Dr. Niels Lindner
ZIB 3007 (Rundbau)
lindner@zib.de

Problem Set 6

due: June 4, 2018

Exercise 1 **5 points**

Fix an integer $k \geq 4$. Give a polynomial-time reduction from 3-VERTEX-COLORING to k-VERTEX-COLORING.

Hint: Use a complete graph K_{k-3}.

Exercise 2 **5 points**

Let G be a weakly connected directed graph with cyclomatic number μ. Let $\mathcal{B} = \{\gamma_1, \dots, \gamma_\mu\}$ be a cycle basis for G such that for each $i = 1, \dots, \mu$, there is an edge $e_i \in E(G)$ such that $\gamma_{i,e_i} = 1$ and $\gamma_{j,e_i} = 0$ for all $j \neq i$.

(a) Prove that $\mathcal{B}$ is a strictly fundamental basis.

(b) Show that an arbitrary cycle basis is strictly fundamental if and only if its cycle matrix can be permuted in such a way that it has the $\mu \times \mu$-identity matrix in its last columns.

Exercise 3 **10 points**

Consider for $n \in \mathbb{N}$ the following directed graph G_n:

$$\begin{aligned} V(G_n) &:= \{I_k \mid k = 0, \dots, n-1\} \cup \{O_k \mid i = 0, \dots, n-1\}, \\ E(G_n) &:= \{(O_k, O_{[k+1]_n}) \mid k = 0, \dots, n-1\} \cup \{(I_k, I_{[k+2]_n}) \mid k = 0, \dots, n-1\} \\ &\quad \cup \{(O_k, I_k) \mid k = 0, \dots, n-1\}. \end{aligned}$$

(a) Draw G_5.

(b) Compute the cyclomatic number $\mu(G_n)$ for all $n \geq 5$.

(c) For $k = 0, \dots, n-1$, let C_k denote the oriented cycle

$$(I_k, I_{[k+2]_n}, O_{[k+2]_n}, O_{[k+3]_n}, O_{[k+4]_n}, \dots, O_k, I_k).$$

Show that

$$\mathcal{B}_n := \{C_k \mid k = 0, \dots, n-1\} \cup \{(I_0, I_2, I_4, \dots, I_0)\}$$

is a cycle basis for G_n, where $n \geq 5$ is odd.

(d) Compute the determinant of $\mathcal{B}_5$.

(e) **[5 additional points]** Compute the determinant of $\mathcal{B}_n$ for any odd $n \geq 5$.