Dr. Niels Lindner (lindner@zib.de)
Pedro Maristany de las Casas (maristany@zib.de)

Problem Set 9

due: December 16, 2019

Exercise 1 **10 points**

Consider a the linear program $\min\{c^T x \mid Ax = b,\ x \geq 0\}$ for a matrix $A \in \mathbb{R}^{(m,n)}$, and vectors $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$. The set of feasible points/solutions build a polyhedron, which we denote by $P(A, b)$.
The support $\operatorname{supp}(x)$ of a vector $x \in \mathbb{R}^n$ is the set of indices corresponding to the non-zero entries of x, i.e.,

$$\operatorname{supp}(x) := \{i \in [n] \mid x_i \neq 0\}.$$

(a) Prove that the following statements are equivalent for $x \in P(A, b)$:

(1) x is a vertex of the polyhedron $P(A, b)$.

(2) $|\operatorname{supp}(x)| = rk(A_{\cdot,\operatorname{supp}(x)})$.

(3) There is a regular submatrix $B \in \mathbb{R}^{r,r}$ of A, $r = rk(A)$, s.t. $Bx_B = b$ and x_B contains all non-zero entries of x. Here, $x_B \in \mathbb{R}^r$ is the vector built out of the original vector x by picking the entries corresponding to the indices of the columns of A that are present in B.

Hint: To prove (1) $\Rightarrow$ (2) you may consider the matrix $\begin{pmatrix} A \\ I_{J,\cdot} \end{pmatrix}$, where $J := [n] \setminus \operatorname{supp}(x)$.

For the rest of the exercise, assume that $rk(A) = m$ and that $x \in P(A, b)$ is a vertex with a matrix B as in (3).

(b) Let $\hat{x} \in P(A, b)$ be feasible. Then $B^{-1}A\hat{x} = x$.

(c) Assume that x has a non-integer entry x_i for some $i \in \{1, \ldots, n\}$. Let $\hat{x} \in P(A, b)$ be an integer point. For $y \in \mathbb{R}$ define the *fractional part* of y as $\{y\} := y - \lfloor y \rfloor \in [0, 1)$. Prove that the inequality

$$\left\{(B^{-1}A)_{i,\cdot}\right\} \hat{x} \geq \{x_i\}$$

defines a cutting plane separating x from the convex hull of all integer points in $P(A, b)$. These are the *Gomory cuts*.

Exercise 2 **10 points**

Choose your favorite public transportation network and model it as an undirected graph: each station is represented by a vertex and vertices representing directly connected stations are linked via an edge. The travel times – the costs on the edges – c have to be positive integers and should be realistic.
The resulting graph $G = (V, E)$ has to have at least 25 nodes, and at least 10 of them need to have an odd degree of at least 3. You are also allowed to take subnetworks of real-world networks if they meet the size conditions.
The task is to solve the Chinese Postman Problem on (G, c).

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(a) Describe which original network you used and provide a link to a picture showing the original network.

(b) Write an .LP file modeling the Chinese Postman Problem on (G, c) as an integer program. The format has been shown in the last exercise session, and is also described here: `http://lpsolve.sourceforge.net/5.0/CPLEX-format.htm`.

(c) Upload your file to the NEOS Server: `https://neos-server.org/neos/solvers/index.html` and solve the instance with your favorite solver (specify which one).

(d) Write down the optimal Chinese Postman tour and its cost.

Hint: Use the following IP formulation, where $b_v := \deg(v) \bmod 2$, so that $b_v \in \{0, 1\}$:

$$\begin{array}{lll} \text{Minimize} & \sum_{e \in E} c_e x_e & \\ \text{s.t.} & \sum_{e \in \delta(v)} x_e - 2y_v = b_v, & v \in V, \\ & x_e \in \{0, 1\}, & e \in E, \\ & y_v \in \mathbb{Z}_{\geq 0}, & v \in V. \end{array}$$