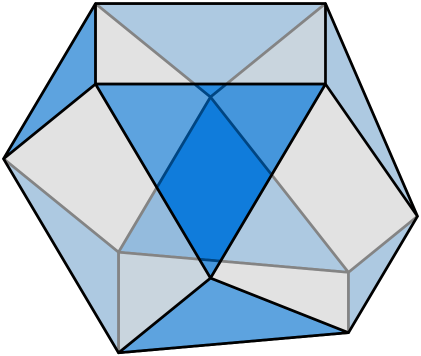

Determining All Integer Vertices of the PESP Polytope by Flipping Arcs

ATMOS 2020 · September 7, 2020

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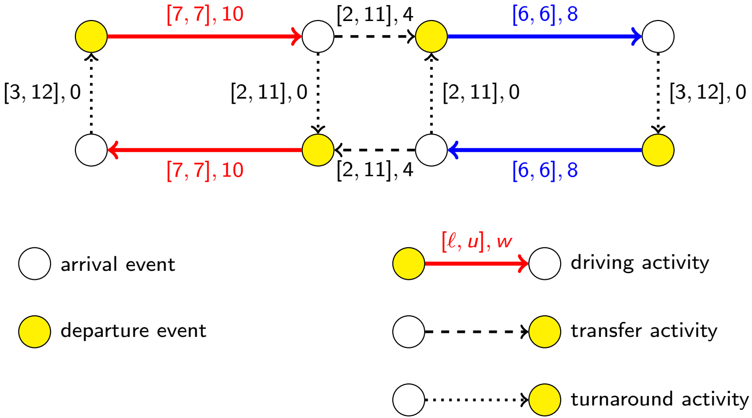
Periodic Timetabling

... is classically modeled by the *Periodic Event Scheduling Problem* (PESP):



PESP instance, period time $T = 10$

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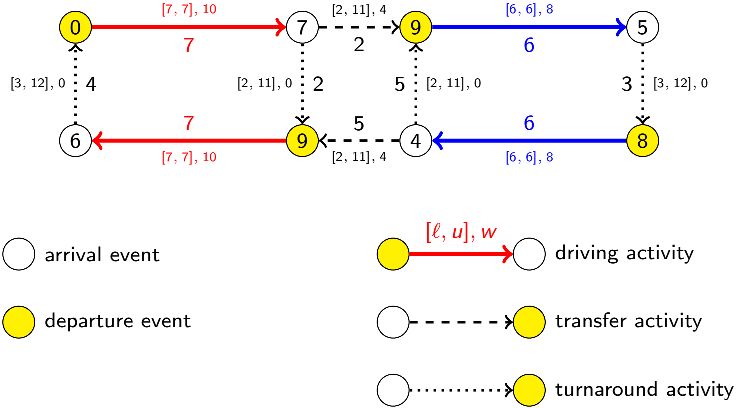

periodic timetable, period time $T = 10$

Periodic Event Scheduling

(Serafini and Ukovich, 1989)

Given

- a weakly connected digraph ("event-activity network") $G = (V, A)$,
- a period time $T \in \mathbb{N}$,
- lower and upper bounds $\ell, u \in \mathbb{Z}^A_{\geq 0}$ with $\ell \leq u$,
- weights $w \in \mathbb{Z}^A_{\geq 0}$,

the **Periodic Event Scheduling Problem (PESP)** is to solve the MIP

$$\begin{aligned}
&\text{Minimize} && \sum_{a \in A} w_a x_a && \\
&\text{s.t.} && x_{ij} = \pi_j - \pi_i + T p_{ij}, && ij \in A, \\
& && \ell_{ij} \leq x_{ij} \leq u_{ij}, && ij \in A, \\
& && 0 \leq \pi_i < T, && i \in V, \\
& && p_{ij} \in \mathbb{Z}, && ij \in A.
\end{aligned}$$

We call any feasible $\pi \in \mathbb{R}^V$ a *periodic timetable*, $x \in \mathbb{R}^A$ a *periodic tension*, and $y := x - \ell$ a *periodic slack*.

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$$\begin{aligned}
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We call any feasible $\pi \in \mathbb{R}^V$ a *periodic timetable*, $x := y + \ell \in \mathbb{R}^A$ a *periodic tension*, and y a *periodic slack*.

