Smoothness and factoriality of projective hypersurfaces

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$$\varphi : \mathbb{A}^{\binom{n+d}{d}} \times \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \quad (f, P) \mapsto \left(\frac{\partial f}{\partial x_0}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \right).$$

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- The locus of singular hypersurfaces is the image of the base locus of φ under projection onto the first factor.
- Since $\mathbb{P}^n$ is complete, this is closed.

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Theorem (Bertini)

The locus of smooth hypersurfaces of degree d in $\mathbb{P}^n$ is open and dense in $\mathbb{A}^{\binom{n+d}{d}}$.

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Example ($K = \mathbb{C}$)

As proper Zariski-closed subspace of $\mathbb{C}^{\binom{n+d}{d}}$, the locus of singular hypersurfaces has Lebesgue measure 0.
So the probability that $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ defines a smooth hypersurface equals 1.

Example ($K = \mathbb{F}_2$, $d = 3$, $n = 2$)

Out of the $1024 = 2^{\binom{2+3}{3}}$ elements of $\mathbb{F}_2[x_0, x_1, x_2]_3$, only 336 define smooth plane cubics.

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Theorem (Poonen's Bertini theorem)

$$\lim_{d\to\infty} \frac{\#\{f \in \mathbb{F}_q[x_0, \ldots, x_n]_d \mid \{f = 0\} \text{ smooth}\}}{\#\mathbb{F}_q[x_0, \ldots, x_n]_d} = \prod_{i=1}^{n+1} \left(1 - \frac{1}{q^i}\right).$$

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Example ($K = \mathbb{F}_2$, $n = 2$)

For $q = 2$ and $n = 2$ this limit equals $21/64 = 336/1024$.

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- *hypersurfaces f with*

$$\deg \operatorname{Proj}\left(\mathbb{F}_q[x_0,\ldots,x_n]\Big/\left\langle \frac{\partial f}{\partial x_0},\ldots,\frac{\partial f}{\partial x_n}\right\rangle\right) < c \deg f$$

for some fixed constant $c > 0$.

Factorial hypersurfaces

Let K be a field. Fix integers $d \geq 1$, $n \geq 4$.
Take $f \in K[x_0, \ldots, x_n]_d$ s. t. $X := \{f = 0\}$ defines a hypersurface in $\mathbb{P}^n_K$ with $\dim X_{\text{sing}} = 0$.

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Definition

X is called *factorial* if one of the above holds.

Example

Smooth $\Rightarrow$ factorial, as there is no difference between Weil and Cartier divisors.

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Theorem (Grothendieck)

Any hypersurface $X \subseteq \mathbb{P}^n$ with $\dim X_{sing} < \dim X - 3$ is factorial.

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This leaves us with the case $\dim X = 3$.

$\mathbb{Q}$-factorial hypersurfaces

Definition

X is called $\mathbb{Q}$-*factorial* if any Weil divisor is $\mathbb{Q}$-Cartier, i. e. if $\mathrm{Cl}(X) \otimes \mathbb{Q} = \mathrm{Pic}(X) \otimes \mathbb{Q}$.

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Example

Let $X = \{x_0x_1 - x_2x_3 = 0\} \subseteq \mathbb{P}^4_K$. Then $\mathrm{Cl}(X) \cong \mathbb{Z}^2$, whereas $\mathrm{Pic}(X) \cong \mathbb{Z}$. $\Rightarrow$ X is not $\mathbb{Q}$-factorial.

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Observation

As $\operatorname{Pic}(X) \cong \mathbb{Z}$,

$$\begin{aligned} X \text{ not } \mathbb{Q}\text{-factorial} &\Leftrightarrow \operatorname{rk} \operatorname{Cl}(X) - \operatorname{rk} \operatorname{Pic}(X) \geq 1 \\ &\Leftrightarrow \operatorname{rk} \operatorname{Cl}(X) \geq 2. \end{aligned}$$

Cohomology

Consider a nice cohomology theory H^* (e. g. étale, rigid, algebraic de Rham) with coefficients in some field of characteristic 0.

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Fact

Let X be a smooth projective K-variety. There is a cycle class map

$$c_1 : \mathrm{Pic}(X) \to H^2(X).$$

If $H^1(X) = 0$, then the induced map

$$c_1 : \mathrm{Pic}(X) \otimes \mathbb{Q} \to H^2(X)$$

is injective.

Cohomology

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Computations

Suppose that $H^1(E) = H^3(E) = 0$. Then:

- $\operatorname{rk} \operatorname{Pic}(\widetilde{X}) = \operatorname{rk} \operatorname{Cl}(X) + e$.
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Conclusion

As $c_1 : \operatorname{Pic}(\widetilde{X}) \otimes \mathbb{Q} \to H^2(\widetilde{X})$ is injective,

$$\operatorname{rk} Cl(X) = \operatorname{rk} Pic(\widetilde{X}) - e \leq h^2(\widetilde{X}) - e = h^4(X).$$

Theorem

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Suppose that X has only ordinary multiple points as singularities. Then blowing up gives a resolution s. t. the exceptional divisor E is a sum of pairwise non-intersecting smooth surfaces in $\mathbb{P}^3$.
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Theorem (Polizzi, Rapagnetta, Sabatino)

The conclusion of the above theorem holds for any X with only isolated singularities in characteristic 0.

