Chapter 3

Periodic Timetabling

§3.1 Overview
3.1 Overview

Periodic Event Scheduling Problem (PESP)

Input

- event-activity network $\mathcal{E} = (V, E)$,
- period time $T \in \mathbb{N}$,
- lower bound vector $\ell \in (\mathbb{R}_{\geq 0})^E$, $0 \leq \ell < T$,
- upper bound vector $u \in (\mathbb{R}_{\geq 0})^E$, $\ell \leq u < T - \ell$,
- weight vector $w \in (\mathbb{R}_{\geq 0})^E$

PESP

Find a periodic timetable $\pi \in [0, T)^V$ and periodic tensions $x \in \mathbb{R}^E$, $\ell \leq x \leq u$, such that

$$x_{ij} = [\pi_j - \pi_i - \ell_{ij}]T + \ell_{ij} \quad \text{for all } ij \in E$$

and $\sum_{ij \in E} w_{ij} x_{ij}$ is minimal.
Complexity of PESP

Theorem
For fixed $T \geq 3$, the PESP feasibility problem is NP-complete.

Remark
This means that

- $T$ is not regarded as part of the input data,
- finding a single feasible solution $(\pi, x)$ is NP-hard.

Strategy of the proof (Odijk, 1994)
We will reduce the Vertex Coloring problem to PESP feasibility.
3.1 Overview

Vertex Coloring

Definition
Given a graph $G = (V, E)$ and $k \in \mathbb{N}$, the $k$-Vertex Coloring problem is to decide whether there is a map $f : V \rightarrow \{1, \ldots, k\}$ such that for all edges $vw \in E$ holds $f(v) \neq f(w)$.

A 3-colorable graph (not 2-colorable)
3.1 Overview

Complexity of $k$-Vertex Coloring

Theorem (Karp, 1972)

$k$-Vertex Coloring is NP-complete.

SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE
INPUT: Clauses $D_1, D_2, \ldots, D_r$, each consisting of at most 3 literals from the set $\{u_1, u_2, \ldots, u_m\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\}$
PROPERTY: The set $\{D_1, D_2, \ldots, D_r\}$ is satisfiable.

CHROMATIC NUMBER
INPUT: graph $G$, positive integer $k$
PROPERTY: There is a function $\phi: N \to \mathbb{Z}_k$ such that, if $u$ and $v$ are adjacent, then $\phi(u) \neq \phi(v)$.

SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE
$\alpha$ CHROMATIC NUMBER

Assume without loss of generality that $m \geq 4$.

$N = \{u_1, u_2, \ldots, u_m\} \cup \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m\} \cup \{v_1, v_2, \ldots, v_m\}$
$\cup \{D_1, D_2, \ldots, D_r\}$
$A = \{(v_i, \bar{x}_j) | i = 1, 2, \ldots, n\} \cup \{(v_i, v_j) | i \neq j\} \cup \{(v_i, x_j) | i \neq j\}$
$\cup \{(v_i, \bar{x}_j) | i \neq j\} \cup \{(u_1, D_f) | u_1 \notin D_f\} \cup \{\bar{u}_i, D_f | \bar{u}_i \in D_f\}$

$k = r + 1$
3-SAT

Definition

Let $u_1, \ldots, u_m$ be variables.

- A literal is a symbol of the form $u_i$ or $\overline{u_i}$ (“not $u_i$”).
- A clause is a disjunction $D_j = \ell_{j_1} \lor \cdots \lor \ell_{j_k}$ of literals.
- A formula in conjunctive normal form (CNF) is a conjunction $F = D_1 \land \cdots \land D_r$ of clauses.
- A formula is in 3-CNF if every clause contains at most three literals.

Given a formula $F$ in 3-CNF, the 3-SAT problem is to decide whether there is a map $a : i \rightarrow \{true, false\}$ (truth assignment) such that $F$ evaluates to true when each variable $u_i$ is set to the truth value $a(i)$.

