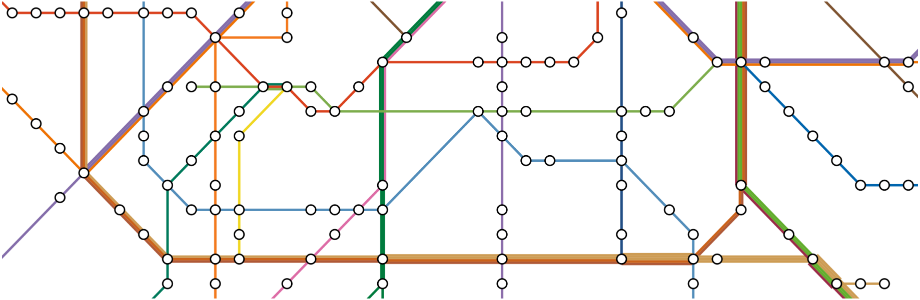
Mathematical Aspects of Public Transportation Networks

Niels Lindner



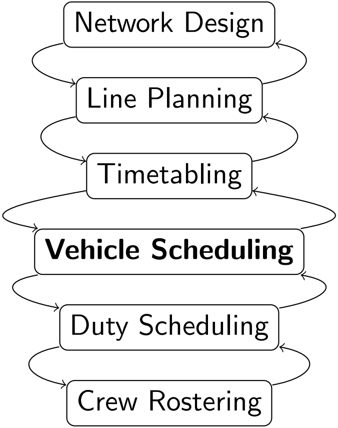
June 18, 2018

Chapter 4

Vehicle Scheduling

§4.1 Periodic Vehicle Scheduling

Public transport planning cycle



strategic planning

operational planning

Vehicle Scheduling: Overview

Main Question

Given a line network with a timetable, how many vehicles are required to operate the timetable?

Scope

- basic: periodic timetables
- better: aperiodic timetables (e.g., for a day)
- realistic: several depots, vehicle types, capacities, ...

All these versions lead to network flow problems. In some scenarios, the minimal number of vehicles is replaced by a more general cost function.

Periodic Vehicle Scheduling

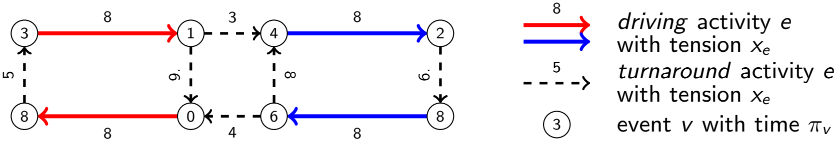
Input data

- event-activity network $\mathcal{E} = (V, E)$
- period time $T \in \mathbb{N}$
- periodic timetable $\pi : V \to [0, T)$
- periodic tensions $x : E \to \mathbb{R}_{\geq 0}$ such that $x_{ij} \equiv_T \pi_j - \pi_i$ for all $ij \in E$

Question

How many vehicles are required to operate the periodic timetable π on $\mathcal{E}$?

Example



Periodic vehicle schedules

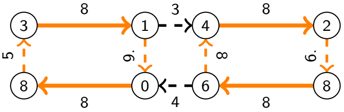
Definition

A **periodic vehicle schedule** is a collection S of directed cycles in $\mathcal{E}$ such that each driving activity is contained in at least (exactly) one cycle of S. For a periodic vehicle schedule S, its **number of vehicles** $\nu(S)$ is given by

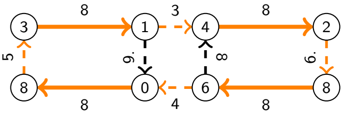
$$\nu(S) := \frac{1}{T} \sum_{e \in E} \sum_{i=1}^{k} \gamma_{i,e} x_e,$$

where $\gamma_1, \dots, \gamma_k \in \{0,1\}^E$ are the incidence vectors of the cycles in S.

Example



$30 + 30$ minutes $\rightarrow$ 6 vehicles



50 minutes $\rightarrow$ 5 vehicles

Remark

By the cycle periodicity property of periodic timetabling, $\nu(S) \in \mathbb{Z}_{\geq 0}$.

Minimum tension circulation

Definition

The **periodic vehicle scheduling problem** is to find a periodic vehicle schedule S with minimal $\nu(S)$.

