Mathematical Aspects of Public Transportation Networks

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Chapter 3

Periodic Timetabling

§3.2 Cycle Spaces
Cycle basis names

Let $G$ be a directed graph.

**Corollary**

*If $G$ has $n$ vertices, $m$ edges and $c$ weakly connected components, then $\mu(G) = \mu(|G|) = m - n + c$.***

**Definition**

- A cycle basis in $G$ coming from a cycle basis in $|G|$ is called an **undirected cycle basis**.
- A cycle basis in $G$ coming from a spanning tree is called a **strictly fundamental basis**.

**Definition**

Let $\mathcal{B} = (\gamma_1, \ldots, \gamma_{\mu(G)})$ be a cycle basis. The $(\mu(G) \times m)$-matrix $\Gamma$ whose rows are given by $\gamma_i, i = 1, \ldots, \mu(G)$, is called the **cycle matrix** of $\mathcal{B}$. 
Consider the following digraph $G$ with red spanning tree $T$:

We produce a strictly fundamental cycle basis by taking the oriented cycle for each co-tree edge of $T$:

The cycles $C_1$ and $C_3$ use only forward edges, whereas $C_2$ uses two backward edges.
§3.2 Cycle Spaces

Cycle basis example

Label the edges by 1, \ldots, 10:

\[
\begin{array}{c}
\circ \rightarrow \circ \ \\
\circ \rightarrow \circ \ \\
\circ \rightarrow \circ \ \\
\circ \rightarrow \circ \ \\
\circ \rightarrow \circ \end{array}
\]

Collecting the incidence vectors of $C_1$, $C_2$, $C_3$ yields the $3 \times 10$-cycle matrix:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\gamma_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
\gamma_2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \\
\gamma_3 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Note that the matrix has full row rank. The part corresponding to the co-tree edges 5, 6, 7 of $T$ is a permutation of the identity matrix.
§3.2 Cycle Spaces

Determinant of a cycle basis

Let $G$ be a directed graph and let $B$ be a cycle basis with cycle matrix $\Gamma$.

Definition

The **determinant** of $B$ is defined as

$$\det(B) := \begin{vmatrix} \text{($\mu(G) \times \mu(G)$)-submatrix of $\Gamma$ corresponding to the co-tree edges of some spanning tree of $G$} \end{vmatrix}.$$  

This is well-defined:

**Theorem (Liebchen, 2003)**

Let $T_1, T_2$ be two spanning trees of $G$. For $i = 1, 2$, denote by $A_i$ the ($\mu(G) \times \mu(G)$)-submatrix of $\Gamma$, where exactly the columns corresponding to $e \notin E(T_i)$ are selected. Then $A_1$ and $A_2$ are both invertible and $\det(A_1) = \pm \det(A_2)$. 

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§3.2 Cycle Spaces

Determinant of a cycle basis

Proof.
Let $\Phi$ be the cycle matrix of a strictly fundamental cycle basis of $G$ coming from the spanning tree $T_1$. The rows of $\Phi$ are indexed by the $\mu := \mu(G)$ co-tree edges of $T_1$. We have

$$\Phi_{e,e'} = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e', \end{cases} \quad \text{for all } e, e' \notin E(T).$$

Note that we can always lift a fundamental cycle in such a way that the co-tree edge becomes a forward edge. In particular, if $\Phi_1$ denotes the restriction of $\Phi$ to the columns corresponding to co-tree edges of $T_1$, then $\Phi_1$ is the identity matrix.

Since $\Phi$ and $B$ are bases, there is an invertible $(\mu \times \mu)$-matrix $S$ such that $\Gamma = S \cdot \Gamma_\Phi$. It follows that $A_1 = S \cdot \Phi_1$ is invertible. This holds analogously for $A_2$. 
Proof (cont.)

Let $\Phi_2$ denote the restriction of $\Phi$ to the columns corresponding to the co-tree edges of $T_2$. Then $A_2 = S \cdot \Phi_2$, so it remains to show that $\det(\Phi_2) = \pm \det(\Phi_1) = \pm 1$. We use induction on $#E(T_1) \Delta E(T_2)$.

$#E(T_1) \Delta E(T_2) = 0$: This is equivalent to $E(T_1) = E(T_2)$, where obviously $\det(\Phi_2) = \det(\Phi_1)$.