Cycle-based MIP Formulation

Theorem (Liebchen, Peeters, 2009)

If $\Gamma \in \mathbb{Z}^{B \times A}$ is an *integral cycle matrix B* of G, i.e., a matrix whose rows are incidence vectors of oriented cycles making up a $\mathbb{Z}$-basis of the cycle space of G, then PESP is equivalent to:

$$\begin{aligned} \text{Minimize} \quad & w^t y \\ \text{s.t.} \quad & \Gamma(y + \ell) = Tz, \\ & 0 \leq y \leq u - \ell, \\ & z \in \mathbb{Z}^B. \end{aligned}$$

Related Polytopes

$$P_{\mathrm{IP}} := \mathrm{conv}\{(y, z) \in \mathbb{R}^A \times \mathbb{Z}^B \mid \Gamma(y + \ell) = Tz, 0 \leq y \leq u - \ell\},$$

convex hull of feasible solutions

$$P_{\mathrm{LP}} := \{(y, z) \in \mathbb{R}^A \times \mathbb{R}^B \mid \Gamma(y + \ell) = Tz, 0 \leq y \leq u - \ell\}.$$

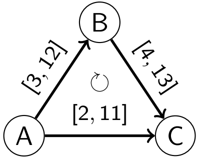
polytope of LP relaxation

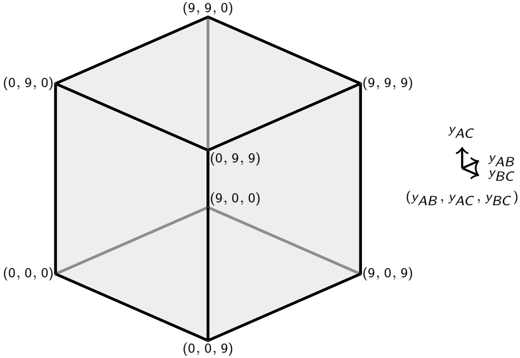
LP Relaxation Polytope

Lemma

$P_{LP} = \{(y, \Gamma(y+\ell)/T) \mid 0 \leq y \leq u - \ell\}$, so the projection to the slack space is combinatorially equivalent to an $|A|$-dimensional cube.

Example





Change-Cycle Inequalities

Theorem (Nachtigall, 1996, 1998)

For any feasible periodic slack y and any oriented cycle γ with positive part γ^+ and negative part γ^- holds the **change-cycle inequality**

$$(T-\alpha)\,\gamma_+^t y + \alpha\,\gamma_-^t y \geq \alpha(T-\alpha), \quad \text{where } \alpha := [-\gamma^t \ell]_T.$$

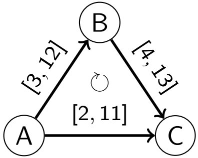
Here, $[\cdot]_T$ denotes the modulo T operator with values in $[0, T)$. The change-cycle inequalities are facet-defining if $\alpha > 0$.

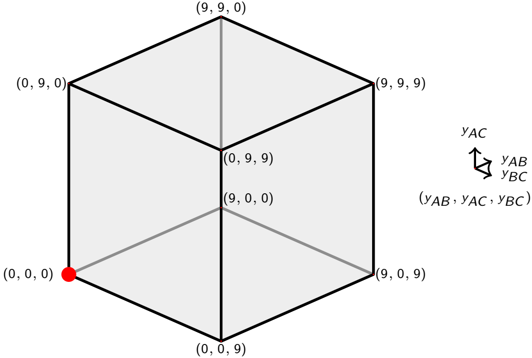
Observation

The optimal solution to the LP relaxation is $y^* = 0$. This is a feasible periodic slack if and only if the change-cycle inequality for $y^* = 0$ holds for any oriented cycle γ.

$\Rightarrow$ Either $y^* = 0$ is optimal, or it is cut off by a change-cycle inequality.

Change-Cycle Inequality: Example

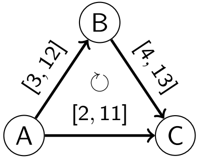


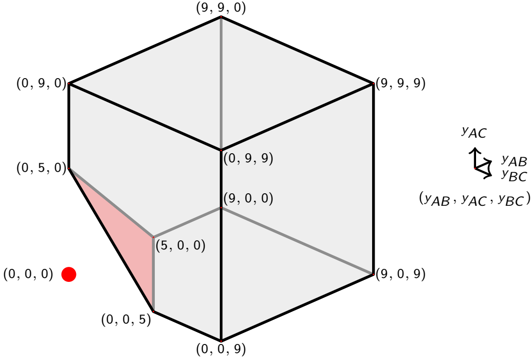


The vector $y^* = (0, 0, 0)$ is an infeasible slack, and it is cut off by the change-cycle inequality

$$5y_{AB} + 5y_{BC} + 5y_{AC} \geq 25.$$

Change-Cycle Inequality: Example





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Flipping Arcs

Flipping an arc $a \in A$: Replace a by an arc $\overline{a}$ in the opposite direction, and set