Tension-based integer programming formulation

$$
\begin{aligned}
&\text{Minimize} && \frac{1}{T}\sum_{e \in E} x_e f_e && \\
&\text{s.t.} && \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0, && v \in V, \\
& && f_e \geq 1, && e \in E \text{ driving activity}, \\
& && f_e \in \mathbb{Z}_{\geq 0}, && e \in E.
\end{aligned}
$$

Observation

The periodic vehicle scheduling problem can be formulated as a mininum cost circulation problem.

Minimum offset circulation

Definition (Recall from periodic timetabling)

For an edge $ij \in E$, define its **periodic offset** as

$$p_{ij} := \frac{x_{ij} - \pi_j + \pi_i}{T} \in \mathbb{Z}_{\geq 0}.$$

Cycle periodicity property

For all incidence vectors γ of oriented cycles in $\mathcal{E}$ holds $\gamma^t x = T \cdot \gamma^t p$.

Offset-based integer programming formulation

$$\begin{array}{lrr}
\text{Minimize} & \displaystyle\sum_{e \in E} p_e f_e & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0, & v \in V, \\
 & f_e \geq 1, & e \in E \text{ driving activity}, \\
 & f_e \in \mathbb{Z}_{\geq 0}, & e \in E.
\end{array}$$

Perfect turnaround matching

Let E_d and E_t be the set of driving and turnaround activities, respectively.

Theorem

There is a one-to-one correspondence

$$\left\{ \begin{array}{c} \textit{circulations covering all} \\ \textit{driving activities exactly once} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{perfect matchings} \\ \textit{in } (V, E_t) \end{array} \right\},$$

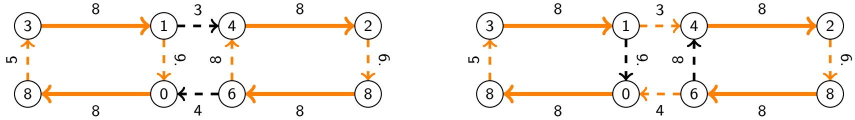
$$(f_e)_{e \in E} \mapsto (f_e)_{e \in E_t}.$$

Moreover, a circulation of cost c w.r.t. p (or x) corresponds to a perfect matching of cost $c - \sum_{e \in E_d} p_e$ w.r.t. p (or $c - \sum_{e \in E_d} x_e$ w.r.t. x).

Proof.

The correspondence has been an exercise. For the cost comparison, note that restricting a circulation to the turnaround activities removes the cost of all driving activities. $\square$

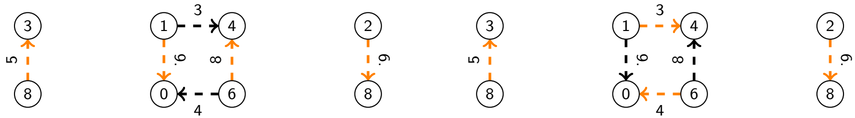
Perfect turnaround matching: Example



circulation cost w.r.t. x: 60

circulation cost w.r.t. x: 50

$\updownarrow$ $\updownarrow$



matching cost w.r.t. x: 28

driving act. cost w.r.t. x: 32

matching cost w.r.t. x: 18

driving act. cost w.r.t. x: 32

Summary

The periodic vehicle scheduling problem has the following interpretations:

- minimum cost circulation w.r.t. periodic tension x covering all driving activities
- minimum cost circulation w.r.t. periodic offset p covering all driving activities
- minimum weight perfect matching w.r.t. periodic tension x of turnaround activities
- minimum weight perfect matching w.r.t. periodic offset p of turnaround activities

The graph (V, E_t) usually decomposes into many small components, so that the perfect matching problem decomposes into smaller problems as well.

Chapter 4

Vehicle Scheduling

§4.2 Aperiodic Vehicle Scheduling

Single-Depot Vehicle Scheduling

Input Data

- set $\mathcal{T}$ of trips $(\tau_{\mathsf{dep}}, \tau_{\mathsf{arr}}) \in \mathbb{R} \times \mathbb{R}$ with $\tau_{\mathsf{dep}} < \tau_{\mathsf{arr}}$
- relation $\preceq$ on $\mathcal{T} \times \mathcal{T}$, where $t_1 \preceq t_2$ holds if a vehicle can use trip t_2 after t_1

Definition

A **vehicle schedule** is a collection $S = \{s_1, \ldots, s_k\}$ of chains $s_i = t_{i,1} \preceq t_{i,2} \preceq \cdots \preceq t_{i,r_i}$ such that each trip in $\mathcal{T}$ occurs in at least (exactly) one chain s_i. The number $\nu(S) := k$ is the **number of vehicles** of S.