$#E(T_1) \Delta E(T_2) > 0$: Let $e_1 \in E(T_1) \setminus E(T_2)$. On the unique path in $T_2$ connecting the endpoints of $e_1$, there must be an edge $e_2 \notin E(T_1)$, as otherwise $T_1$ would contain a cycle. The fundamental cycle of $e_1$ in $T_1$ uses $e_2$, so that $\Phi_{e_1,e_2} = \pm 1$. Since there is only one fundamental cycle for $T_1$ using the co-tree edge $e_2$, this means that $\Phi_{e,e_2} = 0$ for $e \neq e_1$. Use Laplace expansion along the column $e_2$. \qed
§3.2 Cycle Spaces

Characterization by determinant

Let $G$ be a digraph with cyclomatic number $\mu$ and cycle basis $B$.

Theorem (Liebchen/Rizzi, 2007)

1. $B$ is undirected if and only if $\det(B)$ is odd.
2. $B$ is strictly fundamental if and only if the cycle matrix of $B$ can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last $\mu$ columns.

Proof.

(2) Exercise. (1) Let $\Gamma$ be the cycle matrix of $B$. Write $\Gamma = S \cdot \Phi$, where $S$ is an invertible $\mu \times \mu$-matrix and $\Phi$ is the matrix of a strictly fundamental basis for some spanning tree $T$. Restricting to the co-tree edges, we obtain $\Gamma|_{\text{co-tree}} = S \cdot \Phi|_{\text{co-tree}} = S$, so $\det(B) = \det(S)$. If $\det(B)$ is odd, then $S$ is invertible over $\mathbb{F}_2$, so the rows of $\Gamma$ mod 2 form a cycle basis for $|G|$. Conversely, if $B$ is undirected, then $\Gamma|_{\text{co-tree}}$ is invertible mod 2, so that also $S$ is invertible mod 2 and hence $\det(B)$ is odd.
More on the determinant

Let \( G \) be a digraph with cyclomatic number \( \mu \).

**Lemma (Liebchen/Peeters, 2003)**

Let \( \Gamma \) be the cycle matrix of a cycle basis for \( G \), and let \( A \) be any \( \mu \times \mu \)-submatrix of \( \Gamma \). Then \( A \) is invertible if and only if the columns of \( A \) correspond to the co-tree edges of some spanning tree of \( G \).

**Proof.**

\[(\Leftarrow)\] Let \( \Phi \) be the cycle matrix of a strictly fundamental basis for some spanning tree \( T \). As before, \( \Gamma = S \cdot \Phi \) for some invertible \( \mu \times \mu \)-matrix \( S \). Let \( A \) be the submatrix of \( \Gamma \) corresponding to the co-tree edges of \( T \). Then \( A = \Gamma|_{\text{co-tree}} = S \cdot \Phi_{\text{co-tree}} = S \), so that \( A \) is invertible.

\[(\Rightarrow)\] Suppose that \( A \) is invertible. Let \( H = \{e_1, \ldots, e_\mu\} \subseteq E(G) \) such that the columns of \( A \) correspond to \( H \). Then any cycle \( \gamma \) can be written as \( \gamma^t = \lambda^t \Gamma \) for some \( \lambda \in \mathbb{Q}^\mu \), as \( \Gamma \) is a cycle basis. If \( \gamma \) contains no edge of \( H \), then \( 0 = (\gamma_{e_1}, \ldots, \gamma_{e_\mu}) = \lambda^T A \), so that \( \lambda = 0 \) as \( A \) is invertible, and \( \gamma = 0 \). In particular, \( E(G) \setminus H \) has no cycle and is thus a spanning tree. \( \square \)
Integral cycle bases

Let $G$ be a digraph with cyclomatic number $\mu$.

**Definition**

A cycle basis $\mathcal{B} = \{\gamma_1, \ldots, \gamma_\mu\}$ is called integral if every incidence vector $\gamma$ of an oriented cycle in $G$ can be written as

$$\gamma = \sum_{i=1}^{\mu} \lambda_i \gamma_i,$$

where $\lambda_1, \ldots, \lambda_\mu \in \mathbb{Z}$.