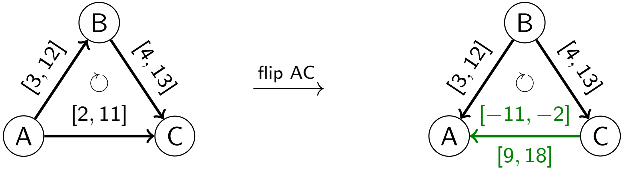
$$\ell_{\overline{a}} := -u_a, \quad u_{\overline{a}} := -\ell_a.$$

plus a suitable integer multiple of T so that $\ell \geq 0$

Observation

A vector $y \in \mathbb{R}^A$ is a feasible periodic slack for the original PESP instance if and only if the vector $\overline{y}$ with $\overline{y}_{\overline{a}} = u_a - \ell_a - y_a$ and agreeing with y otherwise is a feasible periodic slack for the PESP instance where a is flipped.

Example



$y = (y_{AB}, y_{AC}, y_{BC}) = (0, 5, 0)$ $\quad$ $\overline{y} = (\overline{y}_{AB}, \overline{y}_{CA}, \overline{y}_{BC}) = (0, 4, 0)$

Flip Inequalities

Theorem (L&L, 2020)

For any feasible periodic slack y, any oriented cycle γ, and any subset $F \subseteq A$, the following **flip inequality** *is valid:*

$$(T - \alpha_F) \sum_{\substack{a \in A \setminus F: \\ \gamma_a = 1}} y_a + \alpha_F \sum_{\substack{a \in A \setminus F: \\ \gamma_a = -1}} y_a$$

$$+ \alpha_F \sum_{\substack{a \in F: \\ \gamma_a = 1}} (u_a - \ell_a - y_a) + (T - \alpha_F) \sum_{\substack{a \in F: \\ \gamma_a = -1}} (u_a - \ell_a - y_a) \quad \geq \quad \alpha_F (T - \alpha_F),$$

where

$$\alpha_F := \left[- \sum_{a \in A \setminus F} \gamma_a \ell_a - \sum_{a \in F} \gamma_a u_a \right]_T .$$

The flip inequalities are facet-defining if $\alpha_F > 0$.

Proof: Flip all arcs in F and transform the change-cycle inequality back. □

Spanning Tree Solutions

Separating Cube Vertices

Recall that the change-cycle inequalities separate the cube vertex $y^* = 0$ from P_{IP}. Any vertex y^* of P_{LP} yields a flip $F := \{a \in A \mid y^*_a = u_a - \ell_a\}$. In the flipped PESP instance, y^* maps to $\overline{y^*} = 0$. The flip inequalities for F hence separate y^* from P_{IP}.

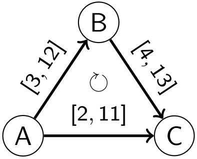
Spanning Tree Solutions

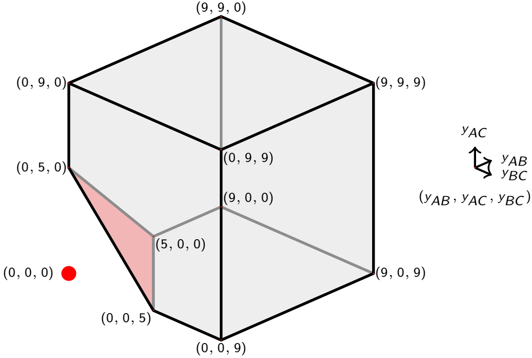
We call a point $(y^*, z^*) \in P_{\text{LP}}$ a **spanning tree solution** if there is a spanning tree S s.t. $y^*_a \in \{0, u_a - \ell_a\}$ for all $a \in S$. The vertices of P_{IP} (Nachtigall, 1998) and the vertices of P_{LP} (trivially) are always spanning tree solutions.

Theorem (L&L, 2020)

Let $(y^*, z^*) \in P_{LP} \setminus P_{IP}$ be a spanning tree solution. Then (y^*, z^*) is separated from P_{IP} by at least one of $2(|A| - |V| + 1)$ explicit flip inequalities.

