Mathematical Aspects of Public Transportation Networks

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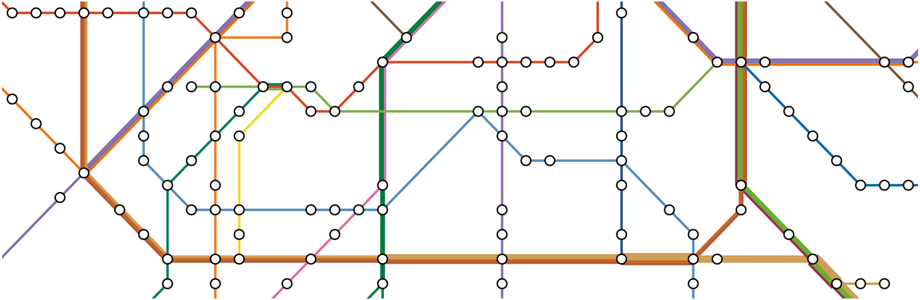

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Chapter 3

Periodic Timetabling

§3.2 Cycle Spaces

Cycle basis names

Let G be a directed graph.

Corollary

If G has n vertices, m edges and c weakly connected components, then $\mu(G) = \mu(|G|) = m - n + c$.

Definition

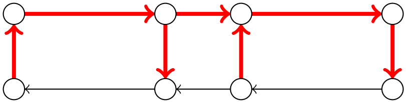
- A cycle basis in G coming from a cycle basis in $|G|$ is called an **undirected cycle basis**.
- A cycle basis in G coming from a spanning tree is called a **strictly fundamental basis**.

Definition

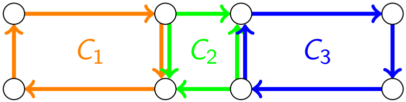
Let $\mathcal{B} = (\gamma_1, \ldots, \gamma_{\mu(G)})$ be a cycle basis. The $(\mu(G) \times m)$-matrix Γ whose rows are given by γ_i, $i = 1, \ldots, \mu(G)$, is called the **cycle matrix** of $\mathcal{B}$.

Cycle basis example

Consider the following digraph G with red spanning tree T:



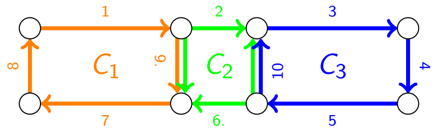
We produce a strictly fundamental cycle basis by taking the oriented cycle for each co-tree edge of T:



The cycles C_1 and C_3 use only forward edges, whereas C_2 uses two backward edges.

Cycle basis example

Label the edges by $1, \ldots, 10$:



Collecting the incidence vectors of C_1, C_2, C_3 yields the 3×10-cycle matrix:

	1	2	3	4	5	6	7	8	9	10
γ_1	1	0	0	0	0	0	1	1	1	0
γ_2	0	1	0	0	0	1	0	0	-1	-1
γ_3	0	0	1	1	1	0	0	0	0	1

Note that the matrix has full row rank. The part corresponding to the co-tree edges $5, 6, 7$ of T is a permutation of the identity matrix.

Determinant of a cycle basis

Let G be a directed graph and let $\mathcal{B}$ be a cycle basis with cycle matrix Γ.

Definition

The **determinant** of $\mathcal{B}$ is defined as

$$\det(\mathcal{B}) := \left| \begin{array}{c} (\mu(G) \times \mu(G))\text{-submatrix of } \Gamma \text{ corresponding to the} \\ \text{co-tree edges of some spanning tree of } G \end{array} \right|.$$

This is well-defined:

Theorem (Liebchen, 2003)

Let T_1, T_2 be two spanning trees of G. For $i = 1, 2$, denote by A_i the $(\mu(G) \times \mu(G))$-submatrix of Γ, where exactly the columns corresponding to $e \notin E(T_i)$ are selected. Then A_1 and A_2 are both invertible and $\det(A_1) = \pm \det(A_2)$.

Determinant of a cycle basis

Proof.