**Theorem (Liebchen/Peeters, 2003)**

The following are equivalent for a cycle basis $\mathcal{B}$ with cycle matrix $\Gamma$:

1. $\mathcal{B}$ is integral,
2. every $\mu \times \mu$-submatrix of $\Gamma$ has determinant $0$ or $\pm 1$,
3. $\det(\mathcal{B}) = 1$. 

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Proof.

(2) $\Leftrightarrow$ (3): by preceding lemma.

(1) $\Rightarrow$ (2): Let $T$ be a spanning tree, giving rise to a strictly fundamental cycle basis with matrix $\Phi$. Then $\Phi = S \cdot \Gamma$ for some invertible $\mu \times \mu$-matrix $S$. Since $\mathcal{B}$ is integral, $S$ has integer entries. Let $A$ be the $\mu \times \mu$-submatrix of $\Gamma$ restricted to the co-tree edges of $T$. Then $S \cdot A$ is the identity matrix. Since $S$ and $A$ have both integer determinants multiplying to 1, we have $\det(A) = \pm 1$.

(3) $\Rightarrow$ (1): For an arbitrary incidence vector $\gamma$ there is a $\lambda \in \mathbb{Q}^\mu$ such that $\gamma^t = \lambda^t \Gamma$ (cycle basis property). Restricting to the co-tree edges $\{e_1, \ldots, e_\mu\}$ of a spanning tree yields $(\gamma_{e_1}, \ldots, \gamma_{e_\mu}) = \lambda^t A$ for the suitable submatrix $A$ of $\Gamma$. Since $A$ has determinant $\pm 1$ by (3), it has an integer inverse and hence $\lambda^t = (\gamma_{e_1}, \ldots, \gamma_{e_\mu}) A^{-1}$ is integer.
§3.2 Cycle Spaces

Summary

Let $G$ be a directed graph.

**Classes of cycle bases**

- **arbitrary**
  
  $\det \neq 0$

- **undirected**

  $\det \equiv_2 1$

- **integral**

  $\det = 1$

- **strictly fundamental**

  $\det = 1 + \text{identity matrix condition}$

Examples for the strict inclusion: Last tutorial and Problem Set 6.
Chapter 3

Periodic Timetabling

§3.3 Cycles in Periodic Timetabling
§3.3 Cycles in Periodic Timetabling

Back to PESP

Input

- event-activity network $\mathcal{E} = (V, E)$,
- period time $T \in \mathbb{N}$,
- lower bound vector $\ell \in (\mathbb{R}_{\geq 0})^E$, $0 \leq \ell < T$,
- upper bound vector $u \in (\mathbb{R}_{\geq 0})^E$, $\ell \leq u < T - \ell$,
- weight vector $w \in (\mathbb{R}_{\geq 0})^E$

MIP formulation

Minimize $\sum_{ij \in E} w_{ij}x_{ij}$

s.t.

$\quad x_{ij} = \pi_j - \pi_i + p_{ij}T$, \hspace{1cm} ij \in E,

$\quad \ell_{ij} \leq x_{ij} \leq u_{ij}$, \hspace{1cm} ij \in E, \hspace{1cm} \text{(periodic tension)}

$\quad 0 \leq \pi_i \leq T - 1$, \hspace{1cm} i \in V, \hspace{1cm} \text{(periodic timetable)}

$\quad p_{ij} \in \mathbb{Z}$, \hspace{1cm} ij \in E. \hspace{1cm} \text{(periodic offset)}
§3.3 Cycles in Periodic Timetabling

Cycle periodicity constraints

Theorem (Nachtigall, 1994; Liebchen/Peeters, 2002)

Consider a PESP instance, and let $x \in \mathbb{R}^E$. The following are equivalent:

1. There exists a periodic timetable $\pi \in [0, T)^V$ such that for all $ij \in E$ exist $p_{ij} \in \mathbb{Z}$ such that $x_{ij} = \pi_j - \pi_i + p_{ij}T$.

2. For each oriented cycle $\gamma$ in $E$ exists $z_\gamma \in \mathbb{Z}$ such that $\gamma^t x = z_\gamma T$.

3. For each integral cycle basis $\{\gamma_1, \ldots, \gamma_\mu\}$ of $E$, there are $z_1, \ldots, z_\mu \in \mathbb{Z}$ such that $\gamma_i^t x = z_i T$ for all $i = 1, \ldots, \mu$.

Proof.