Theorem (Karp, 1972)

3-SAT is NP-complete.

Proof.

Transformation from SAT - the first known NP-complete problem.
3-SAT ≤ 3-Vertex Coloring

Theorem (Garey/Johnson/Stockmeyer, 1976)

3-Vertex Coloring is NP-complete.

Proof.
Consider the following clause gadget:

- If at least one of $a, b, c$ has color 1, then this extends to a coloring of the gadget where $y$ is colored with 1.
- If $a, b, c$ all have the same color $i$, then $y$ must be colored with $i$. 
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3-SAT \leq 3-Vertex Coloring

Theorem (Garey/Johnson/Stockmeyer, 1976)

3-Vertex Coloring is NP-complete.

Proof.
Consider the following clause gadget:

- If at least one of \( a, b, c \) has color 1, then this extends to a coloring of the gadget where \( y \) is colored with 1.
- If \( a, b, c \) all have the same color \( i \), then \( y \) must be colored with \( i \).
Given a formula $F$ in 3-CNF, we construct a graph $G$ as follows:

- Start with a truth gadget and a variable gadget:

  ![Diagram](image)

- For each clause $a_i \lor b_i \lor c_i$ in $F$, insert the clause gadget, by replacing $a$, $b$, $c$ with the corresponding vertex $u_i$ or $\overline{u_i}$ of the variable gadget.

- Add edges $\{F, y_i\}$ and $\{X, y_i\}$ for each clause.
Suppose that \( F \) is satisfiable by some truth assignment.

- Color \( T \) with 1, \( F \) with 2 and \( X \) with 3.
- If \( u_i \) is \textit{true}, then color the vertex \( u_i \) with 1 and \( \overline{u_i} \) with 2. Otherwise, color \( u_i \) with 2 and \( \overline{u_i} \) with 1.
- Since \( F \) is satisfied, for each clause, at least one of the literals \( a_i, b_i, c_i \) has color 1, so this extends to a coloring where all \( y_i \) have color 1.
- This coloring is compatible with the truth and variable gadget.
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3-SAT \leq 3-Vertex Coloring

Graph for \( F = u_1 \lor u_2 \lor \overline{u_3} \)

Proof (\( \Rightarrow \)).

Suppose that \( F \) is satisfiable by some truth assignment.

- Color \( T \) with 1, \( F \) with 2 and \( X \) with 3.
- If \( u_i \) is true, then color the vertex \( u_i \) with 1 and \( \overline{u_i} \) with 2. Otherwise, color \( u_i \) with 2 and \( \overline{u_i} \) with 1.
- Since \( F \) is satisfied, for each clause, at least one of the literals \( a_i, b_i, c_i \) has color 1, so this extends to a coloring where all \( y_i \) have color 1.
- This coloring is compatible with the truth and variable gadget.
3-SAT \leq 3-Vertex Coloring

Proof ($\Rightarrow$).

Suppose that $F$ is satisfiable by some truth assignment.

- Color $T$ with 1, $F$ with 2 and $X$ with 3.
- If $u_i$ is true, then color the vertex $u_i$ with 1 and $\overline{u_i}$ with 2. Otherwise, color $u_i$ with 2 and $\overline{u_i}$ with 1.
- Since $F$ is satisfied, for each clause, at least one of the literals $a_i, b_i, c_i$ has color 1, so this extends to a coloring where all $y_i$ have color 1.
- This coloring is compatible with the truth and variable gadget.
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3-SAT $\leq$ 3-Vertex Coloring

Proof ($\Rightarrow$).

Suppose that $F$ is satisfiable by some truth assignment.