Definition

The **(single-depot) vehicle scheduling problem** is to find a vehicle schedule S minimizing $\nu(S)$.

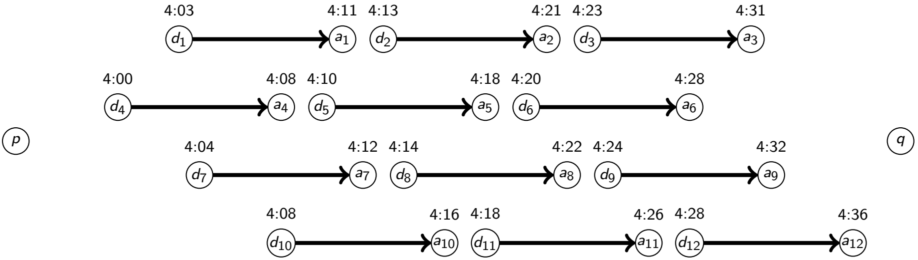
Network flow model

Build an event-activity network $\mathcal{E}$ as follows:

1. Create two events p and q.
2. Create activities $(p, d_t), (d_t, a_t), (a_t, q)$ for each trip $t \in \mathcal{T}$ (*pull-out, driving, pull-in*).
3. For each pair $t_1 \preceq t_2$, add an activity (a_{t_1}, d_{t_2}) (*turnaround*).

The events p and q are *depot* vertices.

Example



Network flow model

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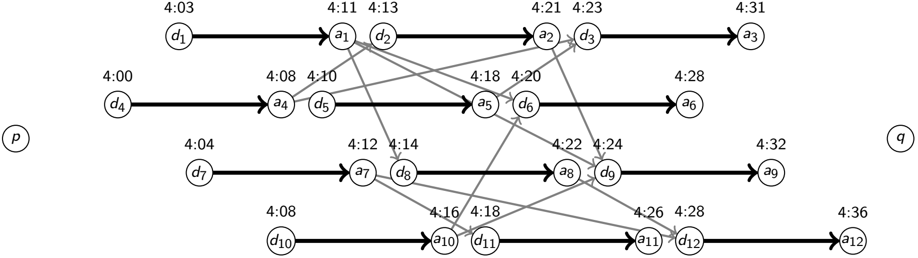

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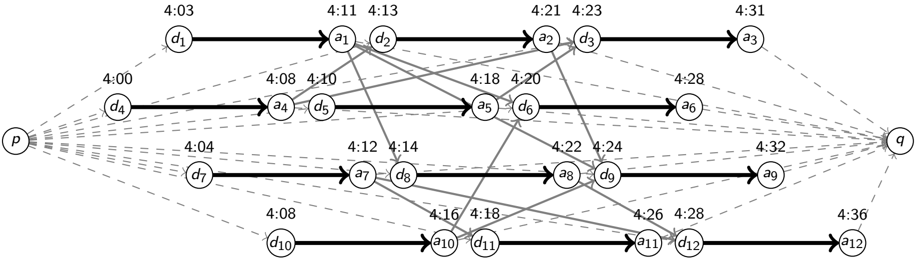
Example



Network flow model

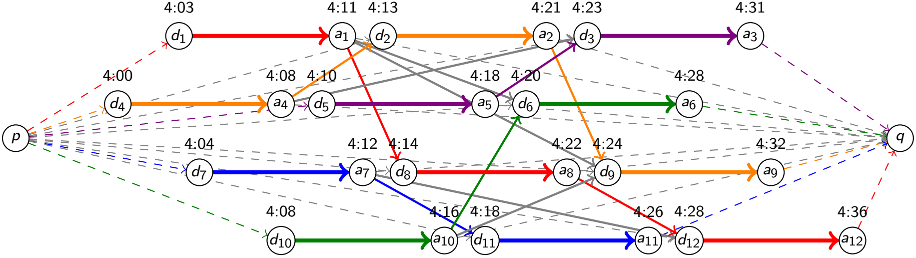
Observation

The single-depot vehicle scheduling problem is solved by finding a minimum value p-q-flow on $\mathcal{E} = (V, E)$ covering each driving activity at least once:

$$\begin{aligned}
\text{Minimize} \quad & \sum_{e \in \delta^+(p)} f_e & & \\
\text{s.t.} \quad & \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0, & & v \in V \setminus \{p, q\}, \\
& f_e \geq 1, & & e \in E \text{ driving activity}, \\
& f_e \in \mathbb{Z}_{\geq 0}, & & e \in E.
\end{aligned}$$

Network flow model: Example

Example



This is an optimal p-q-flow covering each driving activity exactly once. The value of flow (= number of vehicles) is 5.

Notation

Let E_d and E_t denote the set of driving and turnaround activities, respectively.

Matching interpretation

Lemma

The following numbers are equal:

(a) *The minimum value of an p-q-flow covering each $e \in E_d$ exactly once.*

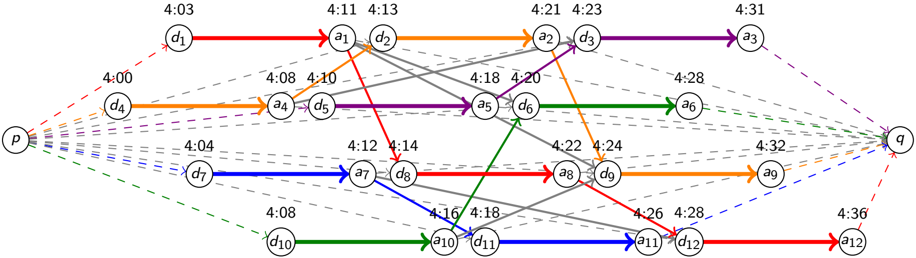
(b) $|E_d| - |M|$, *where M is a maximum cardinality matching of (V, E_t).*

Proof.

A feasible flow with value ν decomposes into ν edge-disjoint p-q-paths, where the activity types along each path have the pattern (pull-out, driving, turnaround, driving, turnaround, ..., driving, pull-in). So each path with k driving activities uses $k-1$ turnaround activities. Summing over all paths yields a matching M of the turnaround activities with $|M| = |E_d| - \nu$.

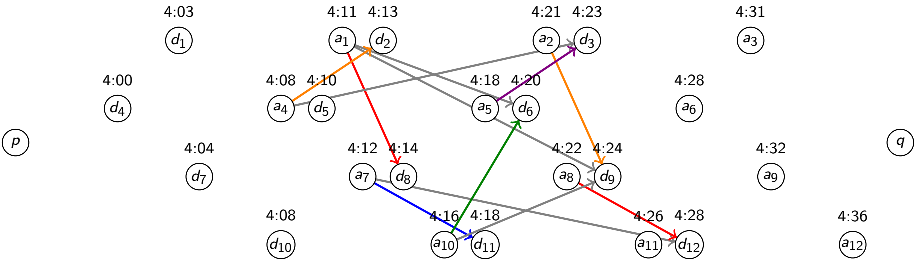
Conversely, let M be a matching of (V, E_t). Consider the flow of value $|E_d|$ using the $|E_d|$ paths (p, d_t, a_t, q) for all trips t. For each edge in M, connect the corresponding trips, thereby reducing the flow value by 1. Repeating this process yields a feasible flow of value $|E_d| - |M|$. □

Matching interpretation: Example



In (V, E_t), only the 7 arrival vertices $a_1, a_2, a_4, a_5, a_7, a_8, a_9, a_{10}$ are non-isolated. All of these vertices are matched, so that we obtain a maximum cardinality (in fact, even perfect) matching. As there are 12 driving activities, the minimal number of vehicles equals $12 - 7 = 5$.

Matching interpretation: Example



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Single-depot case: Summary

Summary

The single-depot vehicle scheduling can be solved by computing a ...

- minimum value network flow covering all driving activities
- maximum cardinality matching of the turnaround activities

Remarks

- The actual timetable and the actual travel times are not important, only the feasible sequences of trips matter.
- When costs for trips or turnarounds come into play, this generalizes to a minimum *cost* network flow or a *weighted* matching problem.