(1) $\Rightarrow$ (2): Let $\gamma \in \{-1, 0, 1\}^E$ be the incidence vector of an oriented cycle $(v_1, \ldots, v_k, v_1)$. If $\gamma$ uses $(v_i, v_{i+1}) \in E$ forward, then

$$\gamma_{v_i,v_{i+1}} x_{v_i,v_{i+1}} = \pi_{v_{i+1}} - \pi_{v_i} + p_{v_i,v_{i+1}} T.$$ 

Otherwise, if $\gamma$ uses $(v_{i+1}, v_i)$ backward, then

$$\gamma_{v_{i+1},v_i} x_{v_{i+1},v_i} = \pi_{v_{i+1}} - \pi_{v_i} - p_{v_{i+1},v_i} T.$$ 

Hence $\gamma^t x = T\gamma^t p$, and clearly $\gamma^t p \in \mathbb{Z}$. 

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§3.3 Cycles in Periodic Timetabling

Cycle periodicity constraints

Proof (cont.)

(2) ⇒ (3): Trivial. (3) ⇒ (2): Let $\gamma$ be the incidence vector of an arbitrary oriented cycle. Since $\{\gamma_1, \ldots, \gamma_\mu\}$ is an integral cycle basis, there are $\lambda_1, \ldots, \lambda_\mu \in \mathbb{Z}$ such that $\gamma = \sum_{i=1}^{\mu} \lambda_i \gamma_i$. In particular

$$\gamma^t \mathbf{x} = \sum_{i=1}^{\mu} \lambda_i \gamma_i^t \mathbf{x} = \sum_{i=1}^{\mu} \lambda_i z_i T = \left( \sum_{i=1}^{\mu} \lambda_i z_i \right) \cdot T \quad \in \mathbb{Z} \cdot T.$$

(2) ⇒ (1): Let $T$ be a spanning tree of $E$, and pick a vertex $s \in V(T)$. Then there is a unique oriented path from $s$ to each other vertex $v \in V(T)$. Each oriented path in $E$ can be expressed as an incidence vector in $\{-1, 0, 1\}^E$ as in the case of cycles. Set $\pi_s := 0$ and $\pi_v := \mathbf{p}_{sv}^t \mathbf{x}$ for all $v \in V(T) \setminus \{s\}$, where $\mathbf{p}_{sv}$ is the unique oriented $s$-$v$-path in $T$. If $ij \in E(T)$, then $\mathbf{p}_{sj} = \mathbf{p}_{si} + \mathbf{e}_{ij}$, so that

$$\pi_j - \pi_i = \mathbf{e}_{ij}^t \mathbf{x} = x_{ij} = x_{ij} + 0 \cdot T.$$
3.3 Cycles in Periodic Timetabling

Cycle periodicity constraints

Proof (cont.)

if \( ij \in E \setminus E(T) \) is a co-tree edge, then this yields a fundamental cycle \( \gamma \).

The cycle \( \gamma \) uses the edge \( ij \) and then the unique path from \( j \) to \( i \) in \( T \).

The incidence vector of this path is simply given by \( p_{si} - p_{sj} \), so that \( \gamma = p_{si} - p_{sj} + e_{ij} \). Hence

\[
\pi_j - \pi_i = p_{sj}^t x - p_{si}^t x = e_{ij}^t x - \gamma^t x = x_{ij} + z_\gamma T,
\]

and we can set \( p_{ij} := z_\gamma \).

Finally, reduce \( \pi \) modulo \( T \).

Corollary

A feasible periodic timetable \( \pi \) can be constructed from a feasible periodic tension \( x \) using a spanning tree.
3.3 Cycles in Periodic Timetabling

**Cycle-based PESP MIP formulation**

In the PESP MIP formulation, we can now replace the constraints

\[
x_{ij} = \pi_j - \pi_i + p_{ij} T, \quad p_{ij} \in \mathbb{Z}
\]

by choosing an integral cycle basis \( \{\gamma_1, \ldots, \gamma_\mu\} \) and requiring

\[
\gamma_i^T x = z_i T, \quad z_i \in \mathbb{Z}.
\]

**New MIP formulation (cycle & tension)**

Let \( \Gamma \) be the cycle matrix of an integral cycle basis for \( \mathcal{E} \).