Let Φ be the cycle matrix of a strictly fundamental cycle basis of G coming from the spanning tree T_1. The rows of Φ are indexed by the $\mu := \mu(G)$ co-tree edges of T_1. We have

$$\Phi_{e,e'} = \begin{cases} 1 & \text{if } e = e', \\ 0 & \text{if } e \neq e', \end{cases} \qquad \text{for all } e, e' \notin E(T).$$

Note that we can always lift a fundamental cycle in such a way that the co-tree edge becomes a forward edge. In particular, if Φ_1 denotes the restriction of Φ to the columns corresponding to co-tree edges of T_1, then Φ_1 is the identity matrix.

Since Φ and $\mathcal{B}$ are bases, there is an invertible $(\mu \times \mu)$-matrix S such that $\Gamma = S \cdot \Gamma_\Phi$. It follows that $A_1 = S \cdot \Phi_1$ is invertible. This holds analogously for A_2.

Determinant of a cycle basis

Proof (cont.)

Let Φ_2 denote the restriction of Φ to the columns corresponding to the co-tree edges of T_2. Then $A_2 = S \cdot \Phi_2$, so it remains to show that $\det(\Phi_2) = \pm\det(\Phi_1) = \pm 1$. We use induction on $\#E(T_1)\Delta E(T_2)$.

$\#E(T_1)\Delta E(T_2) = 0$: This is equivalent to $E(T_1) = E(T_2)$, where obvioulsy $\det(\Phi_2) = \det(\Phi_1)$.

$\#E(T_1)\Delta E(T_2) > 0$: Let $e_1 \in E(T_1) \setminus E(T_2)$. On the unique path in T_2 connecting the endpoints of e_1, there must be an edge $e_2 \notin E(T_1)$, as otherwise T_1 would contain a cycle. The fundamental cycle of e_1 in T_1 uses e_2, so that $\Phi_{e_1,e_2} = \pm 1$. Since there is only one fundamental cycle for T_1 using the co-tree edge e_2, this means that $\Phi_{e,e_2} = 0$ for $e \neq e_1$. Use Laplace expansion along the column e_2. □

Characterization by determinant

Let G be a digraph with cyclomatic number μ and cycle basis $\mathcal{B}$.

Theorem (Liebchen/Rizzi, 2007)

(1) *$\mathcal{B}$ is undirected if and only if* $\det(\mathcal{B})$ *is odd.*

(2) *$\mathcal{B}$ is strictly fundamental if and only if the cycle matrix of $\mathcal{B}$ can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last μ columns.*

Proof.

(2) Exercise. (1) Let Γ be the cycle matrix of $\mathcal{B}$. Write $\Gamma = S \cdot \Phi$, where S is an invertible $\mu \times \mu$-matrix and Φ is the matrix of a strictly fundamental basis for some spanning tree T. Restricting to the co-tree edges, we obtain $\Gamma|_{\text{co-tree}} = S \cdot \Phi|_{\text{co-tree}} = S$, so $\det(\mathcal{B}) = \det(S)$. If $\det(\mathcal{B})$ is odd, then S is invertible over $\mathbb{F}_2$, so the rows of Γ mod 2 form a cycle basis for $|G|$. Conversely, if $\mathcal{B}$ is undirected, then $\Gamma|_{\text{co-tree}}$ is invertible mod 2, so that also S is invertible mod 2 and hence $\det(\mathcal{B})$ is odd. □

More on the determinant

Let G be a digraph with cyclomatic number μ.

Lemma (Liebchen/Peeters, 2003)

Let Γ be the cycle matrix of a cycle basis for G, and let A be any $\mu \times \mu$-submatrix of Γ. Then A is invertible if and only if the columns of A correspond to the co-tree edges of some spanning tree of G.

Proof.

($\Leftarrow$) Let Φ be the cycle matrix of a strictly fundamental basis for some spanning tree T. As before, $\Gamma = S \cdot \Phi$ for some invertible $\mu \times \mu$-matrix S. Let A be the submatrix of Γ corresponding to the co-tree edges of T. Then $A = \Gamma|_{\text{co-tree}} = S \cdot \Phi_{\text{co-tree}} = S$, so that A is invertible.

($\Rightarrow$) Suppose that A is invertible. Let $H = \{e_1, \ldots, e_\mu\} \subseteq E(G)$ such that the columns of A correspond to H. Then any cycle γ can be written as $\gamma^t = \lambda^t \Gamma$ for some $\lambda \in \mathbb{Q}^\mu$, as Γ is a cycle basis. If γ contains no edge of H, then $0 = (\gamma_{e_1}, \ldots, \gamma_{e\mu}) = \lambda^T A$, so that $\lambda = 0$ as A is invertible, and $\gamma = 0$. In particular, $E(G) \setminus H$ has no cycle and is thus a spanning tree. □

Integral cycle bases

Let G be a digraph with cyclomatic number μ.

