Smoothness and factoriality of projective hypersurfaces

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Let $K$ be a field. Fix integers $d \geq 1, n \geq 2$. 

Identify $K[x_0, \ldots, x_n]_d \sim K^{(n+d)\binom{d}{d}}$ as $K$-vector spaces.

Suppose $\text{char } K \nmid d$ and consider the rational map $\phi: A^{(n+d)\binom{d}{d}} \times \mathbb{P}^n \rightarrow \mathbb{P}^n, (f, P) \mapsto \left(\frac{\partial f}{\partial x_0}(P), \ldots, \frac{\partial f}{\partial x_n}(P)\right)$.

The locus of singular hypersurfaces is the image of the base locus of $\phi$ under projection onto the first factor. Since $\mathbb{P}^n$ is complete, this is closed.
Let $K$ be a field. Fix integers $d \geq 1$, $n \geq 2$.

**Question**

Choose $f \in K[x_0, \ldots, x_n]_d$ uniformly at random. What is the probability that $\{f = 0\} \subseteq \mathbb{P}_K^n$ is smooth?
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A first approach:

- Identify $K[x_0, \ldots, x_n]_d \cong K^{n+d\choose d}$ as $K$-vector spaces.
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\varphi : \mathbb{A}^{(n+d)} \times \mathbb{P}^n \to \mathbb{P}^n, \quad (f, P) \mapsto \left( \frac{\partial f}{\partial x_0}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \right).
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- The locus of singular hypersurfaces is the image of the base locus of $\varphi$ under projection onto the first factor.
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- The locus of singular hypersurfaces is the image of the base locus of $\varphi$ under projection onto the first factor.
- Since $\mathbb{P}^n$ is complete, this is closed.
Density of smooth hypersurfaces

Theorem (Bertini)

The locus of smooth hypersurfaces of degree $d$ in $\mathbb{P}^n$ is open and dense in $\mathbb{A}^{\binom{n+d}{d}}$. 

Example ($K = \mathbb{C}$)
As proper Zariski-closed subspace of $\mathbb{C}^{\binom{n+d}{d}}$, the locus of singular hypersurfaces has Lebesgue measure 0. So the probability that $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ defines a smooth hypersurface equals 1.
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Example ($K = \mathbb{F}_2$, $d = 3$, $n = 2$)

Out of the $1024 = 2^{\left(\frac{2+3}{3}\right)}$ elements of $\mathbb{F}_2[x_0, x_1, x_2]_3$, only 336 define smooth plane cubics.
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Theorem (Poonen’s Bertini theorem)

$$\lim_{d \to \infty} \frac{\# \{ f \in \mathbb{F}_q[x_0, \ldots, x_n]_d \mid \{ f = 0 \} \text{ smooth} \}}{\# \mathbb{F}_q[x_0, \ldots, x_n]_d} = \prod_{i=1}^{n+1} \left( 1 - \frac{1}{q^i} \right).$$
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Example ($K = \mathbb{F}_2$, $n = 2$)

For $q = 2$ and $n = 2$ this limit equals $21/64 = 336/1024$. 
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**Theorem (Poonen, L.)**

*The following sets have density 1:*

1. Geometrically integral hypersurfaces,
2. Hypersurfaces with 0-dimensional singular locus,
3. Hypersurfaces $f$ with $\deg \text{Proj}(\mathbb{F}_q[x_0, \ldots, x_n]/\langle \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n} \rangle) < c \deg f$ for some fixed constant $c > 0$. 
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for some fixed constant $c > 0$. 
Let $K$ be a field. Fix integers $d \geq 1$, $n \geq 4$. Take $f \in K[x_0, \ldots, x_n]$ s. t. $X := \{f = 0\}$ defines a hypersurface in $\mathbb{P}^n_K$ with $\dim X_{\text{sing}} = 0$. Theorem

The following are equivalent:

- $K[x_0, \ldots, x_n]/\langle f \rangle$ is a UFD,
- The natural restriction map of Weil divisor class groups $\text{Cl}(\mathbb{P}^n_K) \to \text{Cl}(X)$ is an isomorphism,
- $\text{Cl}(X) = \mathbb{Z} \cdot \text{hyperplane class},$
- Every Weil divisor on $X$ is a Cartier divisor.

