A NOTE ON BERTINI IRREDUCIBILITY THEOREMS FOR SIMPLICIAL TORIC VARIETIES OVER FINITE FIELDS

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Abstract. This short note deals with extending the Bertini irreduciblity theorems obtained by Charles and Poonen in [1] to simplicial toric varieties.

1. Goal

Consider a finite field $\mathbb{F}_q$ and fix an algebraic closure $\mathbb{F}$. Let $\mathbb{P}$ be a projective normal simplicial toric variety over $\mathbb{F}_q$ with singular locus $\mathbb{P}_{\text{sing}}$. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$.

We follow the notation in [1]: Let X be a scheme of finite type over $\mathbb{F}$, $Y \subseteq X$ a subscheme. Let further $\phi : X \to \mathbb{P}_{\mathbb{F}}$ be an $\mathbb{F}$-morphism.

- Y is called *horizontal* if $\dim \overline{\phi(Y)} \geq 1$ and $\overline{\phi(Y)}$ is not contained in $(\mathbb{P}_{\mathbb{F}})_{\text{sing}}$,
- $\operatorname{Irr} Y$ denotes the set of irreducible components of Y,
- $\operatorname{Irr}_{\text{horiz}} Y$ is the set of all horizontal irreducible components of Y,
- Y_{horiz} denotes the union of all horizontal irreducible components of Y.

For sections $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + kE))$, where k is an integer, define $X_f := \phi^{-1}(V(f))$.

Theorem 1.1. *Suppose that $\phi : X \to \mathbb{P}_{\mathbb{F}}$ is an $\mathbb{F}$-morphism such that $\dim \overline{\phi(C)} \geq 2$ and $\dim \overline{\phi(C)} \cap (\mathbb{P}_{\mathbb{F}})_{\text{sing}} \leq \dim \overline{\phi(C)} - 2$ for each $C \in \operatorname{Irr} X$. Then*

$$\lim_{k \to \infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + kE)) \mid \operatorname{Irr} X \to \operatorname{Irr}_{\text{horiz}} X_f, C \mapsto (C \cap X_f)_{\text{horiz}} \text{ is a bijection}\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + kE))} = 1.$$

Remarks.

(1) Suppose that $D = 0$ and E is a very ample Cartier divisor defining a closed immersion $i : \mathbb{P} \hookrightarrow \mathbb{P}^n$. This gives a linear map

$$i^* : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(kE)),$$

which is surjective for $k \gg 0$. In particular, Theorem 1.1 is obtained from [1, Theorem 1.6], as $V(i^*(g)) = i^{-1}(V(g))$ for $g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$. Moreover, this holds for any projective variety $\mathbb{P}$ over $\mathbb{F}_q$, and the conditions on $(\mathbb{P}_{\mathbb{F}})_{\text{sing}}$ may be dropped as well.

(2) The codimension two condition on the intersection with the singular locus is necessary if D is not trivial. For example, consider $\mathbb{P} = \mathbb{P}(1, 2, 3, 6)$ with coordinates x_0, x_1, x_2, x_3. Let $X := V(x_0) \subseteq \mathbb{P}_{\mathbb{F}}$ and $\phi : X \hookrightarrow \mathbb{P}_{\mathbb{F}}$ be the inclusion. X is an irreducible surface in $\mathbb{P}_{\mathbb{F}}$ and

$$X \cap (\mathbb{P}_{\mathbb{F}})_{\text{sing}} = V(x_0, x_1) \cup V(x_0, x_2)$$

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is one-dimensional. For $k \geq 1$, let f be a weighted homogeneous polynomial of degree $6k+1$. One finds that f can be written as

$$f = x_1^2 x_2 \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} c_{ij} x_1^{3i} x_2^{2j} x_3^{k-i-j-1} + \text{ terms divisible by } x_0, \quad c_{ij} \in \mathbb{F}_q.$$

Thus if $X \cap V(f)$ is irreducible, then f lies in a subspace of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6k+1))$ of codimension

$$\sum_{i=0}^{k-1} (k-1-i) = \frac{k(k-1)}{2}.$$

As a consequence, the fraction of $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6k+1))$ such that $X \cap V(f)$ is irreducible is at most $q^{-k(k-1)/2}$. In particular, the density of f for which the map $\operatorname{Irr} X \to \operatorname{Irr} X_f$ is a bijection is bounded from above by $\lim_{k\to\infty} q^{-k(k-1)/2} = 0$.