- Color $T$ with 1, $F$ with 2 and $X$ with 3.
- If $u_i$ is true, then color the vertex $u_i$ with 1 and $\overline{u_i}$ with 2. Otherwise, color $u_i$ with 2 and $\overline{u_i}$ with 1.
- Since $F$ is satisfied, for each clause, at least one of the literals $a_i, b_i, c_i$ has color 1, so this extends to a coloring where all $y_i$ have color 1.
- This coloring is compatible with the truth and variable gadget.
3-SAT ≤ 3-Vertex Coloring

Proof (⇐).

Conversely, suppose that $G$ has a 3-coloring.

- W.l.o.g. $T$ has color 1, $F$ has color 2 and $X$ has color 3.
- This yields a truth assignment on the variables (1: true, 2: false).
- Moreover, $y_i$ is colored with 1 for all clauses.
- For a clause, not all of $a_i, b_i, c_i$ can have color 2, because this would imply that $y_i$ has color 2.
- In particular, $F$ is satisfiable.
Corollary
For fixed $k \geq 3$, $k$-Vertex Coloring is NP-complete.

Proof.
Probably an exercise.
3.1 Overview

**T-Vertex Coloring ≤ T-PESP**

**Theorem (Odijk, 1994)**

Fix an integer $T$. Then $T$-Vertex Coloring can be reduced to PESP feasibility with period time $T$.

**Proof.**
Let $G = (V, E)$ be an arbitrary graph (w.l.o.g. directed). Define a PESP instance on $G$ as follows (weights are unimportant for feasibility):

$$\ell_e := 1, \quad u_e := T - 1, \quad e \in E.$$  

Suppose that $G$ has a $T$-coloring $f : V \rightarrow \{1, 2, \ldots, T\}$. Define $\pi_v := f(v) - 1$ for all $v \in V$. Then $\pi$ takes values in $\{0, 1, \ldots, T - 1\}$. Set

$$x_{ij} := \begin{cases} 
\pi_j - \pi_i & \text{if } \pi_j \geq \pi_i, \\
\pi_j - \pi_i + T & \text{otherwise,}
\end{cases} \quad ij \in E.$$  

Clearly $x_{ij} \geq 0$. Since $f$ is a coloring, also $x_{ij} \geq 1 = \ell_{ij}$. Moreover $x_{ij} \leq T - 1 = u_{ij}$. Hence $(\pi, x)$ is feasible for PESP.
3.1 Overview

**T-Vertex Coloring ≤ T-PESP**

**Proof.**
Conversely, let \((\pi, x)\) be feasible for PESP on the graph \(G\). As lower and upper bounds are integer, we can assume that this holds for \(\pi\) and \(x\) as well (total unimodularity). Then

\[
f(v) := \pi_v + 1 \in \{1, 2, \ldots, T\}, \quad v \in V,
\]

is a \(T\)-vertex coloring for \(G\). □
Chapter 3

Periodic Timetabling

§3.2 Cycle Spaces
§3.2 Cycle Spaces

Motivation: PESP MIP formulation

So far, we considered the following MIP formulation of PESP:

Minimize  \[ \sum_{ij \in E} w_{ij} x_{ij} \]

s.t.  \[ x_{ij} = \pi_j - \pi_i + p_{ij} T, \quad ij \in E, \]

\[ \ell_{ij} \leq x_{ij} \leq u_{ij}, \quad ij \in E, \]  (periodic tension)

\[ 0 \leq \pi_i \leq T - 1, \quad i \in V, \]  (periodic timetable)

\[ p_{ij} \in \mathbb{Z}, \quad ij \in E. \]  (periodic offset)

If the event-activity network has \( n \) events and \( m \) activities, then this formulation uses \( m \) constraints, \( m + n \) continuous variables, and \( m \) integer variables.

We will now construct a formulation with \( m - n + 1 \) constraints, \( m \) continuous variables, and \( m - n + 1 \) integer variables. This variant behaves much better in practice.
Let $G = (V, E)$ be an undirected graph.