Multi-Depot Vehicle Scheduling

Input Data

- set $\mathcal{T}$ of trips $(\tau_{\text{dep}}, \tau_{\text{arr}}) \in \mathbb{R} \times \mathbb{R}$ with $\tau_{\text{dep}} < \tau_{\text{arr}}$
- relation $\preceq$ on $\mathcal{T} \times \mathcal{T}$, where $t_1 \preceq t_2$ holds if a vehicle can use trip t_2 after t_1
- set $\mathcal{D}$ of depots
- assignment $D : \mathcal{T} \to \mathcal{P}(\mathcal{D})$ of feasible depots for each trip

Definition

The **multi-depot vehicle scheduling problem** is to find a vehicle schedule S minimizing $\nu(S)$ such that for each chain $t_1 \preceq \cdots \preceq t_r$ in S holds $D(t_1) \cap \cdots \cap D(t_r) \neq \emptyset$.

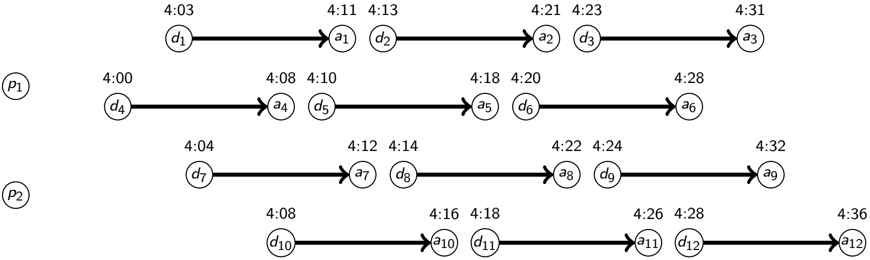
I.e., the trips served by a vehicle must be feasible for a common depot. In particular, we can assume that the pull-out and pull-in depots of each vehicle are the same.

Network flow model

Build an event-activity network $\mathcal{E}$ as follows:

1. Create two *depot* vertices p_d and q_d for each depot $d \in \mathcal{D}$.
2. Add *driving* activities (d_t, a_t) for each trip $t \in \mathcal{T}$.
3. Add *pull-out* activities (p_d, d_t) for each trip $t \in \mathcal{T}$ and each $d \in D(t)$.
4. Add *pull-in* activities (a_t, q_d) for each trip $t \in \mathcal{T}$ and each $d \in D(t)$.
5. For each pair $t_1 \preceq t_2$, add a *turnaround* activity (a_{t_1}, d_{t_2}).

Example (2 depots)



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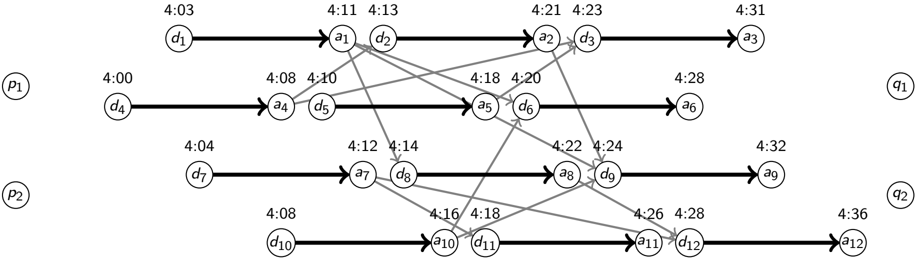

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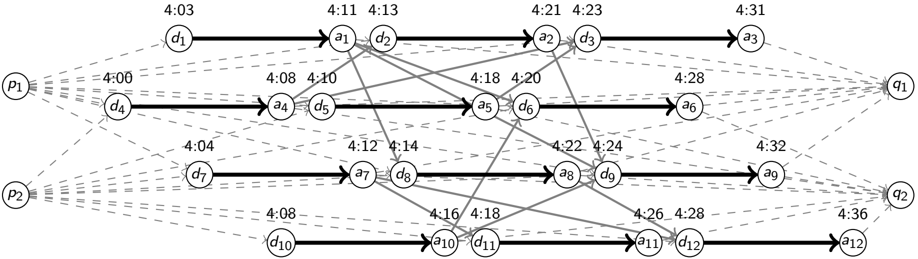


