Problem Set 5

due: May 28, 2018

Exercise 1

Fix \( T \in \mathbb{N} \). For \( x \in \mathbb{R} \), define \([x]_T\) as the unique \( y \in [0,T)\) with \( x \equiv y \mod T\). Show that:

(a) \([x]_T = x - T\lfloor x/T \rfloor\) for all \( x \in \mathbb{R} \),

(b) \([[x]_T]_T = [x]_T\) for all \( x \in \mathbb{R} \),

(c) \([x + nT]_T = [x]_T\) for all \( n \in \mathbb{Z}, x \in \mathbb{R} \),

(d) \([x + y]_T \leq [x]_T + [y]_T\) for all \( x, y \in \mathbb{R} \),

(e) \(0 \leq \sum_{i=1}^n [x_i]_T - \left[ \sum_{i=1}^n x_i \right]_T \leq (n - 1)T\) for all \( n \in \mathbb{N}, x_1, \ldots, x_n \in \mathbb{R} \).

Exercise 2

Let \( E = (V, E) \) be an event-activity network. Suppose that

(1) \( V = V_{\text{arr}} \cup V_{\text{dep}} \) and \( \#V_{\text{arr}} = \#V_{\text{dep}} \),

(2) \( \#\delta^-(v) = 1 \) for all \( v \in V_{\text{arr}} \) and \( \#\delta^+(v) = 1 \) for all \( v \in V_{\text{dep}} \).

Let \( E_t := E \cap (V_{\text{arr}} \times V_{\text{dep}}) \). Moreover, define a circulation in \( E \) as a disjoint union of directed circuits in \( E \). Prove that there is a bijective map

\[ \{\text{perfect matchings of } (V, E_t)\} \rightarrow \{\text{circulations in } (V, E) \text{ visiting each vertex } v \in V\}. \]

Exercise 3

Find a \( 2 \times 2\)-matrix \( A \) over \( \mathbb{Z}/10\mathbb{Z} \) such that \( \#\{x \in (\mathbb{Z}/10\mathbb{Z})^2 \mid Ax = 0\} = 2 \).

Please turn over!
Exercise 4

Let $\mathcal{E} = (V, E)$ denote the following event-activity network with period time $T = 10$:

Driving activities have a solid line, transfer activities are dashed. The lower bounds $\ell$ and upper bounds $u$ on each activity are indicated as an interval $[\ell_e, u_e]$.

(a) Compute all solutions $\pi \in (\mathbb{Z}/10\mathbb{Z})^8$ satisfying

$$\pi_j - \pi_i \equiv 10 \ell_{ij} \quad \text{for all driving activities } ij \in E.$$ 

(b) Compute an optimal solution to the PESP on $\mathcal{E}$, where each transfer activity has weight 1 and each driving activity has weight 0.