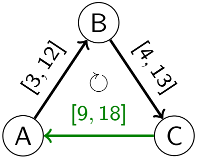
For each co-tree arc a' of the spanning tree S, one can pick the corresponding fundamental cycle, and the two sets $F_1 := \{a \in S \mid y_a = u_a - \ell_a\}$, $F_2 := F_1 \cup \{a'\}$. In particular, infeasible spanning tree sols in P_{LP} can be separated in linear time.

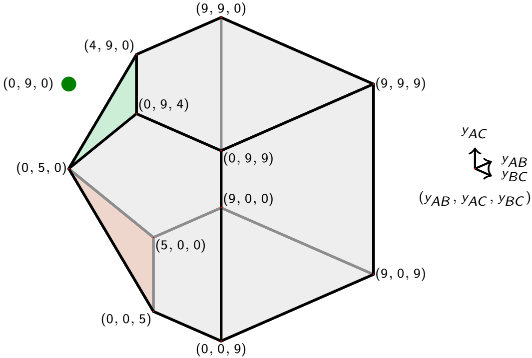




The change-cycle inequality is the flip inequality for $F = \emptyset$ and cuts off $(0, 0, 0)$.

Truncating the Cube



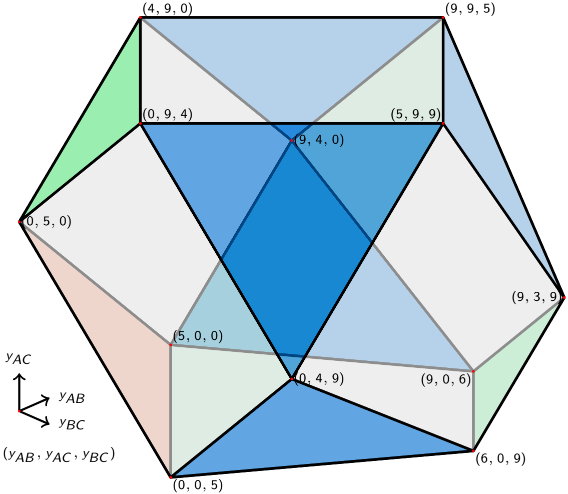


Flip AC: In the flipped PESP instance w.r.t. $F = \{AC\}$, the change-cycle inequality is $6\overline{y}_{AB} + 6\overline{y}_{BC} + 6\overline{y}_{CA} \geq 24$, which in the original instance translates to the flip inequality $6y_{AB} + 6y_{BC} + 6(9 - y_{AC}) \geq 24$. Simplifying, we obtain

$$y_{AB} + y_{BC} - y_{AC} \geq -5.$$

This is one of the two *cycle inequalities* (Odijk, 1994). It cuts off $(0, 9, 0)$.

Truncating the Cube



This is the (projected) *flip polytope* of our example: All 8 cube vertices have been cut off by the corresponding 8 flip inequalities. It is combinatorially equivalent to a *cuboctahedron* with 12 vertices, 24 edges, 14 facets:

- 6 bound ineq. (← cube)
- 2 cycle ineq.
- 1 change-cycle ineq.
- 5 other flip ineq.

Here, the flip inequalities already determine P_{IP}: The vertices of the flip polytope are spanning tree solutions.

The set of feasible slacks is the union of the 3 green polygons, which arise as intersection with the planes $z = \frac{\gamma^t(y+\ell)}{10} = 0, 1, 2$.

Two Theorems on the Flip Polytope

The **flip polytope** P_{flip} is the subpolytope of P_{LP} consisting of all (y, z) such that y satisfies the flip inequalities for all oriented cycles γ and all $F \subseteq A$. Clearly $P_{\text{IP}} \subseteq P_{\text{flip}} \subseteq P_{\text{LP}}$.

Theorem (L&L, 2020)

The vertices of P_{IP} are precisely the integer vertices of P_{flip}.

P_{flip} hence shows a remarkable structure, as it determines all integer vertices: Every vertex of P_{IP} appears as a vertex of P_{flip}, but P_{flip} might contain more vertices.

Theorem (L&L, 2020)

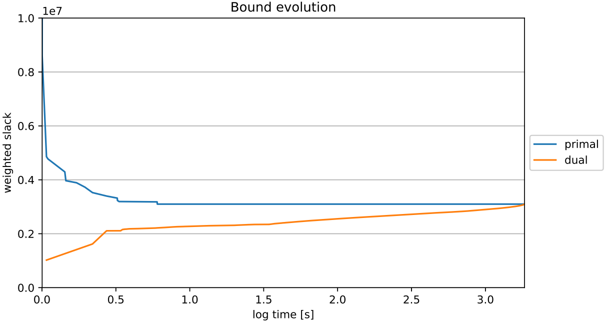
Suppose that each arc is contained in at most one (undirected) cycle. Then $P_{flip} = P_{IP}$.