Minimize

\[
\sum_{ij \in E} w_{ij} x_{ij}
\]

s.t.

\[
\Gamma x = z^T, \quad \text{(cycle periodicity)}
\]

\[
\ell \leq x \leq u, \quad \text{(periodic tension)}
\]

\[
z \in \mathbb{Z}^\mu. \quad \text{(cycle offset)}
\]

This uses less constraints and variables than the original formulation.
3.3 Cycles in Periodic Timetabling

**Cycle-and-slack-based PESP MIP formulation**

**Definition**

The **periodic slack** is \( y := x - \ell \).

**Remark**

If a periodic timetable \( \pi \) is given, then \( y_{ij} = [\pi_j - \pi_i - \ell_{ij}]_T \).

This gives rise to an equivalent MIP formulation, minimizing the total slack:

**New MIP formulation (cycle & slack)**

Minimize \( \sum_{ij \in E} w_{ij} y_{ij} \)

s.t. \( \Gamma(y + \ell) = zT \), \hspace{1cm} \text{(cycle periodicity)}

\( 0 \leq y \leq u - \ell, \) \hspace{1cm} \text{(periodic slack)}

\( z \in \mathbb{Z}^\mu. \) \hspace{1cm} \text{(cycle offset)}
§3.3 Cycles in Periodic Timetabling

Example

Consider the following PESP instance \((T = 10)\):
Example

The cycle basis is integral (even strictly fundamental). In the cycle & slack-formulation, this yields the following:

Minimize \( y_2 + y_4 + y_6 + y_8 + y_9 + y_{10} \)

s.t.

\[
\begin{align*}
y_1 + y_7 + y_8 + y_9 - 10z_1 &= -19, \quad \text{(cycle periodicity for } \gamma_1) \\
y_2 + y_6 - y_9 - y_{10} - 10z_2 &= 0, \quad \text{(cycle periodicity for } \gamma_2) \\
y_3 + y_4 + y_5 + y_{10} - 10z_3 &= -17, \quad \text{(cycle periodicity for } \gamma_3) \\
0 &\leq y_2, y_4, y_6, y_8, y_9, y_{10} \leq 9, \quad \text{(periodic slack, transfer)} \\
z_1, z_2, z_3 &\in \mathbb{Z}. \quad \text{(cycle offset)}
\end{align*}
\]

We may omit the fixed \( y \)-variables (i.e., the ones for the driving activities), giving a MIP with 3 integer and 6 continuous variables, and 3 constraints.

Optimal sol.: \( y_2 = y_6 = y_9 = y_{10} = z_2 = 0, y_4 = 3, y_8 = 1, z_1 = z_3 = 2 \),

minimal slack: 4.
§3.3 Cycles in Periodic Timetabling

Offset variable bounds

Question

Recall that in the old timetable-based formulation, we could w.l.o.g. achieve that the periodic offsets satisfy $p_{ij} \in \{0, 1, 2\}$. What about the cycle offsets in the cycle-based formulation?

Definition

For a PESP instance, define the offset space as

$$P_{\text{offset}} := \{ z \in \mathbb{Z}^\mu \mid \exists y \in \mathbb{R}^E : 0 \leq y \leq u - \ell, \Gamma(y + \ell) = Tz \}.$$  

Theorem (Odijk, 1996)

If $z \in P_{\text{offset}}$, then any cycle $\gamma^t = \lambda^t \Gamma$ satisfies the cycle inequality

$$\left[ \frac{\gamma_+^t \ell - \gamma_-^t u}{T} \right] \leq \lambda^t z \leq \left[ \frac{\gamma_+^t u - \gamma_-^t \ell}{T} \right].$$

Conversely, if for given $z \in \mathbb{Z}^\mu$, the cycle inequality holds for each oriented cycle $\gamma$, then $z \in P_{\text{offset}}$. 
§3.3 Cycles in Periodic Timetabling

**Cycle inequality**

**Notation**
Each incidence vector $\gamma$ of an oriented cycle decomposes as $\gamma = \gamma_+ - \gamma_-$, where $\gamma_+ \in \{0, 1\}^E$ ("forward part") and $\gamma_- \in \{0, 1\}^E$ ("backward part").

Example: $(1, 1, 0, 0, -1, -1) = (1, 1, 0, 0, 0, 0) - (0, 0, 0, 0, 1, 1)$.

