Problem Set 6

due: June 4, 2018

Exercise 1 5 points

Fix an integer $k \geq 4$. Give a polynomial-time reduction from 3-VERTEX-COLORING to $k$-VERTEX-COLORING.

*Hint:* Use a complete graph $K_{k-3}$.

Exercise 2 5 points

Let $G$ be a weakly connected directed graph with cyclomatic number $\mu$. Let $B = \{\gamma_1, \ldots, \gamma_\mu\}$ be a cycle basis for $G$ such that for each $i = 1, \ldots, \mu$, there is an edge $e_i \in E(G)$ such that $\gamma_{i,e_i} = 1$ and $\gamma_{j,e_i} = 0$ for all $j \neq i$.

(a) Prove that $B$ is a strictly fundamental basis.

(b) Show that an arbitrary cycle basis is strictly fundamental if and only if its cycle matrix can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last columns.

Exercise 3 10 points

Consider for $n \in \mathbb{N}$ the following directed graph $G_n$:

$$
V(G_n) := \{I_k \mid k = 0, \ldots, n-1\} \cup \{O_k \mid i = 0, \ldots, n-1\},
$$

$$
E(G_n) := \{(O_k, O_{k+1}) \mid k = 0, \ldots, n-1\} \cup \{(I_k, I_{k+2}) \mid k = 0, \ldots, n-1\}
\cup \{(O_k, I_k) \mid k = 0, \ldots, n-1\}.
$$

(a) Draw $G_5$.

(b) Compute the cyclomatic number $\mu(G_n)$ for all $n \geq 5$.

(c) For $k = 0, \ldots, n-1$, let $C_k$ denote the oriented cycle

$$(I_k, I_{k+2}, O_{k+2}, O_{k+3}, O_{k+4}, \ldots, O_k, I_k).$$

Show that

$$
B_n := \{C_k \mid k = 0, \ldots, n-1\} \cup \{(I_0, I_2, I_4, \ldots, I_0)\}
$$

is a cycle basis for $G_n$, where $n \geq 5$ is odd.

(d) Compute the determinant of $B_5$.

(e) [5 additional points] Compute the determinant of $B_n$ for any odd $n \geq 5$. 