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Problem Set 14

due: February 4, 2020

Exercise 1 **20 points**

Consider the 7×8-matrix

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{pmatrix}.$$

(a) Construct a directed graph $G = (V, E)$ such that A is the vertex-edge incidence matrix of G. Make a planar drawing of G.

(b) Find pairwise distinct vertices $s_1, s_2, t_1, t_2 \in V$ such that the relaxed maximum two-commodity flow problem

$$\begin{aligned} &\text{Maximize} && \sum_{e \in \delta^+(s_1)} f_e^1 + \sum_{e \in \delta^+(s_2)} f_e^2 && \\ &\text{s.t.} && \sum_{e \in \delta^+(v)} f_e^1 - \sum_{e \in \delta^-(v)} f_e^1 = 0, && v \in V \setminus \{s_1, t_1\}, \\ &&& \sum_{e \in \delta^+(v)} f_e^2 - \sum_{e \in \delta^-(v)} f_e^2 = 0, && v \in V \setminus \{s_2, t_2\}, \\ &&& f_e^1 + f_e^2 \leq 1, && e \in E, \\ &&& 0 \leq f_e^1 \leq 1, && e \in E, \\ &&& 0 \leq f_e^2 \leq 1, && e \in E, \end{aligned}$$

on the digraph G from (a) has a non-integral optimal solution. In particular, compute an optimal integral solution (i.e., $f_e^1, f_e^2 \in \{0, 1\}$ for all $e \in E$) and show that the value of flow is less than in your fractional solution.

(c) Compute the dimension of the polytope in $\mathbb{R}^{16}$ defined by the system of linear (in)equalities in (b) for your choice of s_1, s_2, t_1, t_2 satisfying the requirements of (b).

(d) Denote by I the 8×8 identity matrix. Find a regular 16×16 submatrix B of

$$\begin{pmatrix} A & 0 \\ 0 & A \\ I & I \\ I & 0 \\ 0 & I \end{pmatrix}$$

and an integer right-hand side $b \in \{0, 1\}^{16}$ such that $B^{-1}b$ is not integral.

Hint: The fractional solution of (b) defines a vertex of the polytope of (c), which in turn defines a basis matrix similar to Problem Set 11.