(3) Let X and ϕ satisfy the hypotheses of Theorem 1.1 and assume that ϕ is an immersion. If $D \in \operatorname{Irr} X$ and $C \subseteq D$ is irreducible with $\operatorname{codim}_D(C) \leq 1$, then $\overline{\phi(C)}$ is not contained in $(\mathbb{P}_{\mathbb{F}})_{\text{sing}}$: Indeed, the singular locus of $\mathbb{P}_{\mathbb{F}}$ has codimension at least two in $\overline{\phi(D)}$, whereas $\overline{\phi(C)}$ has codimension at most one. Therefore, the subscript "horiz" may be omitted and one obtains:

Corollary 1.2. *Let X be a geometrically irreducible subscheme of a projective normal simplicial toric variety $\mathbb{P}$ over $\mathbb{F}_q$. If $\dim X \geq 2$ and $\dim \overline{X} \cap \mathbb{P}_{\text{sing}} \leq \dim X - 2$, then*

$$\lim_{k\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+kE)) \mid X \cap V(f) \textit{ is geometrically irreducible}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+kE))} = 1.$$

In particular, this holds for $X = \mathbb{P}$.

The proof of Theorem 1.1 follows the outline of [1], some proofs being almost literally the same. However, the presence of singularities in $\mathbb{P}$ rises some technical difficulties.

2. Lemmas

From now on, a statement $S(f)$, where $f \in H^0(X, \mathcal{O}_X(D+kE))$ for some positive integer k, is said to hold for f in a set of density 1 if

$$\lim_{k\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+kE)) \mid S(f) \text{ is true}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+kE))} = 1.$$

Lemma 2.1. *Let X be either*

- *a subscheme of $\mathbb{P}$ over $\mathbb{F}_q$ such that $\dim X \setminus (X \cap \mathbb{P}_{\text{sing}}) \geq 1$, or*
- *a subscheme of $\mathbb{P}_{\mathbb{F}}$ over $\mathbb{F}$ such that $\dim X \setminus (X \cap (\mathbb{P}_{\mathbb{F}})_{\text{sing}}) \geq 1$.*

Then for f in a set of density 1, f does not vanish on X.

Proof. Replacing X by its image under the natural map $\mathbb{P}_{\mathbb{F}} \to \mathbb{P}_{\mathbb{F}_q}$, assume that X is defined over $\mathbb{F}_q$. Now the assertion follows as in the proof of [3, Lemma 4.5 (2)]. □

Lemma 2.2. *Let $X \subseteq \mathbb{P}$ (or $\mathbb{P}_{\mathbb{F}}$) be a subscheme over $\mathbb{F}_q$ (or $\mathbb{F}$) such that $\dim X \geq 1$. Then for f in a set of density 1, $X \cap V(f) \neq \emptyset$.*

Proof. Assume again that X is defined over $\mathbb{F}_q$. Fix a positive integer r and denote by $X_{<r}$ the set of closed points of X whose degree is smaller than r. As in [3, Lemma 4.1], the density of sections f such that $V(f) \cap X_{<r}$ is empty equals

$$\prod_{P \in X_{<r}} \left(1 - q^{-\nu_P(D)}\right),$$

where $\nu_P(D) := \dim_{\mathbb{F}_q} H^0(X_P, \mathcal{O}_X(D)|_{X_P})$ and X_P is the subscheme of X corresponding to the maximal ideal $\mathfrak{m}_P$. If $\nu_P(D) = 0$ for some point $P \in X_{<r}$, then the above product equals zero. Otherwise choose a positive integer m such that mD is Cartier. This is possible since $\mathbb{P}$ is simplicial and hence $\mathbb{Q}$-factorial (see e. g. [2, Proposition 4.2.7]). Using that the sheaf $\mathcal{O}_X(mD)|_{X_P}$ is invertible and hence locally isomorphic to $\mathcal{O}_{X_P}$, there is an injective map

$$H^0(X_P, \mathcal{O}_X(D)|_{X_P}) \to H^0(X_P, \mathcal{O}_X(mD)|_{X_P}) \cong H^0(X_P, \mathcal{O}_{X_P}), \quad g \mapsto g^m,$$

thus $0 < \nu_P(D) \leq \deg P$. As $\deg P$ necessarily divides $\nu_P(D)$, this implies $\nu_P(D) = \deg P$. In particular, the density of f such that $V(f) \cap X_{<r}$ is empty equals

$$\prod_{P \in X_{<r}} \left(1 - q^{-\deg P}\right) = \frac{1}{\zeta_{X_{<r}}(1)}.$$

This diverges to 0 as $r \to \infty$, since $\dim X \geq 1$. □

Lemma 2.3. *Let X be an $\mathbb{F}$-scheme of finite type, $\phi : X \to \mathbb{P}_{\mathbb{F}}$ an $\mathbb{F}$-morphism such that $\dim \overline{\phi(C)} \geq 2$ for all $C \in \operatorname{Irr} X$. Let U be a dense open subscheme of X. Then for f in a set of density* 1*, there is a bijection*

$$\operatorname{Irr}_{\text{horiz}} X_f \to \operatorname{Irr}_{\text{horiz}} U_f, \quad C \mapsto C \cap U.$$

Proof. If every $C \in \operatorname{Irr}_{\text{horiz}} X_f$ meets U, the above map is clearly bijective with its inverse given by taking the closure in X_f.