**Definition**
A cycle in $G$ is an Eulerian subgraph of $G$.

**Remarks**
- In other words, a cycle is a subgraph $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$ such that $\deg_{G'}(v)$ is even for all $v \in V'$.
- Any cycle decomposes as an edge-disjoint union of circuits.
- We will sometimes identify a cycle with its sequence of edges or vertices.
§3.2 Cycle Spaces

Symmetric difference of cycles

Lemma
Let \( C_1, C_2 \) be two cycles in \( G \). Then the symmetric difference

\[
C_1 \Delta C_2 := (C_1 \cup C_2) \setminus (C_1 \cap C_2)
\]

is a cycle in \( G \).

Proof.
Let \( v \in V(C_1 \Delta C_2) \). Then

\[
\deg_{C_1 \Delta C_2}(v) = \deg_{C_1}(v) + \deg_{C_2}(v) - 2 \deg_{C_1 \cap C_2}(v)
\]

is even.
§3.2 Cycle Spaces

Incidence vectors and cycle space

Let $G$ be an undirected graph.

Definition

For a cycle $C$ define its **incidence vector** $\gamma_C \in \{0, 1\}^E$ as

$$
\gamma_e := \begin{cases} 
1 & \text{if } e \in E(C), \\
0 & \text{if } e \notin E(C),
\end{cases} \quad e \in E(G).
$$

The **cycle space** of $G$ is the set

$$
\mathcal{C}(G) := \{\gamma_C \mid C \text{ is a cycle in } G\} \subseteq \{0, 1\}^E.
$$

**Lemma**

$\mathcal{C}(G)$ is an $\mathbb{F}_2$-vector space.

**Proof.**

Addition $\leftrightarrow$ symmetric difference.  \hfill $\square$
§3.2 Cycle Spaces

Cyclomatic number

Let $G$ be an undirected graph.

**Definition**
The cyclomatic number of $G$ is defined as

$$\mu(G) := \dim_{\mathbb{F}_2} \mathcal{C}(G).$$

In other words, the cyclomatic number is the length of any cycle basis.

**Lemma**
*Suppose that $G$ has $n$ vertices, $m$ edges, and $c$ connected components. Then $\mu(G) = m - n + c$.***

**Proof.**
Suppose first that $G$ is connected. Let $T$ be a spanning tree of $G$, i.e., a maximal cycle-free subgraph containing all $n$ vertices.
Cyclomatic number

Proof (cont.)

We call an edge $e \in E(G)$ a co-tree edge if $e \notin E(T)$. Since $T$ contains $n - 1$ edges, there are precisely $m - n + 1$ co-tree edges.

Since $T$ is a spanning tree, adding a single co-tree edge $e$ to $T$ produces a cycle containing $e$. This way, we obtain $m - n + 1$ cycles in $G$, one for each co-tree edge.

The incidence vectors of these cycles are $\mathbb{F}_2$-linearly independent, since each co-tree edge is contained in precisely one cycle. In formulae, if $\gamma_e \in \{0, 1\}^E$ denotes the incidence vector of the cycle produced by the co-tree edge $e$, then

$$
\gamma_{e,e'} = \begin{cases} 
1 & \text{if } e = e', \\
0 & \text{if } e \neq e' \text{ for all co-tree edges } e' \notin E(T).
\end{cases}
$$

This shows $\dim_{\mathbb{F}_2} C(G) \geq m - n + 1$ for connected $G$. 

May 28, 2018
Proof (cont.)

Let $\zeta$ be the incidence vector of an arbitrary cycle of $G$. Let

$$\zeta' := \zeta - \sum_{e \in E(T)} \zeta e \gamma_e \in \mathcal{C}(G).$$

Then for any co-tree edge $e' \notin E(T)$, the corresponding entry of $\zeta'$ is

$$\zeta'_{e'} = \zeta_{e'} - \sum_{e \notin E(T)} \zeta e \gamma_{e,e'} = \zeta_{e'} - \zeta_{e'} = 0,$$

so that $\zeta'$ corresponds to a cycle inside the tree $T$. Since trees cannot have cycles, $\zeta' = 0$ and $\zeta$ is therefore in the $\mathbb{F}_2$-span of $\{\gamma_e \mid e \notin E(T)\}$.