Definition

A cycle basis $\mathcal{B} = \{\gamma_1, \ldots, \gamma_\mu\}$ is called **integral** if every incidence vector γ of an oriented cycle in G can be written as

$$\gamma = \sum_{i=1}^{\mu} \lambda_i \gamma_i, \quad \text{where } \lambda_1, \ldots, \lambda_\mu \in \mathbb{Z}.$$

Theorem (Liebchen/Peeters, 2003)

The following are equivalent for a cycle basis $\mathcal{B}$ with cycle matrix Γ:

(1) *$\mathcal{B}$ is integral,*

(2) *every $\mu \times \mu$-submatrix of Γ has determinant 0 or ± 1,*

(3) $\det(\mathcal{B}) = 1$.

Integral cycle bases

Proof.

(2) $\Leftrightarrow$ (3): by preceding lemma.

(1) $\Rightarrow$ (2): Let T be a spanning tree, giving rise to a strictly fundamental cycle basis with matrix Φ. Then $\Phi = S \cdot \Gamma$ for some invertible $\mu \times \mu$-matrix S. Since $\mathcal{B}$ is integral, S has integer entries. Let A be the $\mu \times \mu$-submatrix of Γ restricted to the co-tree edges of T. Then $S \cdot A$ is the identity matrix. Since S and A have both integer determinants multiplying to 1, we have $\det(A) = \pm 1$.

(3) $\Rightarrow$ (1): For an arbitrary incidence vector γ there is a $\lambda \in \mathbb{Q}^\mu$ such that $\gamma^t = \lambda^t \Gamma$ (cycle basis property). Restricting to the co-tree edges $\{e_1, \ldots, e_\mu\}$ of a spanning tree yields $(\gamma_{e_1}, \ldots, \gamma_{e_\mu}) = \lambda^t A$ for the suitable submatrix A of Γ. Since A has determinant ± 1 by (3), it has an integer inverse and hence $\lambda^t = (\gamma_{e_1}, \ldots, \gamma_{e_\mu}) A^{-1}$ is integer. □

Summary

Let G be a directed graph.

Classes of cycle bases

arbitrary	$\det \neq 0$	
$\supsetneq$		
undirected	$\det \equiv_2 1$	
$\supsetneq$		
integral	$\det = 1$	
$\supsetneq$		
strictly fundamental	$\det = 1$	$+$ identity matrix condition

Examples for the strict inclusion: Last tutorial and Problem Set 6.

Chapter 3

Periodic Timetabling

§3.3 Cycles in Periodic Timetabling

Back to PESP

Input

- event-activity network $\mathcal{E} = (V, E)$,
- period time $T \in \mathbb{N}$,
- lower bound vector $\ell \in (\mathbb{R}_{\geq 0})^E$, $0 \leq \ell < T$,
- upper bound vector $u \in (\mathbb{R}_{\geq 0})^E$, $\ell \leq u < T - \ell$,
- weight vector $w \in (\mathbb{R}_{\geq 0})^E$

MIP formulation

$$
\begin{array}{llll}
\text{Minimize} & \displaystyle\sum_{ij \in E} w_{ij} x_{ij} & & \\
\text{s.t.} & x_{ij} = \pi_j - \pi_i + p_{ij} T, & ij \in E, & \\
 & \ell_{ij} \leq x_{ij} \leq u_{ij}, & ij \in E, & \text{(periodic tension)} \\
 & 0 \leq \pi_i \leq T - 1, & i \in V, & \text{(periodic timetable)} \\
 & p_{ij} \in \mathbb{Z}, & ij \in E. & \text{(periodic offset)}
\end{array}
$$

Cycle periodicity constraints

Theorem (Nachtigall, 1994; Liebchen/Peeters, 2002)

Consider a PESP instance, and let $x \in \mathbb{R}^E$. The following are equivalent:

(1) *There exists a periodic timetable $\pi \in [0, T)^V$ such that for all $ij \in E$ exist $p_{ij} \in \mathbb{Z}$ such that $x_{ij} = \pi_j - \pi_i + p_{ij} T$.*

(2) *For each oriented cycle γ in $\mathcal{E}$ exists $z_\gamma \in \mathbb{Z}$ such that $\gamma^t x = z_\gamma T$.*

(3) *For each integral cycle basis $\{\gamma_1, \ldots, \gamma_\mu\}$ of $\mathcal{E}$, there are $z_1, \ldots, z_\mu \in \mathbb{Z}$ such that $\gamma_i^t x = z_i T$ for all $i = 1, \ldots, \mu$.*

Proof.