This is satisfied, e.g., in our running example. However, we do not have $P_{\text{flip}} = P_{\text{IP}}$ in general: There is an infeasible PESP instance on a wheel graph with $P_{\text{flip}} \neq \emptyset$.

The flip inequalities comprise Nachtigall's change-cycle and Odijk's cycle inequalities. However, they differ from the *multi-circuit cuts* (Liebchen, Swarat, 2008), as the latter detect infeasibility in the aforementioned wheel instance.

Separating Flips in Practice

Typical PESP Branch-and-Cut Bound Evolution



← *logarithmic* time axis

The primal bound on this tiny instance stops moving after 10 seconds, proving optimality takes 30 minutes.

Aim: Improve dual bounds by flip inequalities.

Obstacles

- For each of the potentially exponentially many cycles, there are exponentially many flips.
- For a general point in P_{LP}, a violated flip inequality can be found in $O(T^2|V|^2|A|)$ time (Borndörfer et al., 2020) → too slow, too much memory.

Separating Flips in Practice

Heuristics

We propose several heuristics that, given a point $(y, z) \in P_{\text{LP}}$ of the LP relaxation, consider the fundamental cycles of a minimum spanning tree w.r.t. y:

Strategy	Description
standard	violated cycle & change-cycle ineq. for all fundamental cycles
all-flip	standard + violated single-arc flip ineq. for all fundamental cycles
max-flip-hybrid	standard + if standard does not produce enough cuts: maximally violated single-arc flip inequality per fundamental cycle
+ 4 more...	e.g., precomputing all flip ineq. for all cycles of length $\leq k$

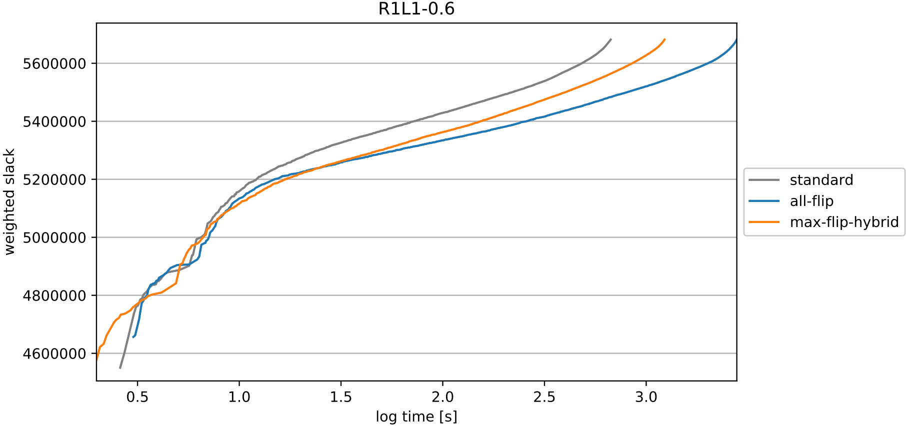
Instances

Instance	Hardness	$\|V\|$	$\|A\|$	$\|A\| - \|V\| + 1$
R1L1-0.6	easy	125	225	101
R4L4-0.6	medium	506	960	455
R1L1	hard	3 664	6 385	2 722
R4L4	extreme	8 384	17 754	9 371

← 4 instances derived from Marc Goerigk's PESPlib

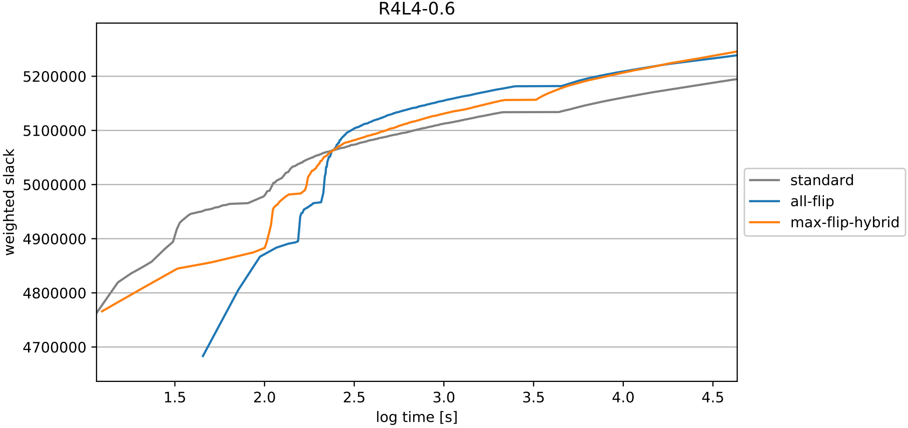
Solver: Concurrent PESP (Borndörfer, Lindner, Roth, 2020) with CPLEX 12.10, 6 threads, Intel Xeon E3-1245 v5 @ 3.5 GHz, 32 GB RAM. Wall time limit: 12 hours.

