0	0	0	1

Note that the matrix has full row rank. The part corresponding to the co-tree edges $5, 6, 7$ of T is a permutation of the identity matrix.

Determinant of a cycle basis

Let G be a directed graph and let $\mathcal{B}$ be a cycle basis with cycle matrix Γ.

Definition

The **determinant** of $\mathcal{B}$ is defined as

$$\det(\mathcal{B}) := \left| \begin{array}{c} (\mu(G) \times \mu(G))\text{-submatrix of } \Gamma \text{ corresponding to the} \\ \text{co-tree edges of some spanning tree of } G \end{array} \right|.$$

This is well-defined:

Theorem (Liebchen, 2003)

Let T_1, T_2 be two spanning trees of G. For $i = 1, 2$, denote by A_i the $(\mu(G) \times \mu(G))$-submatrix of Γ, where exactly the columns corresponding to $e \notin E(T_i)$ are selected. Then A_1 and A_2 are both invertible and $\det(A_1) = \pm \det(A_2)$.

Determinant of a cycle basis

Proof.

Let Φ be the cycle matrix of a strictly fundamental cycle basis of G coming from the spanning tree T_1. The rows of Φ are indexed by the $\mu := \mu(G)$ co-tree edges of T_1. We have

$$\Phi_{e,e'} = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e', \end{cases} \quad \text{for all } e, e' \notin E(T).$$

Note that we can always lift a fundamental cycle in such a way that the co-tree edge becomes a forward edge. In particular, if Φ_1 denotes the restriction of Φ to the columns corresponding to co-tree edges of T_1, then Φ_1 is the identity matrix.

Since Φ and $\mathcal{B}$ are bases, there is an invertible $(\mu \times \mu)$-matrix S such that $\Gamma = S \cdot \Gamma_\Phi$. It follows that $A_1 = S \cdot \Phi_1$ is invertible. This holds analogously for A_2.

Determinant of a cycle basis

Proof.

Let Φ_2 denote the restriction of Φ to the columns corresponding to the co-tree edges of T_2. Then $A_2 = S \cdot \Phi_2$, so it remains to show that $\det(\Phi_2) = \pm \det(\Phi_1) = \pm 1$. We use induction on $\# E(T_1) \Delta E(T_2)$.

$\# E(T_1) \Delta E(T_2) = 0$: This is equivalent to $E(T_1) = E(T_2)$, where obvioulsy $\det(\Phi_2) = \det(\Phi_1)$.

$\# E(T_1) \Delta E(T_2) > 0$: Let $e_1 \in E(T_1) \setminus E(T_2)$. On the unique path in T_2 connecting the endpoints of e_1, there must be an edge $e_2 \notin E(T_1)$, as otherwise T_1 would contain a cycle. The fundamental cycle of e_1 in T_1 uses e_2, so that $\Phi_{e_1,e_2} = \pm 1$. Since there is only one fundamental cycle for T_1 using the co-tree edge e_2, this means that $\Phi_{e,e_2} = 0$ for $e \neq e_1$. Use Laplace expansion along the column e_2. □

Characterization by determinant

Let G be a digraph with cyclomatic number μ and cycle basis $\mathcal{B}$.

Theorem (Liebchen/Rizzi, 2007)

(1) *$\mathcal{B}$ is undirected if and only if $\det(\mathcal{B})$ is odd.*

(2) *$\mathcal{B}$ is strictly fundamental if and only if the cycle matrix of $\mathcal{B}$ can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last μ columns.*

Proof.

(2) Exercise. (1) Let Γ be the cycle matrix of $\mathcal{B}$. Write $\Gamma = S \cdot \Phi$, where S is an invertible $\mu \times \mu$-matrix and Φ is the matrix of a strictly fundamental basis for some spanning tree T. Restricting to the co-tree edges, we obtain $\Gamma|_{\overline{E(T)}} = S \cdot \Phi|_{\overline{E(T)}} = S$, so $\det(\mathcal{B}) = \det(S)$. If $\det(\mathcal{B})$ is odd, then S is invertible over $\mathbb{F}_2$, so the rows of Γ mod 2 form a cycle basis for $|G|$. Conversely, if $\mathcal{B}$ is undirected, then $\Gamma|_{\overline{E(T)}}$ is invertible mod 2, so that also S is invertible mod 2 and hence $\det(\mathcal{B})$ is odd. □