(1) $\Rightarrow$ (2): Let $\gamma \in \{-1, 0, 1\}^E$ be the incidence vector of an oriented cycle $(v_1, \ldots, v_k, v_1)$. If γ uses $(v_i, v_{i+1}) \in E$ forward, then

$$\gamma_{v_i,v_{i+1}} x_{v_i,v_{i+1}} = \pi_{v_{i+1}} - \pi_{v_i} + p_{v_i,v_{i+1}} T.$$

Otherwise, if γ uses (v_{i+1}, v_i) backward, then

$$\gamma_{v_{i+1},v_i} x_{v_{i+1},v_i} = \pi_{v_{i+1}} - \pi_{v_i} - p_{v_{i+1},v_i} T.$$

Hence $\gamma^t x = T \gamma^t p$, and clearly $\gamma^t p \in \mathbb{Z}$.

Cycle periodicity constraints

Proof (cont.)

(2) $\Rightarrow$ (3): Trivial. (3) $\Rightarrow$ (2): Let γ be the incidence vector of an arbitrary oriented cycle. Since $\{\gamma_1, \ldots, \gamma_\mu\}$ is an integral cycle basis, there are $\lambda_1, \ldots, \lambda_\mu \in \mathbb{Z}$ such that $\gamma = \sum_{i=1}^{\mu} \lambda_i \gamma_i$. In particular

$$\gamma^t x = \sum_{i=1}^{\mu} \lambda_i \gamma_i^t x = \sum_{i=1}^{\mu} \lambda_i z_i T = \left(\sum_{i=1}^{\mu} \lambda_i z_i \right) \cdot T \quad \in \mathbb{Z} \cdot T.$$

(2) $\Rightarrow$ (1): Let T be a spanning tree of $\mathcal{E}$, and pick a vertex $s \in V(T)$. Then there is a unique oriented path from s to each other vertex $v \in V(T)$. Each oriented path in $\mathcal{E}$ can be expressed as an incidence vector in $\{-1, 0, 1\}^E$ as in the case of cycles. Set $\pi_s := 0$ and $\pi_v := p_{sv}^t x$ for all $v \in V(T) \setminus \{s\}$, where p_{sv} is the unique oriented s-v-path in T. If $ij \in E(T)$, then $p_{sj} = p_{si} + e_{ij}$, so that

$$\pi_j - \pi_i = e_{ij}^t x = x_{ij} = x_{ij} + 0 \cdot T.$$

Cycle periodicity constraints

Proof (cont.)

if $ij \in E \setminus E(T)$ is a co-tree edge, then this yields a fundamental cycle γ. The cycle γ uses the edge ij and then the unique path from j to i in T. The incidence vector of this path is simply given by $p_{si} - p_{sj}$, so that $\gamma = p_{si} - p_{sj} + e_{ij}$. Hence

$$\pi_j - \pi_i = p_{sj}^t x - p_{si}^t x = e_{ij}^t x - \gamma^t x = x_{ij} + z_\gamma T,$$

and we can set $p_{ij} := z_\gamma$.

Finally, reduce π modulo T. □

Corollary

A feasible periodic timetable π can be constructed from a feasible periodic tension x using a spanning tree.