**Remark**
Oджik’s theorem gives a strategy to generate valid inequalities for PESP: All integer solutions satisfy the cycle inequality for all cycles. If an LP solver finds a fractional solution and there is a cycle $\gamma$ violating the cycle inequality, then we can add the cycle inequality for $\gamma$ as additional constraint and solve again.
§3.3 Cycles in Periodic Timetabling

**Cycle inequality: Example**

Consider the following PESP instance \((T = 10)\):

**Bounds:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<td>7</td>
<td>11</td>
<td>12</td>
<td>12</td>
</tr>
</tbody>
</table>

**Cycle inequalities:**

\[
2 = \left\lceil \frac{(7 + 7 + 2 + 3)}{10} \right\rceil \leq z_1 \leq \left\lfloor \frac{(7 + 7 + 11 + 12)}{10} \right\rfloor = 3
\]

\[
-1 = \left\lceil \frac{(3 + 3 - 12 - 12)}{10} \right\rceil \leq z_2 \leq \left\lfloor \frac{(12 + 12 - 3 - 3)}{10} \right\rfloor = 1
\]

\[
2 = \left\lceil \frac{(6 + 2 + 6 + 3)}{10} \right\rceil \leq z_3 \leq \left\lfloor \frac{(6 + 11 + 6 + 12)}{10} \right\rfloor = 3
\]

→ bounds for the cycle offset variables.
§3.3 Cycles in Periodic Timetabling

Cycle inequality

Proof (⇒).

Let \( z \in P_{\text{offset}} \) and let \( \gamma^t = \lambda^t \Gamma \) be an oriented cycle. Since \( \lambda^t z \) is integer, it suffices to prove

\[
\frac{\gamma^t_+ \ell - \gamma^t_- u}{T} \leq \lambda^t z \leq \frac{\gamma^t_+ u - \gamma^t_- \ell}{T}.
\]

Since \( z \in P_{\text{offset}} \), we find \( 0 \leq y \leq u - \ell \) such that \( \Gamma(y + \ell) = Tz \). This implies \( \gamma^t_+ y \geq 0 \) and \( \gamma^t_- y \leq \gamma^t_- (u - \ell) \), and therefore

\[
\gamma^t(y + \ell) = \gamma^t_+ y - \gamma^t_- y + \gamma^t_+ \ell - \gamma^t_- \ell \geq \gamma^t_- (u - \ell) + \gamma^t_+ \ell - \gamma^t_- \ell = \gamma^t_+ \ell - \gamma^t_- u.
\]

On the other hand, \( \gamma^t_+ y \leq \gamma^t_+ (u - \ell) \) and \( \gamma^t_- y \geq 0 \), so that

\[
\gamma^t(y + \ell) = \gamma^t_+ y - \gamma^t_- y + \gamma^t_+ \ell - \gamma^t_- \ell \leq \gamma^t_+ (u - \ell) + \gamma^t_+ \ell - \gamma^t_- \ell = \gamma^t_+ u - \gamma^t_- \ell.
\]

Putting this together,

\[
\gamma^t_+ \ell - \gamma^t_- u \leq \gamma^t(y + \ell) \leq \gamma^t_+ u - \gamma^t_- \ell.
\]

Finally note \( \lambda^t z = \lambda^t \Gamma(y + \ell)/T = \gamma^t(y + \ell)/T \).
§3.3 Cycles in Periodic Timetabling

**Cycle inequality**

**Proof (⇐).**

Given \( z \in \mathbb{Z}^\mu \) such that the cycle inequality holds for each oriented cycle, we have to show that there is \( 0 \leq y \leq u - \ell \) with \( \Gamma(y + \ell) = Tz \).

Let \( \gamma^t = \lambda^t \Gamma \) and let \( p \in \mathbb{Z}^E \) be an integer solution of \( \Gamma p = z \) (integral cycle basis). By the cycle inequality,

\[
\lambda^t z = \lambda^t \Gamma p = \gamma^t p \leq (\gamma^t_+ u - \gamma^t_- \ell) / T.
\]

Define \( \ell' := \ell - pT \) and \( u' := u - pT \). Then the above inequality reads as

\[
\gamma^t_+ u' - \gamma^t_- \ell' \geq 0.
\]

Let \( E' \) be the network obtained from \( E \) by adding to each edge \( ij \in E \) its anti-parallel edge \( ji \). For each edge \( ij \in E(E') \) set

\[
 w_{ij} := \begin{cases} 
 u'_{ij} & \text{if } ij \in E, \\
 -\ell'_{ji} & \text{if } ji \in E.
\end{cases}
\]
Cycle inequality

Proof (cont.)