Network flow model

Observations

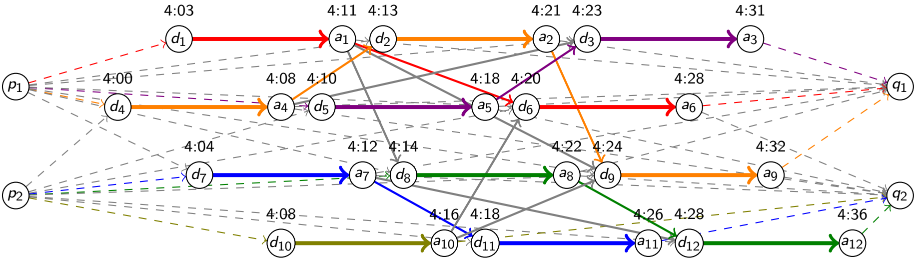
- This is not a normal network flow problem: In a flow with the p_d as sources and the q_d as sinks, a vehicle might pull out from depot 1 and pull in to depot 2.
- Instead, our flow covering all driving activities needs to decompose into p_d-q_d-flows for each depot $d \in \mathcal{D}$.
- This leads to a *multi-commodity flow*.

Multi-commodity flow model

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{d \in \mathcal{D}} \sum_{e \in \delta^+(p_d)} f_e^d \\
\text{s.t.} \quad & \sum_{e \in \delta^+(v)} f_e^d - \sum_{e \in \delta^-(v)} f_e^d = 0, && d \in \mathcal{D}, v \in V \setminus \{p_d, q_d\}, \\
& \sum_{d \in D(t)} f_e^d = 1, && e \in E \text{ driving activity of trip } t, \\
& \sum_{d \notin D(t)} f_e^d = 0, && e \in E \text{ driving activity of trip } t, \\
& f_e^d \in \{0, 1\}, && d \in \mathcal{D}, e \in E.
\end{aligned}
$$

This defines a p_d-q_d flow f^d for each depot $d \in \mathcal{D}$. Each driving activity is covered by exactly one such f^d, and d is feasible for the corresponding trip.

Multi-commodity flow: Example



This optimal 2-commodity flow decomposes into 3 p_1-q_1-paths and 3 p_2-q_2-paths ($\rightarrow$ 6 vehicles required).

General multi-commodity flow

Let $G = (V, E)$ be a digraph with cost functions $c^1, \ldots, c^k : E \to \mathbb{R}$, balances $b^1, \ldots, b^k : V \to \mathbb{R}$, and capacities $u, u^1, \ldots, u^k : E \to \mathbb{R}_{\geq 0}$. The problem

$$
\begin{array}{lrlr}
\text{Minimize} & \displaystyle\sum_{i=1}^{k} \sum_{e \in E} c_e^i f_e^i & & \\
\text{s.t.} & \displaystyle\sum_{e \in \delta^+(v)} f_e^i - \sum_{e \in \delta^-(v)} f_e^i = b_v^i, & & i = 1, \ldots, k, v \in V, \\
 & \displaystyle\sum_{i=1}^{k} f_e^i \leq u_e, & & e \in E, \\
 & f_e^i \in \{0, 1, \ldots, u_e^i\}, & & i = 1, \ldots, k, e \in E.
\end{array}
$$

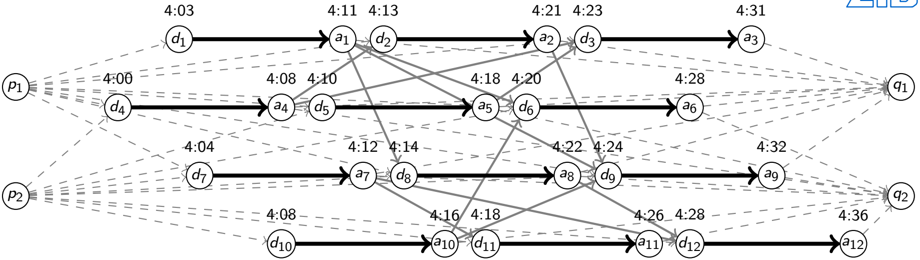
is called an integer **minimum cost k-commodity flow problem**.