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Conjecture

$$\lim_{d\to\infty} \frac{\#\{f \in \mathbb{F}_q[x_0, \ldots, x_4]_d \mid h^4(\{f = 0\}) \leq 1\}}{\#\mathbb{F}_q[x_0, \ldots, x_4]_d} = 1.$$

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I. e.: $\mathbb{Q}$-factorial threefold hypersurfaces should form a set of density 1.

Proof strategy

Assume $f \in K[x_0, \ldots, x_4]_d$ defines a hypersurface $X \subseteq \mathbb{P}^4$ with $\dim X_{\text{sing}} = 0$.

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Goal

Show that $h^4(X) \geq 2$ implies $\mu(f) \geq cd$ for some constant $c > 0$.

Then for $K = \mathbb{F}_q$, the set of f with $h^4(X) \leq 1$ would contain the density 1 set of f with $\mu(f) < cd$.

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The proof of the second theorem pursues the blowup idea and easily carries over to $\mathbb{F}_q$.

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Computation

Suppose that $X_{\text{sing}} \subseteq \{x_0 \neq 0\} \cong \mathbb{A}^4$.
If $h^4(X) \geq 2$, then the natural restriction map

$$H^4(\mathbb{P}^4 \setminus X) \to H^4(\mathbb{A}^4 \setminus (X \cap \mathbb{A}^4))$$

is not surjective.

Proof over $\mathbb{C}$

Fact

There is a differential form Ω such that

$$H^4(\mathbb{P}^4 \setminus X) = \left\{ \frac{g\Omega}{f^k} \middle| g \in \mathbb{C}[x_0, \ldots, x_4]_{kd-5}, k \geq 1 \right\} /_\sim,$$

$$H^4(\mathbb{A}^4 \setminus X) = \left\{ \frac{h\, dx_1 \wedge \cdots \wedge dx_4}{f(1, x_1, \ldots, x_4)^k} \middle| h \in \mathbb{C}[x_1, \ldots, x_4], k \geq 1 \right\} /_\sim,$$

and the natural map $H^4(\mathbb{P}^4 \setminus X) \to H^4(\mathbb{A}^4 \setminus X)$ is induced by

$$g \mapsto g(1, x_1, \ldots, x_4).$$

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Definition

For given $k \geq 1$, define the *pole order filtration* P via

$$P^k H^4(\mathbb{P}^4 \setminus X) = \left\{ \frac{g\Omega}{f^k} \middle| g \in \mathbb{C}[x_0, \ldots, x_4]_{kd-5} \right\} /\sim,$$

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Consequence

For $k \geq 1$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Gr}_P^k H^4(\mathbb{P}^4 \setminus X) \arrow[r] & \mathrm{Gr}_P^k H^4(\mathbb{A}^4 \setminus (X \cap \mathbb{A}^4)) \\
\mathbb{C}[x_0, \ldots, x_4]_{kd-5} \arrow[u, "\text{surj.}"] \arrow[r, "g \mapsto g(1, x_1, \ldots, x_4)"] & \mathbb{C}[x_1, \ldots, x_4] \arrow[u, "\text{surj.}"']
\end{tikzcd}$$

Proof over $\mathbb{C}$

The right map actually factors through the *Tjurina algebra*

$$T(f) := \left(\mathbb{C}[x_1, \ldots, x_4] \Big/ \left\langle f(1,x), \frac{\partial f(1,x)}{\partial x_1}, \ldots, \frac{\partial f(1,x)}{\partial x_4} \right\rangle \right).$$

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Corollary

Suppose $h^4(X) \geq 2$. Then $\exists\, k \geq 1$ s. t.

$$\mathbb{C}[x_0, \ldots, x_4]_{kd-5} \xrightarrow{g \mapsto g(1, x_1, \ldots, x_4)} T(f)$$

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Observation

$T(f)$ defines a zero-dimensional scheme $Z \subseteq \mathbb{P}^4$ of length $\mu(f)$. Moreover, the above map is just the natural map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(kd-5)) \to H^0(Z, \mathcal{O}_Z(kd-5)).$$

Proof over $\mathbb{C}$

Corollary

Suppose $h^4(X) \geq 2$. *Then* $\exists\, k \geq 1$ *s. t.* $H^1(\mathbb{P}^4, \mathcal{I}_Z(kd-5)) \neq 0$.

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Theorem

Let $X = \{f = 0\}$ *be a hypersurface in* $\mathbb{P}^4_{\mathbb{C}}$ *with only isolated singularities. If* $h^4_{dR}(X) \geq 2$, *then* $\mu(f) \geq d-3$.

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Conjecture

Let $X = \{f = 0\}$ be a hypersurface in $\mathbb{P}^4_{\mathbb{F}_q}$ with only isolated singularities. If $h^4_{\mathrm{rig}}(X) \geq 2$, then $\mu(f) \geq d - 3$.

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