We claim that every directed cycle in $E'$ has non-negative weight. Indeed, if $\tilde{\gamma}$ is such a cycle, then

$$w^t \tilde{\gamma} = \sum_{ij \in \tilde{\gamma}: ij \in E} u'_{ij} + \sum_{ij \in \tilde{\gamma}: ji \in E} (-\ell_{ij}) = \gamma^t_+ u' - \gamma^t_- \ell' \geq 0,$$

where $\gamma$ is the corresponding oriented cycle in $E$ using the edges $ij \in E$ forward and the $ji \in E$ backward.

This implies that the shortest path problem in $(E', w)$ behaves well. In particular, there is a potential $\pi \in \mathbb{R}^V$ such that

$$\pi_j - \pi_i \leq w_{ij} \text{ for all } ij \in E(E').$$

Taking $\pi$ to $E$, we have

$$\pi_j - \pi_i \leq u'_{ij} \quad \text{and} \quad \pi_i - \pi_j \leq -\ell'_{ij} \quad \text{for all } ij \in E.$$
3.3 Cycles in Periodic Timetabling

Cycle inequality

Proof (cont.)

This means

$$\ell_{ij} \leq \pi_j - \pi_i + p_{ij} T \leq u_{ij} \quad \text{for all } ij \in E.$$

In particular, if we set

$$y_{ij} := \pi_j - \pi_i + p_{ij} T - \ell_{ij}, \quad ij \in E,$$

then obviously $0 \leq y \leq u - \ell$. Moreover

$$\Gamma(y + \ell) = T \cdot \Gamma p = Tz,$$

as for each oriented cycle $\gamma = (v_1, \ldots, v_k, v_1)$, the potential differences $\pi_2 - \pi_1, \ldots, \pi_k - \pi_{k-1}, \pi_1 - \pi_k$ along $\gamma$ sum up to 0.
§3.3 Cycles in Periodic Timetabling

Cycle inequality: Applications

There are two applications for the cycle inequalities to PESP:

- give bounds on the integer variables in the cycle-based MIP, thereby reducing the search space for optimal solutions
  → find an integral cycle basis minimizing the possible values for the integer variables
  → \textit{minimum-weight cycle basis}

- add violated cycle inequalities as cutting planes in a LP-based MIP solving procedure
  → give an algorithm that checks if there is a violated cycle inequality
  → \textit{separation of cycle cuts}
§3.3 Cycles in Periodic Timetabling

**Minimum-weight cycle basis**

Let $G$ be a digraph with a weight vector $d \in \mathbb{R}_{\geq 0}^{E(G)}$.

**Definition**

The **minimum weight cycle basis** problem is to find a cycle basis \( \{\gamma_1, \ldots, \gamma_\mu\} \) of $G$ such that

\[
\sum_{i=1}^{\mu} \sum_{e \in E(G)} \gamma_{i,e} d_e
\]

is minimal.

**Application to PESP**

For a cycle $\gamma$, denote by $a_\gamma$ and $b_\gamma$ the lower and upper bounds of the cycle inequality for $\gamma$, respectively. Then using $\gamma_1, \ldots, \gamma_\mu$ for the MIP formulation produces

\[
\prod_{i=1}^{\mu} (b_{\gamma_i} - a_{\gamma_i} + 1)
\]

possible combinations of values for the integer variables $z_1, \ldots, z_m$. 
§3.3 Cycles in Periodic Timetabling

Minimum-weight cycle basis

However, this is not a weight vector.