Cycle-based PESP MIP formulation

In the PESP MIP formulation, we can now replace the constraints

$$x_{ij} = \pi_j - \pi_i + p_{ij} T, \quad p_{ij} \in \mathbb{Z}$$

by choosing an integral cycle basis $\{\gamma_1, \ldots, \gamma_\mu\}$ and requiring

$$\gamma_i^t x = z_i T, \quad z_i \in \mathbb{Z}.$$

New MIP formulation (cycle & tension)

Let Γ be the cycle matrix of an integral cycle basis for $\mathcal{E}$.

$$\begin{aligned} &\text{Minimize} && \sum_{ij \in E} w_{ij} x_{ij} && \\ &\text{s.t.} && \Gamma x = zT, && \text{(cycle periodicity)} \\ &&& \ell \leq x \leq u, && \text{(periodic tension)} \\ &&& z \in \mathbb{Z}^\mu. && \text{(cycle offset)} \end{aligned}$$

This uses less constraints and variables than the original formulation.

Cycle-and-slack-based PESP MIP formulation

Definition

The **periodic slack** is $y := x - \ell$.

Remark

If a periodic timetable π is given, then $y_{ij} = [\pi_j - \pi_i - \ell_{ij}]_T$.

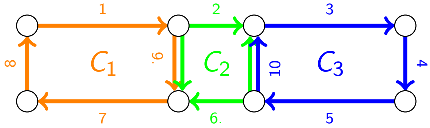
This gives rise to an equivalent MIP formulation, minimizing the total slack:

New MIP formulation (cycle & slack)

$$
\begin{aligned}
&\text{Minimize} && \sum_{ij \in E} w_{ij} y_{ij} && \\
&\text{s.t.} && \Gamma(y + \ell) = zT, && \text{(cycle periodicity)} \\
& && 0 \leq y \leq u - \ell, && \text{(periodic slack)} \\
& && z \in \mathbb{Z}^{\mu}. && \text{(cycle offset)}
\end{aligned}
$$

Example

Consider the following PESP instance ($T = 10$):



Bounds and weights:

	1	2	3	4	5	6	7	8	9	10
ℓ	7	3	6	2	6	3	7	2	3	3
u	7	12	6	11	6	12	7	11	12	12
w	0	1	0	1	0	1	0	1	1	1

Cycle matrix:

	1	2	3	4	5	6	7	8	9	10
γ_1	1	0	0	0	0	0	1	1	1	0
γ_2	0	1	0	0	0	1	0	0	-1	-1
γ_3	0	0	1	1	1	0	0	0	0	1

Example

The cycle basis is integral (even strictly fundamental). In the cycle & slack-formulation, this yields the following:

$$
\begin{aligned}
\text{Minimize} \quad & y_2 + y_4 + y_6 + y_8 + y_9 + y_{10} \\
\text{s.t.} \quad & y_1 + y_7 + y_8 + y_9 - 10z_1 = -19, && \text{(cycle periodicity for } \gamma_1) \\
& y_2 + y_6 - y_9 - y_{10} - 10z_2 = 0, && \text{(cycle periodicity for } \gamma_2) \\
& y_3 + y_4 + y_5 + y_{10} - 10z_3 = -17, && \text{(cycle periodicity for } \gamma_3) \\
& y_1, y_3, y_5, y_7 = 0, && \text{(periodic slack, driving)} \\
& 0 \leq y_2, y_4, y_6, y_8, y_9, y_{10} \leq 9, && \text{(periodic slack, transfer)} \\
& z_1, z_2, z_3 \in \mathbb{Z}. && \text{(cycle offset)}
\end{aligned}
$$

We may omit the fixed y-variables (i.e., the ones for the driving activities), giving a MIP with 3 integer and 6 continuous variables, and 3 constraints.

Optimal sol.: $y_2 = y_6 = y_9 = y_{10} = z_2 = 0$, $y_4 = 3$, $y_8 = 1$, $z_1 = z_3 = 2$, minimal slack: 4.

Offset variable bounds

Question

Recall that in the old timetable-based formulation, we could w.l.o.g. achieve that the periodic offsets satisfy $p_{ij} \in \{0, 1, 2\}$. What about the cycle offsets in the cycle-based formulation?

Definition

For a PESP instance, define the **offset space** as

$$P_{\text{offset}} := \{z \in \mathbb{Z}^{\mu} \mid \exists y \in \mathbb{R}^{E} : 0 \leq y \leq u - \ell,\ \Gamma(y + \ell) = Tz\}.$$

Theorem (Odijk, 1996)

If $z \in P_{offset}$, then any cycle $\gamma^t = \lambda^t \Gamma$ satisfies the **cycle inequality**

$$\left\lceil \frac{\gamma_+^t \ell - \gamma_-^t u}{T} \right\rceil \leq \lambda^t z \leq \left\lfloor \frac{\gamma_+^t u - \gamma_-^t \ell}{T} \right\rfloor .$$

Conversely, if for given $z \in \mathbb{Z}^{\mu}$, the cycle inequality holds for each oriented cycle γ, then $z \in P_{offset}$.