This finishes the proof for $c = 1$. If $G$ has several connected components, then add the cyclomatic numbers of all components.
§3.2 Cycle Spaces

Fundamental cycles

Definition

Let $T$ be a spanning tree of an undirected graph $G$.

- A cycle created by adding a co-tree edge to $T$ is called fundamental cycle.
- A fundamental cycle basis is a cycle basis consisting of fundamental cycles.

Remark

The following is an algorithm to construct a fundamental cycle basis: Compute first a spanning tree (Prim, Kruskal, ...) and then take the fundamental cycle for each co-tree edge.
3.2 Cycle Spaces

Example: Undirected cycle basis

Consider the following graph $G$:

$G$ has 8 vertices, 10 edges, 1 connected component and hence $\mu(G) = 10 - 8 + 1 = 3$.

Fundamental cycle basis:
Example: Undirected cycle basis

Consider the following graph $G$:

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Consider the following graph $G$:

$G$ has 8 vertices, 10 edges, 1 connected component and hence $\mu(G) = 10 - 8 + 1 = 3$.

Fundamental cycle basis:
§3.2 Cycle Spaces

Directed cycles

Let $G$ be a directed graph.

Definition
A directed cycle in $G$ is an Eulerian subgraph of $G$.
An oriented cycle in $G$ is a cycle of the undirected graph $|G|$ underlying $G$.

Remarks

► Any directed cycle is an oriented cycle.
► An oriented cycle uses edges either in forward or in backward direction.
► Any directed/oriented cycle decomposes as an edge-disjoint union of directed/oriented circuits.
§3.2 Cycle Spaces

Incidence vectors and cycle space

Definition

Let $C$ be an oriented cycle in $G$. Then its **incidence vector** $\gamma_C \in \{-1, 0, 1\}^E$ is defined as

$$\gamma_e := \begin{cases} 
1 & \text{if } C \text{ uses } e \text{ as forward edge}, \\
-1 & \text{if } C \text{ uses } e \text{ as backward edge}, \\
0 & \text{otherwise}
\end{cases}$$

$e \in E(G)$.

The **$\mathbb{Q}$-cycle space** of $G$ is the $\mathbb{Q}$-vector space

$$\mathcal{C}_\mathbb{Q}(G) := \text{span}_\mathbb{Q} \{\gamma_C \mid C \text{ oriented cycle of } G\}.$$

A basis consisting of incidence vectors of true oriented cycles is called a **cycle basis** of $G$.

The **cyclomatic number** of $G$ is defined as $\mu(G) := \dim_\mathbb{Q} \mathcal{C}_\mathbb{Q}(G)$. 
Lemma
Let $B$ be a cycle basis for $|G|$. Then lifting each cycle in $B$ to an oriented cycle in $G$ yields a $\mathbb{Q}$-basis of $C_{\mathbb{Q}}(G)$.

Proof.
Let $B = \{\gamma_1, \ldots, \gamma_\mu\}$ and let $\gamma'_i$ be the incidence vector of an oriented cycle in $G$ projecting to $\gamma_i$ in $|G|$, $i = 1, \ldots, \mu := \mu(|G|)$.

Linear independence: Suppose $\sum_{i=1}^\mu \lambda_i \gamma'_i = 0$ for some $\lambda_1, \ldots, \lambda_\mu \in \mathbb{Q}$. Clearing denominators, we can assume that $\lambda_1, \ldots, \lambda_\mu \in \mathbb{Z}$ and $\gcd(\lambda_1, \ldots, \lambda_\mu) = 1$. Reducing modulo 2, we have $\sum_{i=1}^\mu [\lambda_i]_2 \gamma_i \equiv_2 0$, which implies that $\lambda_i \equiv_2 0$ for all $i$, as $B$ is an $\mathbb{F}_2$-basis. Since all $\lambda_i$ were coprime, this means that $\lambda_1 = \cdots = \lambda_\mu = 0$. 
§3.2 Cycle Spaces

Undirected cycle bases

Proof (cont.)