Multi-commodity flow: Complexity

Remarks

- When the flows f^i can be relaxed to rational numbers in $[0, u^i]$, then there are polynomial-time algorithms (linear programming).
- In fact, for rational f^i, there are strongly polynomial-time algorithms. I.e., the running time does not depend on cost, balance or capacities.
- However, for $k \geq 2$ commodities, the total unimodularity property of single-commodity flows gets lost. In particular, we cannot use linear programming to obtain integer flows.
- There are non-integral minimum cost 2-commodity flows with integer costs, balances and capacities.
- Finding an integer k-commodity flow is NP-hard for every fixed $k \geq 2$ (Even/Itai/Shamir 1974: SAT $\leq$ integer 2-commodity flow).

Path-based multi-commodity flow/Set partitioning



Observation

A feasible multi-commodity flow consists certain p_d-q_d-paths. Let $\mathcal{P}_d$ denote the set of all p_d-q_d-paths for a depot d. Then

- Every driving activity of some trip t must be covered by exactly one path $p \in \mathcal{P}_d$ with $d \in D(t)$.
- We want to minimize the number of required paths.

Path-based multi-commodity flow model

For a driving activity e of trip t, let $\mathcal{P}_e := \bigcup_{d \in D(t)} \{p \in \mathcal{P}_d \mid e \in p\}$ denote the set of p_d-q_d-paths using e coming from a feasible depot $d \in D(t)$ for t.

Integer program

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{d \in \mathcal{D}} \sum_{p \in \mathcal{P}_d} f_p \\
\text{s.t.} \quad & \sum_{p \in \mathcal{P}_e} f_p = 1, && e \in E \text{ driving activity of trip } t, \\
& f_p \in \{0, 1\}, && p \in \bigcup_{d \in \mathcal{D}} \mathcal{P}_d.
\end{aligned}
$$

Remarks

- Any multi-commodity flow problem has a path-based formulation.
- The number of paths is enormous. $\rightarrow$ column generation (pricing: shortest path problems).

Extensions

- depot capacities κ_d: In the path-based multi-commodity flow formulation, these are modeled as

$$\sum_{p \in \mathcal{P}_d} f_p \leq \kappa_d, \quad d \in \mathcal{D}.$$

- operational costs (e.g., fuel)
- fixed costs (e.g., maintenance, investment)
- multiple vehicle types: one commodity for each feasible combination of a depot and a vehicle type
- time windows: regular trips (e.g., according to line frequency) vs. irregular trips (e.g., school trips)
- route constraints (e.g., battery vehicles)

Real-world examples

	Berlin (BVG)	Hamburg (HHA)	Hamburg (VHH)
Depots	10	14	10
Vehicle types	9	9	9
Combinations	44	40	19
Trips	25 000	16 000	5 500
Turnarounds	70 000 000	15 100 000	10 000 000

Löbel, Optimal Vehicle Scheduling in Public Transit, 1997

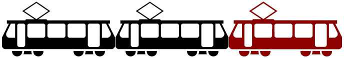
Chapter 4

Vehicle Scheduling

§4.3 Railway Stock Rotation Planning

Terminology

- Find an optimal *periodic* vehicle schedule for a standard week (*rotation*).
- A *vehicle configuration* is a multiset of *vehicles*.



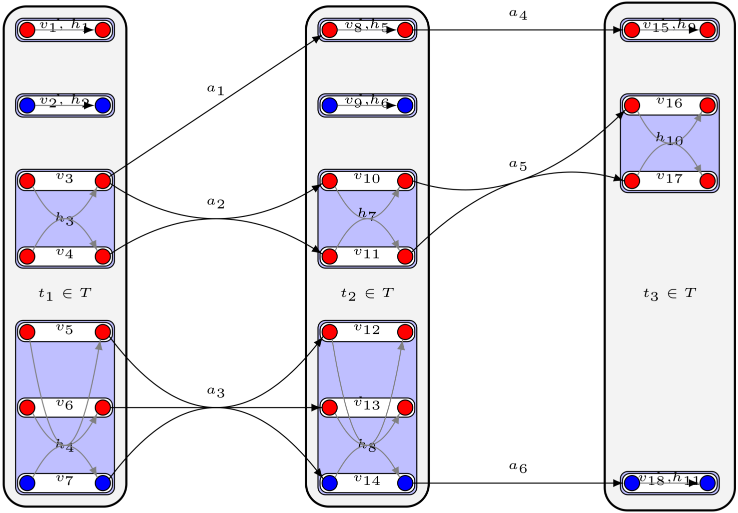
- For each trip, there is a feasible set of *vehicle configurations*.
- Before or after a trip, a vehicle configuration can be changed by *coupling*.
- A *train* is a set of at most seven trips haven the same departure and arrival stops and the same departure and arrival times, but on different days.