Cycle inequality

Notation

Each incidence vector γ of an oriented cycle decomposes as $\gamma = \gamma_+ - \gamma_-$, where $\gamma_+ \in \{0,1\}^E$ ("forward part") and $\gamma_- \in \{0,1\}^E$ ("backward part").

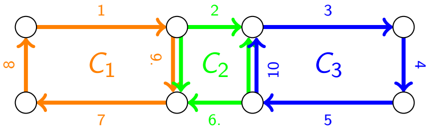
Example: $(1,1,0,0,-1,-1) = (1,1,0,0,0,0) - (0,0,0,0,1,1)$.

Remark

Odjik's theorem gives a strategy to generate valid inequalities for PESP: All integer solutions satisfy the cycle inequality for all cycles. If an LP solver finds a fractional solution and there is a cycle γ violating the cycle inequality, then we can add the cycle inequality for γ as additional constraint and solve again.

Cycle inequality: Example

Consider the following PESP instance ($T = 10$):



Bounds:

	1	2	3	4	5	6	7	8	9	10
ℓ	7	3	6	2	6	3	7	2	3	3
u	7	12	6	11	6	12	7	11	12	12

Cycle inequalities:

$$2 = \lceil (7+7+2+3)/10 \rceil \leq z_1 \leq \lfloor (7+7+11+12)/10 \rfloor = 3$$
$$-1 = \lceil (3+3-12-12)/10 \rceil \leq z_2 \leq \lfloor (12+12-3-3)/10 \rfloor = 1$$
$$2 = \lceil (6+2+6+3)/10 \rceil \leq z_3 \leq \lfloor (6+11+6+12)/10 \rfloor = 3$$

$\rightarrow$ bounds for the cycle offset variables.

Cycle inequality

Proof (⇒).

Let $z \in P_{\text{offset}}$ and let $\gamma^t = \lambda^t \Gamma$ be an oriented cycle. Since $\lambda^t z$ is integer, it suffices to prove

$$\frac{\gamma_+^t \ell - \gamma_-^t u}{T} \leq \lambda^t z \leq \frac{\gamma_+^t u - \gamma_-^t \ell}{T}.$$

Since $z \in P_{\text{offset}}$, we find $0 \leq y \leq u - \ell$ such that $\Gamma(y + \ell) = Tz$. This implies $\gamma_+^t y \geq 0$ and $\gamma_-^t y \leq \gamma_-^t (u - \ell)$, and therefore

$$\gamma^t(y+\ell) = \gamma_+^t y - \gamma_-^t y + \gamma_+^t \ell - \gamma_-^t \ell \geq \gamma_-^t(\ell - u) + \gamma_+^t \ell - \gamma_-^t \ell = \gamma_+^t \ell - \gamma_-^t u.$$

On the other hand, $\gamma_+^t y \leq \gamma_+^t (u - \ell)$ and $\gamma_-^t y \geq 0$, so that

$$\gamma^t(y+\ell) = \gamma_+^t y - \gamma_-^t y + \gamma_+^t \ell - \gamma_-^t \ell \leq \gamma_+^t(u - \ell) + \gamma_+^t \ell - \gamma_-^t \ell = \gamma_+^t u - \gamma_-^t \ell.$$

Putting this together,

$$\gamma_+^t \ell - \gamma_-^t u \leq \gamma^t(y + \ell) \leq \gamma_+^t u - \gamma_-^t \ell.$$

Finally note $\lambda^t z = \lambda^t \Gamma(y + \ell)/T = \gamma^t(y + \ell)/T$.

Cycle inequality

Proof ($\Leftarrow$).

Given $z \in \mathbb{Z}^\mu$ such that the cycle inequality holds for each oriented cycle, we have to show that there is $0 \leq y \leq u - \ell$ with $\Gamma(y + \ell) = Tz$.