It remains to show that $\dim_{\mathbb{Q}} C_{\mathbb{Q}}(G) = \mu(|G|)$. Consider a spanning tree $T$ of $|G|$ with its fundamental cycle basis $B = \{ \gamma_e \mid e \notin E(T) \}$.

Let $\zeta \in \{-1, 0, 1\}^{E(G)}$ be the incidence vector of an arbitrary oriented cycle in $G$ and suppose that $\zeta$ does not lie in the span of the lifts $\{ \gamma'_e \mid e \notin E(T) \}$ of the vectors in $B$ to $G$. Then also

$$
\zeta' := \zeta - \sum_{e \notin E(T)} \zeta_e \cdot \gamma'_{e,e} \notin \text{span}\{ \gamma'_e \mid e \notin E(T) \}
$$

Then $\zeta'_e = 0$ for any edge in $E(G)$ corresponding to a co-tree edge of $T$, so that $\zeta$ has support only in the tree edges. But $T$ is a tree and hence cannot contain a cycle, so $\zeta' = 0$ (contradiction).

Remark

In particular, fundamental cycle bases work as in the undirected case.
§3.2 Cycle Spaces

Cycle basis names

Let $G$ be a directed graph.

**Corollary**

If $G$ has $n$ vertices, $m$ edges and $c$ weakly connected components, then 
$\mu(G) = \mu(|G|) = m - n + c$.

**Definition**

- A cycle basis in $G$ coming from a cycle basis in $|G|$ is called an **undirected cycle basis**.
- A cycle basis in $G$ coming from a spanning tree is called a **strictly fundamental basis**.

**Definition**

Let $\mathcal{B} = (\gamma_1, \ldots, \gamma_{\mu(G)})$ be a cycle basis. The $(\mu(G) \times m)$-matrix $\Gamma$ whose rows are given by $\gamma_i$, $i = 1, \ldots, \mu(G)$, is called the **cycle matrix** of $\mathcal{B}$. 
§3.2 Cycle Spaces

Cycle basis example

Consider the following digraph $G$ with red spanning tree $T$:

![Diagram of digraph G with red spanning tree T]

We produce a strictly fundamental cycle basis by taking the oriented cycle for each co-tree edge of $T$:

![Diagram of cycle basis with cycles C1, C2, C3]

The cycles $C_1$ and $C_3$ use only forward edges, whereas $C_2$ uses two backward edges.
§3.2 Cycle Spaces

Cycle basis example

Label the edges by 1, \ldots, 10:

Collecting the incidence vectors of $C_1$, $C_2$, $C_3$ yields the $3 \times 10$-cycle matrix:

\[
\begin{array}{cccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
 \gamma_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
 \gamma_2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
 \gamma_3 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Note that the matrix has full row rank. The part corresponding to the co-tree edges 5, 6, 7 of $T$ is a permutation of the identity matrix.
§3.2 Cycle Spaces

Determinant of a cycle basis

Let $G$ be a directed graph and let $\mathcal{B}$ be a cycle basis with cycle matrix $\Gamma$.