Hypergraph model

Define a *hypergraph* $G = (V, H, A)$ as follows:

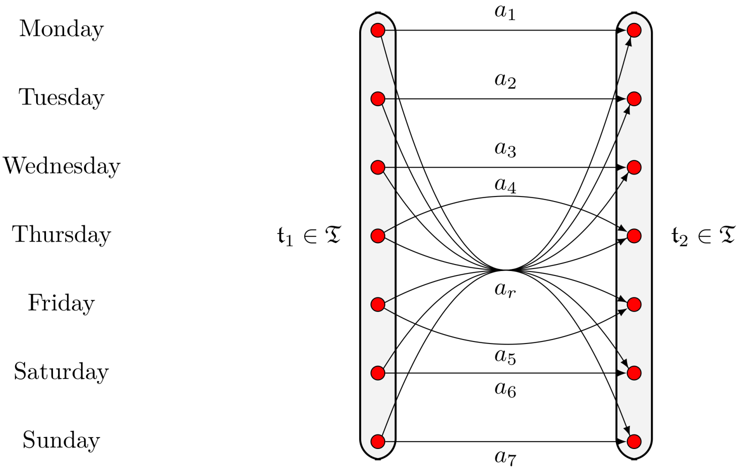
- A *node* $v \in V$ is a tuple (t, c, f, m), where t is a trip, c is a feasible vehicle configuration for t, and f is a vehicle used m times by c.
- A *hypernode* $h \in H$ is a collection $V(t, c)$ of all nodes (t, c, f, m) for a given trip t and a given configuration c.
- A *hyperarc* $a \in A$ is a non-empty set of pairs $(v, w) \in V \times V$, constructed as follows:
  - *Configuration conserving arcs*: If a vehicle can go from trip t_1 to trip t_2 with the same configuration, then add an hyperarc consisting of $|V(t_1, c)| = |V(t_2, c)|$ arcs connecting them.
  - *Coupling arcs*: Connect trips with different configurations.
  - *Regularity hyperarcs*: Let T_1, T_2 be trains, let c be a configuration and let $o \in \{0, \dots, 6\}$. Let $\mathfrak{a}$ be the set of all arcs connecting any trip from T_1 with any trip of T_2 with configuration c such that midnight is passed o times between the arrival of t_1 and the departure of t_2. If $|\mathfrak{a}| \geq 2$, then add a hyperarc $\{(v, w) \in V \times V \mid \exists a \in \mathfrak{a} : (v, w) \in a\}$.

Hypergraph model: Conservation and coupling



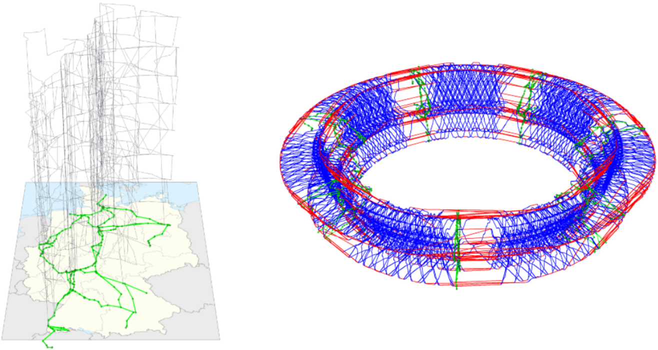
Borndörfer et. al., A Hypergraph Model for Railway Vehicle Rotation Planning, 2011

Hypergraph model: Regularity



Borndörfer et. al., A Hypergraph Model for Railway Vehicle Rotation Planning, 2011

Hypergraph model: Torus



ICE-A network, HyDraw output