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Problem Set 7

due: December 2, 2019

Exercise 1 **6 points**

Let $G = (V, E)$ be a directed simple graph. Pick two distinct vertices $s, t \in V$. Further let $c \in \mathbb{R}^E$ be an *arbitrary* cost vector. Define $b \in \{-1, 0, 1\}^V$ via

$$b_v := \begin{cases} 1 & \text{if } v = s, \\ -1 & \text{if } v = t, \\ 0 & \text{otherwise,} \end{cases}$$

and consider the integer program

$$(\star) \qquad \begin{aligned} &\text{Minimize} && \sum_{e \in E} c_e x_e \\ &\text{s.t.} && \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} x_e = b_v, && v \in V, \\ &&& x_e \in \{0, 1\}, && e \in E. \end{aligned}$$

Here, we define for an arbitrary set $S \subseteq V$ the edge sets

$$\delta^+(S) := \{(v, w) \in E \mid v \in S, w \notin S\},$$
$$\delta^-(S) := \{(v, w) \in E \mid v \notin S, w \in S\},$$

and for a vertex $v \in V$, we write $\delta^\pm(v)$ for $\delta^\pm(\{v\})$.

(a) Characterize all feasible solutions to the integer program $(\star)$.

(b) Why is an optimal solution to $(\star)$ not necessarily an incidence vector of a shortest s-t-path?

(c) Prove that a feasible solution x to $(\star)$ is an incidence vector of an s-t-path if and only if

$$\sum_{e \in \delta^+(s)} x_e = 1,$$
$$\sum_{e \in \delta^+(t)} x_e = 0,$$
$$\sum_{e \in \delta^+(S)} x_e \geq \sum_{e \in \delta^+(v)} x_e \quad \text{for all } S \subsetneq V \text{ s.t. } s \in S, \text{ for all } v \in V \setminus S.$$

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Exercise 2 **8 points**

Let $G = (V, E)$ be an undirected graph with cost vector $c \in \mathbb{R}^E_{\geq 0}$. A *perfect 2-matching* in G is a feasible solution to the integer program

$$
(\triangle) \qquad \begin{aligned} \text{Minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(v)} x_e = 2, && v \in V, \\ & x_e \in \{0, 1, 2\}, && e \in E, \end{aligned}
$$

where $\delta(v)$ is the set of edges in E incident to $v \in V$. An optimal solution to $(\triangle)$ is called a *minimum cost perfect 2-matching.*

(a) Prove that the cost of a minimum cost perfect 2-matching on a complete graph is a lower bound on the cost of an optimal TSP tour.

(b) Let x be a feasible solution to $(\triangle)$. Show that $\{e \in E \mid x_e = 1\}$ is a union of vertex-disjoint circuits.

(c) Prove that if G is bipartite and $(\triangle)$ is feasible, then there is always a minimum cost perfect 2-matching x with $x_e \in \{0, 2\}$ for all $e \in E$, i.e., x is a "double perfect matching".

(d) Construct a non-bipartite graph G with a cost vector $c \in \mathbb{R}^E_{\geq 0}$ such that G has a perfect matching, and no perfect 2-matching x with $x_e \in \{0, 2\}$ for all $e \in E$ is of minimum cost.

Exercise 3 **6 points**

Let $G = (V, E)$ be a connected undirected graph with cost vector $c \in \mathbb{R}^E_{\geq 0}$ and let $T \subseteq V$ be of even cardinality. Formulate an integer linear program with $|V| + |E|$ integer variables that solves the *minimum cost T-join problem* on (G, c), and prove its correctness.