**Definition**

The **determinant** of $\mathcal{B}$ is defined as

$$\det(\mathcal{B}) := \left| (\mu(G) \times \mu(G))\text{-submatrix of } \Gamma \text{ corresponding to the co-tree edges of some spanning tree of } G \right|.$$

This is well-defined:

**Theorem (Liebchen, 2003)**

Let $T_1, T_2$ be two spanning trees of $G$. For $i = 1, 2$, denote by $A_i$ the $(\mu(G) \times \mu(G))$-submatrix of $\Gamma$, where exactly the columns corresponding to $e \notin E(T_i)$ are selected. Then $A_1$ and $A_2$ are both invertible and $\det(A_1) = \pm \det(A_2)$. 
Determinant of a cycle basis

Proof.
Let $\Phi$ be the cycle matrix of a strictly fundamental cycle basis of $G$ coming from the spanning tree $T_1$. The rows of $\Phi$ are indexed by the $\mu := \mu(G)$ co-tree edges of $T_1$. We have

$$
\Phi_{e,e'} = \begin{cases} 
1 & \text{if } e = e', \\
0 & \text{if } e \neq e', 
\end{cases}
$$

for all $e, e' \notin E(T)$.

Note that we can always lift a fundamental cycle in such a way that the co-tree edge becomes a forward edge. In particular, if $\Phi_1$ denotes the restriction of $\Phi$ to the columns corresponding to co-tree edges of $T_1$, then $\Phi_1$ is the identity matrix.

Since $\Phi$ and $B$ are bases, there is an invertible $(\mu \times \mu)$-matrix $S$ such that $\Gamma = S \cdot \Gamma_\Phi$. It follows that $A_1 = S \cdot \Phi_1$ is invertible. This holds analogously for $A_2$. 
§3.2 Cycle Spaces

Determinant of a cycle basis

Proof.

Let \( \Phi_2 \) denote the restriction of \( \Phi \) to the columns corresponding to the co-tree edges of \( T_2 \). Then \( A_2 = S \cdot \Phi_2 \), so it remains to show that \( \det(\Phi_2) = \pm \det(\Phi_1) = \pm 1 \). We use induction on \#E(T_1)\Delta E(T_2).

\#E(T_1)\Delta E(T_2) = 0: This is equivalent to \( E(T_1) = E(T_2) \), where obviously \( \det(\Phi_2) = \det(\Phi_1) \).

\#E(T_1)\Delta E(T_2) > 0: Let \( e_1 \in E(T_1) \setminus E(T_2) \). On the unique path in \( T_2 \) connecting the endpoints of \( e_1 \), there must be an edge \( e_2 \notin E(T_1) \), as otherwise \( T_1 \) would contain a cycle. The fundamental cycle of \( e_1 \) in \( T_1 \) uses \( e_2 \), so that \( \Phi_{e_1,e_2} = \pm 1 \). Since there is only one fundamental cycle for \( T_1 \) using the co-tree edge \( e_2 \), this means that \( \Phi_{e,e_2} = 0 \) for \( e \neq e_1 \). Use Laplace expansion along the column \( e_2 \).
§3.2 Cycle Spaces

Characterization by determinant

Let $G$ be a digraph with cyclomatic number $\mu$ and cycle basis $B$.

**Theorem (Liebchen/Rizzi, 2007)**

1. $B$ is undirected if and only if $\det(B)$ is odd.
2. $B$ is strictly fundamental if and only if the cycle matrix of $B$ can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last $\mu$ columns.

**Proof.**

(2) Exercise. (1) Let $\Gamma$ be the cycle matrix of $B$. Write $\Gamma = S \cdot \Phi$, where $S$ is an invertible $\mu \times \mu$-matrix and $\Phi$ is the matrix of a strictly fundamental basis for some spanning tree $T$. Restricting to the co-tree edges, we obtain $\Gamma|_{\overline{E(T)}} = S \cdot \Phi|_{\overline{E(T)}} = S$, so $\det(B) = \det(S)$. If $\det(B)$ is odd, then $S$ is invertible over $\mathbb{F}_2$, so the rows of $\Gamma$ mod 2 form a cycle basis for $|G|$. Conversely, if $B$ is undirected, then $\Gamma|_{\overline{E(T)}}$ is invertible mod 2, so that also $S$ is invertible mod 2 and hence $\det(B)$ is odd.