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Problem Set 8

due: June 18, 2018

Exercise 1 **7 points**

Let $G = (V, E)$ be a weakly connected directed graph with incidence matrix $A \in \{-1, 0, 1\}^{V \times E}$. Further let Γ be a cycle matrix of an arbitrary cycle basis of G.

(a) Show that $\Gamma \cdot A^T = 0$.

(b) Let $a_1, \dots, a_k$ be the columns of A corresponding to the edges $e_1, \dots, e_k \in E$. Prove that $a_1, \dots, a_k$ are $\mathbb{Q}$-linearly dependent if and only if the subgraph of G induced by $e_1, \dots, e_k$ contains an oriented cycle.

(c) Conclude that, as a $\mathbb{Q}$-vector space, $\mathbb{Q}^E$ is isomorphic to the direct sum of the row spaces of A and Γ.

Exercise 2 **7 points**

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Define the polyhedra

$$P(A, b) := \{x \in \mathbb{R}^n \mid Ax \leq b\}, \text{ and}$$
$$Q(A, b) := \{x \in \mathbb{R}^n \mid \lambda^T Ax \leq \lfloor \lambda^T b \rfloor \text{ for all } \lambda \geq 0 \text{ with } \lambda^T A \in \mathbb{Z}^n\}.$$

(a) Prove that $Q(A, b) \subseteq P(A, b)$.

(b) Show that $P(A, b) \cap \mathbb{Z}^n \subseteq Q(A, b)$.

(c) Let $(\mathcal{E}, T, \ell, u, w)$ be a PESP instance on n events and m activities. Let $B \in \{-1, 0, 1\}^{m \times n}$ be the transpose of the incidence matrix of $\mathcal{E}$, and denote by I_m the $m \times m$-identity matrix. Define

$$A := \begin{pmatrix} B & T \cdot I_m \\ -B & -T \cdot I_m \end{pmatrix} \in \mathbb{R}^{2m \times (n+m)}, \quad b := \begin{pmatrix} u \\ -\ell \end{pmatrix} \in \mathbb{R}^{2m}.$$

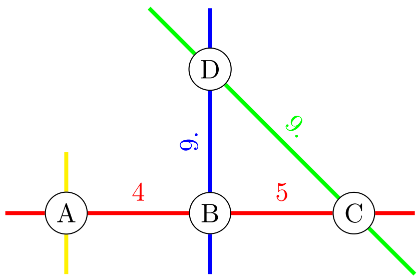
Show that for any $(\pi, p) \in Q(A, b)$ and any oriented cycle γ in $\mathcal{E}$ holds the cycle inequality

$$\gamma^t p \leq \left\lfloor \frac{\gamma_+^t u - \gamma_-^t \ell}{T} \right\rfloor.$$

Please turn over!

Exercise 3 **6 points**

Consider the following line network detail, defining a PESP instance I as follows:



- Each of the line operates in both directions with a frequency of 10 minutes each.
- The travel times between two stations are indicated on the edges.
- The waiting times in A and D should be exactly 2 minutes, the waiting times in B and C should take 0 minutes.
- The minimum transfer time for all 32 transfer activities is 2 minutes.
- Suppose that all transfer activities have weight 1, and all other activities have weight 0.

(a) Show that the weighted periodic slack is at least 88 for every *symmetric* periodic timetable feasible for I.

(b) Find an asymmetric periodic timetable for I where the weighted periodic slack at most 87.