Lemma

\[ \sum_{e \in E(G)} |\gamma_e| \frac{(u_e - l_e)}{T} \leq b_\gamma - a_\gamma < 2 + \sum_{e \in E(G)} |\gamma_e| \frac{(u_e - l_e)}{T} \]

Proof.

\[
\begin{align*}
    b_\gamma - a_\gamma + 1 &= \left\lceil \frac{\gamma^t_u - \gamma^t_l}{T} \right\rceil - \left\lfloor \frac{\gamma^t_l - \gamma^t_u}{T} \right\rfloor \\
    &< \frac{\gamma^t_u - \gamma^t_l}{T} + 1 - \left( \frac{\gamma^t_l - \gamma^t_u}{T} - 1 \right) \\
    &= 2 + \frac{\gamma^t_+(u - l) + \gamma^t_-(u - l)}{T} = 2 + \sum_{e \in E} |\gamma_e| \frac{(u_e - l_e)}{T},
\end{align*}
\]

\[
\begin{align*}
    b_\gamma - a_\gamma + 1 &\geq \frac{\gamma^t_+(u - l)}{T} - \frac{\gamma^t_+(l - u)}{T} = \sum_{e \in E} |\gamma_e| \frac{(u_e - l_e)}{T}.
\end{align*}
\]
3.3 Cycles in Periodic Timetabling

**Minimum-weight cycle basis**

As a compromise, compute the minimum weight undirected cycle basis for the weight vector $d := u - \ell$.

**Complexity of finding a minimum cycle basis**

<table>
<thead>
<tr>
<th>class</th>
<th>complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>arbitrary</td>
<td>polynomial</td>
</tr>
<tr>
<td>undirected</td>
<td>polynomial</td>
</tr>
<tr>
<td>integral</td>
<td>unknown</td>
</tr>
<tr>
<td>strictly fundamental</td>
<td>NP-complete</td>
</tr>
</tbody>
</table>

**Idea for arbitrary/undirected cycle bases**

The set of all (undirected) cycle bases forms a matroid. In particular, a minimum-weight (undirected) cycle basis can be computed by a greedy algorithm. However, the set of all (undirected) cycle bases is too large.
§3.3 Cycles in Periodic Timetabling

The Horton set

Let $G$ be a connected undirected graph with weights $w : E(G) \rightarrow \mathbb{R}_{\geq 0}$. For $v \in V(G)$, let $T_v$ be a shortest path tree w.r.t. $w$ with root $v$.

Definition

The **Horton set** $\mathcal{H}$ of $G$ consists of the following **Horton cycles** of $G$:

$$v \xrightarrow{p_{vi}} i \rightarrow j \xrightarrow{p_{jv}} v,$$

where $v \in V(G)$, $\{i, j\} \in E(G)$, $p_{vi}$ is the unique $v$-$i$-path in $T_v$, $p_{jv}$ is the unique $j$-$v$-path in $T_v$, and $p_{vi}$ and $p_{jv}$ are edge-disjoint.

Remark

The Horton set consists of $O(|V(G)||E(G)|)$ cycles, and can be computed in polynomial time.

Theorem (Horton, 1987)

$\mathcal{H}$ contains a minimum-weight cycle basis w.r.t. $w$. It is computed by the greedy algorithm on $\mathcal{H}$.
3.3 Cycles in Periodic Timetabling

**Minimum-weight undirected cycle basis algorithm**

Let $G$ be a connected undirected graph with weights $w : E(G) \to \mathbb{R}_{\geq 0}$.

**Horton’s Algorithm**

1. Compute shortest-path trees $T_v$ w.r.t. $w$ for all $v \in V(G)$.
2. Build the Horton set $\mathcal{H}$.
3. Sort $\mathcal{H}$ by weight $w$ in ascending order.
4. Set $B := \emptyset$.
5. For all cycles $\gamma \in \mathcal{H}$ in ascending order:
   - Add $\gamma$ to $B$.
   - If $B$ is linearly dependent over $\mathbb{F}_2$, then remove $C$.
   - If $\#B = \mu(G)$, then return $B$.

**Remark**

This computes a minimum-weight cycle basis in an undirected graph. For directed graphs, the cycle basis may be computed first on the underlying undirected graph $|G|$, and then be lifted to oriented cycles on $G$. 
Consider a PESP instance with period time $T$ on $n$ events and $m$ activities.

**Theorem (Borndörfer/Hoppmann/Karbstein/Lindner, 2015, 2018)**

(1) *There is an algorithm that, given a point $(y, z)$ of the LP relaxation to the cycle & slack-MIP formulation, computes an oriented cycle violating the cycle inequality w.r.t. $(y, z)$ or decides that no such cycle exists. This algorithm runs in $O(Tn^2 m)$ time (i.e., is pseudo-polynomial).*

(2) *There is no strongly polynomial-time algorithm for cycle cut separation unless $P = NP$.***