Let $\gamma^t = \lambda^t \Gamma$ and let $p \in \mathbb{Z}^E$ be an integer solution of $\Gamma p = z$ (integral cycle basis). By the cycle inequality,

$$\lambda^t z = \lambda^t \Gamma p = \gamma^t p \leq (\gamma_+^t u - \gamma_-^t \ell)/T.$$

Define $\ell' := \ell - pT$ and $u' := u - pT$. Then the above inequality reads as

$$\gamma_+^t u' - \gamma_-^t \ell' \geq 0.$$

Let $\mathcal{E}'$ be the network obtained from $\mathcal{E}$ by adding to each edge $ij \in E$ its anti-parallel edge ji. For each edge $ij \in E(\mathcal{E}')$ set

$$w_{ij} := \begin{cases} u'_{ij} & \text{if } ij \in E, \\ -\ell'_{ji} & \text{if } ji \in E. \end{cases}$$

Cycle inequality

Proof (cont.)

We claim that every directed cycle in $\mathcal{E}'$ has non-negative weight. Indeed, if $\tilde{\gamma}$ is such a cycle, then

$$w^t \tilde{\gamma} = \sum_{ij \in \tilde{\gamma}\colon ij \in E} u'_{ij} + \sum_{ij \in \tilde{\gamma}\colon ji \in E} (-\ell_{ij}) = \gamma_+^t u' - \gamma_-^t \ell' \geq 0,$$

where γ is the corresponding oriented cycle in $\mathcal{E}$ using the edges $ij \in E$ forward and the $ji \in E$ backward.

This implies that the shortest path problem in $(\mathcal{E}', w)$ behaves well. In particular, there is a potential $\pi \in \mathbb{R}^V$ such that

$$\pi_j - \pi_i \leq w_{ij} \text{ for all } ij \in E(\mathcal{E}').$$

Taking π to $\mathcal{E}$, we have

$$\pi_j - \pi_i \leq u'_{ij} \quad \text{and} \quad \pi_i - \pi_j \leq -\ell'_{ij} \quad \text{for all } ij \in E.$$

Cycle inequality

Proof (cont.)

This means

$$\ell_{ij} \leq \pi_j - \pi_i + p_{ij} T \leq u_{ij} \quad \text{for all } ij \in E.$$

In particular, if we set

$$y_{ij} := \pi_j - \pi_i + p_{ij} T - \ell_{ij}, \quad ij \in E,$$

then obviously $0 \leq y \leq u - \ell$. Moreover

$$\Gamma(y + \ell) = T \cdot \Gamma p = Tz,$$

as for each oriented cycle $\gamma = (v_1, \ldots, v_k, v_1)$, the potential differences $\pi_2 - \pi_1, \ldots, \pi_k - \pi_{k-1}, \pi_1 - \pi_k$ along γ sum up to 0. □

Cycle inequality: Applications

There are two applications for the cycle inequalities to PESP:

- give bounds on the integer variables in the cycle-based MIP, thereby reducing the search space for optimal solutions
  $\rightarrow$ find an integral cycle basis minimizing the possible values for the integer variables
  $\rightarrow$ *minimum-weight cycle basis*
- add violated cycle inequalities as cutting planes in a LP-based MIP solving procedure
  $\rightarrow$ give an algorithm that checks if there is a violated cycle inequality
  $\rightarrow$ *separation of cycle cuts*

Minimum-weight cycle basis

Let G be a digraph with a weight vector $d \in {\mathbb{R}_{\geq 0}}^{E(G)}$.

Definition

The **minimum weight cycle basis** problem is to find a cycle basis $\{\gamma_1, \ldots, \gamma_\mu\}$ of G such that

$$\sum_{i=1}^{\mu} \sum_{e \in E(G)} \gamma_{i,e} d_e$$

is minimal.

Application to PESP

For a cycle γ, denote by a_γ and b_γ the lower and upper bounds of the cycle inequality for γ, respectively. Then using $\gamma_1, \ldots, \gamma_\mu$ for the MIP formulation produces

$$\prod_{i=1}^{\mu} (b_{\gamma_i} - a_{\gamma_i} + 1)$$

possible combinations of values for the integer variables $z_1, \ldots, z_m$.

Minimum-weight cycle basis

However, this is not a weight vector.