Definition

$X$ is called factorial if one of the above holds.
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Smooth $\Rightarrow$ factorial, as there is no difference between Weil and Cartier divisors.
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The quadric cone $X = \{x_0x_1 - x_2x_3 = 0\} \subseteq \mathbb{P}^4_K$ is not factorial, as $x_0x_1 = x_2x_3$ in $K[x_0, \ldots, x_4]/\langle x_0x_1 - x_2x_3 \rangle$. 

**Theorem (Grothendieck)**

Any hypersurface $X \subseteq \mathbb{P}^n$ with $\dim X_{\text{sing}} < \dim X - 3$ is factorial.
Factorial hypersurfaces

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Theorem (Grothendieck)

Any hypersurface $X \subseteq \mathbb{P}^n$ with $\dim X_{\text{sing}} < \dim X - 3$ is factorial.

This leaves us with the case $\dim X = 3$. 
Definition

$X$ is called $\mathbb{Q}$-factorial if any Weil divisor is $\mathbb{Q}$-Cartier, i.e. if $\text{Cl}(X) \otimes \mathbb{Q} = \text{Pic}(X) \otimes \mathbb{Q}$. 

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Let $X = \{x_0 x_1 - x_2 x_3 = 0\} \subseteq \mathbb{P}^4_K$. Then $\text{Cl}(X) \sim \mathbb{Z}_2$, whereas $\text{Pic}(X) \sim \mathbb{Z}$. $\Rightarrow$ $X$ is not $\mathbb{Q}$-factorial.
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Let $X \subseteq \mathbb{P}^4_K$ be a hypersurface with $\dim X_{\text{sing}} = 0$. 
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**Observation**

As $\text{Pic}(X) \cong \mathbb{Z}$,

$X$ not $\mathbb{Q}$-factorial $\iff \text{rk Cl}(X) - \text{rk Pic}(X) \geq 1$

$\iff \text{rk Cl}(X) \geq 2$. 
Consider a nice cohomology theory $H^*$ (e. g. étale, rigid, algebraic de Rham) with coefficients in some field of characteristic 0.
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**Fact**

Let $X$ be a smooth projective $K$-variety. There is a cycle class map

$$c_1 : \text{Pic}(X) \to H^2(X).$$

If $H^1(X) = 0$, then the induced map

$$c_1 : \text{Pic}(X) \otimes \mathbb{Q} \to H^2(X)$$

is injective.
Let $\tilde{X} \to X$ be a resolution of singularities. Denote by $e$ the number of irreducible components of the exceptional divisor.
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**Computations**

Suppose that $H^1(E) = H^3(E) = 0$. Then:

- $\text{rk Pic}(\tilde{X}) = \text{rk Cl}(X) + e$.
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**Conclusion**

As $c_1 : \text{Pic}(\tilde{X}) \otimes \mathbb{Q} \to H^2(\tilde{X})$ is injective,

$$\text{rk } \text{Cl}(X) = \text{rk } \text{Pic}(\tilde{X}) - e \leq h^2(\tilde{X}) - e = h^4(X).$$
Theorem

Suppose that $X$ has a resolution of singularities as above. Then

$X$ not $\mathbb{Q}$-factorial $\Rightarrow h^4(X) \geq 2.$
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\[ X \text{ not } \mathbb{Q}\text{-factorial} \implies h^4(X) \geq 2. \]

Example

Suppose that $X$ has only ordinary multiple points as singularities. Then blowing up gives a resolution such that the exceptional divisor $E$ is a sum of pairwise non-intersecting smooth surfaces in $\mathbb{P}^3$. In particular $H^1(E) = H^3(E) = 0$. 
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Theorem (Polizzi, Rapagnetta, Sabatino)

The conclusion of the above theorem holds for any $X$ with only isolated singularities in characteristic 0.
Suppose $K = \mathbb{F}_q$. 
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**Question**