Dual Bound Comparison: R1L1-0.6



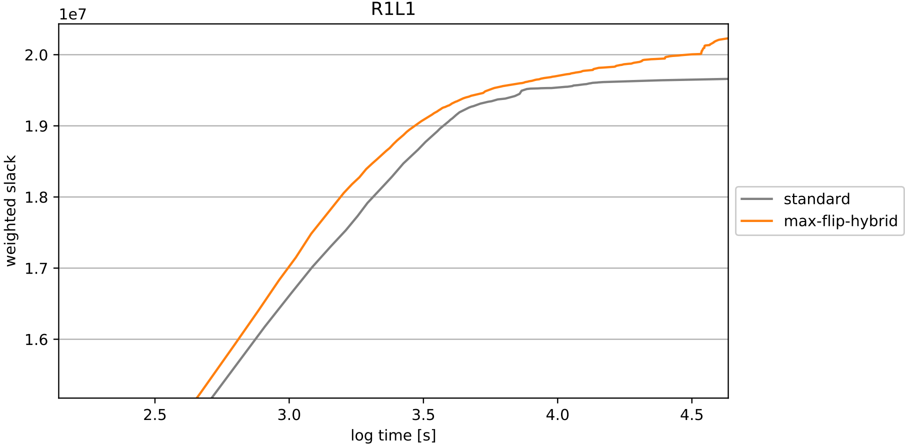
standard > max-flip-hybrid > all-flip, no trade-off from adding flip inequalities

Dual Bound Comparison: R4L4-0.6



max-flip-hybrid > all-flip > standard

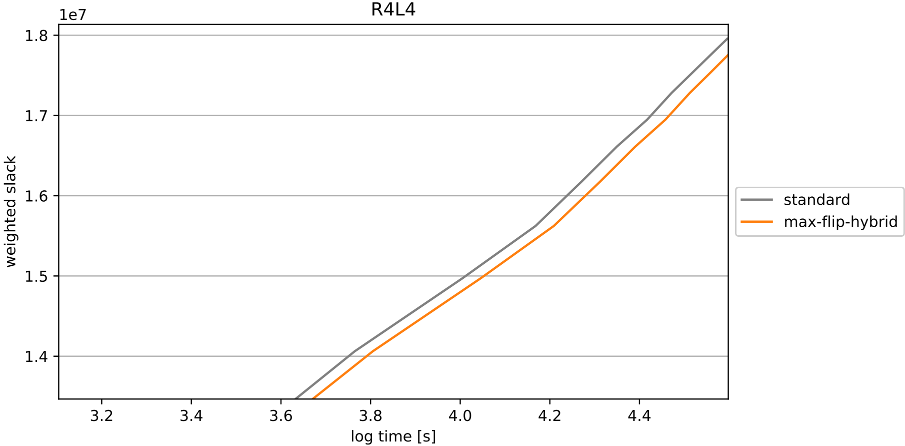
Dual Bound Comparison: R1L1



max-flip-hybrid > standard, all-flip runs out of memory

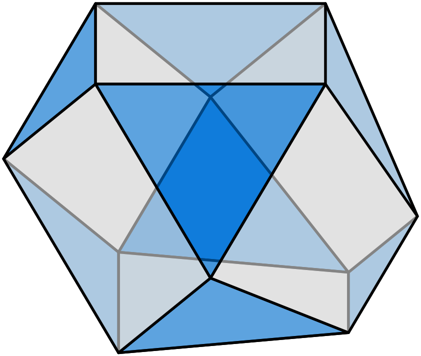
new PESPlib dual bound record 20 230 655 (1.8 % improvement)

Dual Bound Comparison: R4L4



standard > max-flip-hybrid, all methods run out of memory within 12 h

new PESPlib dual bound record 17 961 400 (13.4 % improvement)



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