Lemma

$$\sum_{e \in E(G)} \frac{|\gamma_e|(u_e - \ell_e)}{T} \leq b_\gamma - a_\gamma < 2 + \sum_{e \in E(G)} \frac{|\gamma_e|(u_e - \ell_e)}{T}$$

Proof.

$$\begin{aligned} b_\gamma - a_\gamma + 1 &= \left\lceil \frac{\gamma_+^t u - \gamma_-^t \ell}{T} \right\rceil - \left\lfloor \frac{\gamma_+^t \ell - \gamma_-^t u}{T} \right\rfloor \\ &< \frac{\gamma_+^t u - \gamma_-^t \ell}{T} + 1 - \left(\frac{\gamma_+^t \ell - \gamma_-^t u}{T} - 1 \right) \\ &= 2 + \frac{\gamma_+^t (u - \ell) + \gamma_-^t (u - \ell)}{T} = 2 + \sum_{e \in E} \frac{|\gamma_e|(u_e - \ell_e)}{T}, \end{aligned}$$

$$b_\gamma - a_\gamma + 1 \geq \frac{\gamma_+^t u - \gamma_-^t \ell}{T} - \frac{\gamma_+^t \ell - \gamma_-^t u}{T} = \sum_{e \in E} \frac{|\gamma_e|(u_e - \ell_e)}{T}.$$

Minimum-weight cycle basis

As a compromise, compute the minimum weight undirected cycle basis for the weight vector $d := u - \ell$.

Complexity of finding a minimum cycle basis

class	complexity
arbitrary	polynomial
undirected	polynomial
integral	unknown
strictly fundamental	NP-complete

Idea for arbitrary/undirected cycle bases

The set of all (undirected) cycle bases forms a matroid. In particular, a minimum-weight (undirected) cycle basis can be computed by a greedy algorithm. However, the set of all (undirected) cycle bases is too large.

The Horton set

Let G be a connected undirected graph with weights $w : E(G) \to \mathbb{R}_{\geq 0}$.
For $v \in V(G)$, let T_v be a shortest path tree w.r.t. w with root v.

Definition

The **Horton set** $\mathcal{H}$ of G consists of the following **Horton cycles** of G:

$$v \xrightarrow{p_{vi}} i \to j \xrightarrow{p_{jv}} v,$$

where $v \in V(G)$, $\{i, j\} \in E(G)$, p_{vi} is the unique v-i-path in T_v, p_{jv} is the unique j-v-path in T_v, and p_{vi} and p_{jv} are edge-disjoint.

Remark

The Horton set consists of $\mathcal{O}(|V(G)||E(G)|)$ cycles, and can be computed in polynomial time.

Theorem (Horton, 1987)

$\mathcal{H}$ contains a minimum-weight cycle basis w.r.t. w. It is computed by the greedy algorithm on $\mathcal{H}$.

Minimum-weight undirected cycle basis algorithm

Let G be a connected undirected graph with weights $w : E(G) \to \mathbb{R}_{\geq 0}$.

Horton's Algorithm

1. Compute shortest-path trees T_v w.r.t. w for all $v \in V(G)$.
2. Build the Horton set $\mathcal{H}$.
3. Sort $\mathcal{H}$ by weight w in ascending order.
4. Set $\mathcal{B} := \emptyset$.
5. For all cycles $\gamma \in \mathcal{H}$ in ascending order:
    - Add γ to $\mathcal{B}$.
    - If $\mathcal{B}$ is linearly dependent over $\mathbb{F}_2$, then remove C.
    - If $\#\mathcal{B} = \mu(G)$, then return $\mathcal{B}$.

Remark

This computes a minimum-weight cycle basis in an undirected graph. For directed graphs, the cycle basis may be computed first on the underlying undirected graph $|G|$, and then be lifted to oriented cycles on G.

Separation of cycle cuts

Consider a PESP instance with period time T on n events and m activities.

Theorem (Borndörfer/Hoppmann/Karbstein/Lindner, 2015, 2018)

(1) *There is an algorithm that, given a point (y, z) of the LP relaxation to the cycle & slack-MIP formulation, computes an oriented cycle violating the cycle inequality w.r.t. (y, z) or decides that no such cycle exists.*
This algorithm runs in $\mathcal{O}(Tn^2m)$ time (i.e., is pseudo-polynomial).

(2) *There is no strongly polynomial-time algorithm for cycle cut separation unless $P = NP$.*