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By Poonen’s Bertini theorem, this is $\geq \prod_{i=1}^{5}(1 - q^{-i})$. 
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**Conjecture**

$$\lim_{d \to \infty} \frac{\#\{f \in \mathbb{F}_q[x_0, \ldots, x_4]_d \mid h^4(\{f = 0\}) \leq 1\}}{\#\mathbb{F}_q[x_0, \ldots, x_4]_d} = 1.$$
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I. e.: $\mathbb{Q}$-factorial threefold hypersurfaces should form a set of density 1.
Assume $f \in K[x_0, \ldots, x_4]_d$ defines a hypersurface $X \subseteq \mathbb{P}^4$ with $\dim X_{\text{sing}} = 0$. 
Assume \( f \in K[x_0, \ldots, x_4]_d \) defines a hypersurface \( X \subseteq \mathbb{P}^4 \) with \( \dim X_{\text{sing}} = 0 \). Let \( \mu(f) \) denote the length of

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**Goal**

Show that \( h^4(X) \geq 2 \) implies \( \mu(f) \geq cd \) for some constant \( c > 0 \).
Proof strategy

Assume \( f \in K[x_0, \ldots, x_4]_d \) defines a hypersurface \( X \subseteq \mathbb{P}^4 \) with \( \dim X_{\text{sing}} = 0 \). Let \( \mu(f) \) denote the length of

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Goal

Show that \( h^4(X) \geq 2 \) implies \( \mu(f) \geq cd \) for some constant \( c > 0 \).

Then for \( K = \mathbb{F}_q \), the set of \( f \) with \( h^4(X) \leq 1 \) would contain the density 1 set of \( f \) with \( \mu(f) < cd \).
The following is known over \( \mathbb{C} \):

**Theorem (Cheltsov)**

Assume \( X \) has only ordinary double points as singularities. If \( X \) is not factorial, then \( X \) has at least \( (d - 1)^2 \) singular points.

**Theorem (Polizzi, Rapagnetta, Sabatino)**

Assume \( X \) has only ordinary multiple points as singularities. If \( X \) is not factorial, then

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\sum_{x \in X_{\text{sing}}} (\text{mult}(x) - 1) \geq d.
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The proof of the second theorem pursues the blowup idea and easily carries over to \( \mathbb{F}_q \).
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**Idea (Dimca-Griffiths)**

Investigate the cohomology of hypersurface complements.
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**Computation**

Suppose that $X_{\text{sing}} \subseteq \{x_0 \neq 0\} \cong \mathbb{A}^4$.

If $h^4(X) \geq 2$, then the natural restriction map

$$H^4(\mathbb{P}^4 \setminus X) \rightarrow H^4(\mathbb{A}^4 \setminus (X \cap \mathbb{A}^4))$$

is not surjective.
Proof over $\mathbb{C}$

Fact

There is a differential form $\Omega$ such that

$$H^4(P^4 \setminus X) = \left\{ \frac{g \Omega}{f^k} \middle| g \in \mathbb{C}[x_0, \ldots, x_4]_{kd-5}, k \geq 1 \right\} / \sim,$$

$$H^4(A^4 \setminus X) = \left\{ \frac{h dx_1 \wedge \cdots \wedge dx_4}{f(1, x_1, \ldots, x_4)^k} \middle| h \in \mathbb{C}[x_1, \ldots, x_4], k \geq 1 \right\} / \sim,$$

and the natural map $H^4(P^4 \setminus X) \rightarrow H^4(A^4 \setminus X)$ is induced by

$$g \mapsto g(1, x_1, \ldots, x_4).$$
Definition

For given $k \geq 1$, define the pole order filtration $P$ via

$$P^k H^4(\mathbb{P}^4 \setminus X) = \left\{ \frac{g\Omega}{f^k} \bigg| g \in \mathbb{C}[x_0, \ldots, x_4]kd-5 \right\} / \sim,$$

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Proof over $\mathbb{C}$
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similarly $P^k H^4(\mathbb{A}^4 \setminus X)$.

**Consequence**

For $k \geq 1$, there is a commutative diagram

$$
\begin{array}{ccc}
\text{Gr}_P^k H^4(\mathbb{P}^4 \setminus X) & \longrightarrow & \text{Gr}_P^k H^4(\mathbb{A}^4 \setminus (X \cap \mathbb{A}^4)) \\
\uparrow \text{surj.} & & \uparrow \text{surj.} \\
\mathbb{C}[x_0, \ldots, x_4]_{kd-5} & \xrightarrow{g \mapsto g(1,x_1,\ldots,x_4)} & \mathbb{C}[x_1, \ldots, x_4]
\end{array}
$$
Proof over $\mathbb{C}$

The right map actually factors through the *Tjurina algebra*

\[
T(f) := \left( \mathbb{C}[x_1, \ldots, x_4]/\left\langle f(1, x), \frac{\partial f(1, x)}{\partial x_1}, \ldots, \frac{\partial f(1, x)}{\partial x_4} \right\rangle \right).
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\[ \text{Corollary} \]

\textit{Suppose} \( h^4(X) \geq 2 \). \textit{Then} \( \exists k \geq 1 \) \textit{s. t.}

\[ \mathbb{C}[x_0, \ldots, x_4]_{kd-5} \xrightarrow{g \mapsto g(1, x_1, \ldots, x_4)} T(f) \]

is not surjective.
Proof over $\mathbb{C}$

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$$T(f) := \left( \mathbb{C}[x_1, \ldots, x_4] / \left\langle f(1, x), \frac{\partial f(1, x)}{\partial x_1}, \ldots, \frac{\partial f(1, x)}{\partial x_4} \right\rangle \right).$$

**Corollary**

*Suppose* $h^4(X) \geq 2$. *Then* $\exists k \geq 1$ *s. t.*

$$\mathbb{C}[x_0, \ldots, x_4]_{kd - 5} \xrightarrow{g \mapsto g(1, x_1, \ldots, x_4)} T(f)$$

*is not surjective.*

**Observation**

$T(f)$ defines a zero-dimensional scheme $Z \subseteq \mathbb{P}^4$ of length $\mu(f)$. Moreover, the above map is just the natural map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(kd - 5)) \rightarrow H^0(Z, \mathcal{O}_Z(kd - 5)).$$
Corollary

Suppose $h^4(X) \geq 2$. Then $\exists k \geq 1$ s. t. $H^1(\mathbb{P}^4, \mathcal{I}_Z(kd - 5)) \neq 0$. 
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- This means that the Castelnuovo-Mumford regularity of \( Z \) is at least \( kd - 3 \).
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- Hence $\mu(f) = \text{length}(Z) \geq kd - 3 \geq d - 3$. 

Proof over $\mathbb{C}$

**Corollary**

Suppose $h^4(X) \geq 2$. Then $\exists \; k \geq 1$ s. t. $H^1(\mathbb{P}^4, \mathcal{I}_Z(kd - 5)) \neq 0$.

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- Hence $\mu(f) = \text{length}(Z) \geq kd - 3 \geq d - 3$.

**Theorem**

Let $X = \{ f = 0 \}$ be a hypersurface in $\mathbb{P}^4_{\mathbb{C}}$ with only isolated singularities. If $h^4_{dR}(X) \geq 2$, then $\mu(f) \geq d - 3$. 
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  $\Rightarrow$ rigid cohomology.
- $\mathbb{P}^4 \setminus X$ is smooth and affine
  $\Rightarrow$ Monsky-Washnitzer cohomology
  $\Rightarrow$ Overconvergent power series instead of polynomials.
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- Need to adjust the pole-order filtration.
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- Need to find a replacement for de Rham cohomology $\Rightarrow$ rigid cohomology.
- $\mathbb{P}^4 \setminus X$ is smooth and affine $\Rightarrow$ Monsky-Washnitzer cohomology $\Rightarrow$ Overconvergent power series instead of polynomials.
- Need to adjust the pole-order filtration.

Conjecture

Let $X = \{ f = 0 \}$ be a hypersurface in $\mathbb{P}^4_{\mathbb{F}_q}$ with only isolated singularities. If $h^4_{\text{rig}}(X) \geq 2$, then $\mu(f) \geq d - 3$. 