There is nothing to show if $\operatorname{Irr}_{\text{horiz}}(X \setminus U) = \emptyset$. Otherwise, let $C \in \operatorname{Irr}_{\text{horiz}}(X \setminus U)$. Since $\overline{\phi(C)}$ is of dimension ≥ 1 and is not contained in $(\mathbb{P}_{\mathbb{F}})^{\text{sing}}$, Lemma 2.1 states that the set of f vanishing on $\overline{\phi(C)}$ has density 0. Excluding these f, every $C \in \operatorname{Irr}_{\text{horiz}} X_f$ meets U, because otherwise $C \in \operatorname{Irr}_{\text{horiz}}(X \setminus U)$ and $f(\overline{\phi(C)}) = 0$. □

Lemma 2.4. *Let X and ϕ be as in Theorem 1.1 and suppose further that X is smooth. Let $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + kE)) \setminus \{0\}$ for some $d \geq 0$. Then C_f contains a horizontal component for any $C \in \operatorname{Irr} X$. Moreover, the following are equivalent:*

(1) *There is a bijection $\operatorname{Irr} X \to \operatorname{Irr}_{\text{horiz}} X_f$, $C \mapsto (C_f)_{\text{horiz}}$.*
(2) *For every $C \in \operatorname{Irr} X$, the scheme $(C_f)_{\text{horiz}}$ is irreducible.*

Proof. Let $C \in \operatorname{Irr} X$. Then $\dim \overline{\phi(C)} \geq 2$ and thus

$$\dim \overline{\phi(C_f)} = \dim \overline{\phi(C \cap \phi^{-1}(\{f = 0\}))} = \dim \overline{\phi(C) \cap \{f = 0\}} \geq \dim \overline{\phi(C)} - 1 \geq 1.$$

In particular, C_f has an irreducible component C' such that the codimension of $\overline{\phi(C')}$ in $\overline{\phi(C)}$ is ≤ 1. By hypothesis, the singular locus of $\mathbb{P}_{\mathbb{F}}$ has codimension ≥ 2 in $\overline{\phi(C)}$. Thus $\overline{\phi(C')}$ is not contained in $(\mathbb{P}_{\mathbb{F}})^{\text{sing}}$ and hence C' is horizontal.

Concerning the "moverover" part, (1) $\Rightarrow$ (2) is obvious. For (2) $\Rightarrow$ (1), note that the map is defined and surjective. By smoothness of X, the components of X do not intersect, so the map is also injective. □

Lemma 2.5. *Let X be a subscheme of $\mathbb{P}_{\mathbb{F}}$ such that X is smooth and $X \cap (\mathbb{P}_{\mathbb{F}})_{\mathrm{sing}}$ is finite. For f in a set of density 1, the singular locus $(X_f)_{\mathrm{sing}}$ is finite.*

Proof. In view of [3, Corollary 5.2], the difficulty comes from the larger fields involved. Splitting X into orbits under the action of the absolute Galois group of $\mathbb{F}_q$, we can follow the proof of [1, Lemma 3.5] to obtain a covering of $X \cap (\mathbb{P}_{\mathbb{F}})^{\mathrm{sm}}$ by finitely many open subschemes U and global derivations $D_1, \ldots, D_m : \mathcal{O}_U(U) \to \mathcal{O}_U(U)$ such that

$$P \in U \cap (X_f)^{\mathrm{sing}} \Rightarrow f(P) = D_1(f)(P) = \cdots = D_m(f)(P) = 0.$$

Proceeding as in the proof of [3, Lemma 4.9], $U \cap \{D_1(f) = \cdots = D_m(f)\}$ is finite with probability $1 - o(1)$ as $k \to \infty$. □

3. Surfaces

Proposition 3.1. *Let X be a 2-dimensional closed integral subscheme of $\mathbb{P}$ such that $X \cap \mathbb{P}_{\mathrm{sing}}$ is finite. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\mathbb{F}} \to \operatorname{Irr}(X_f)_{\mathbb{F}}$ sending C to $C \cap X_f$.*

Proof. Since the natural map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + kE)) \to H^0(X, \mathcal{O}_{\mathbb{P}}(D + kE)|_X)$$

is surjective for sufficiently large k, densities may be calculated by counting elements X_f in $\mathbb{P}H^0(X, \mathcal{O}_{\mathbb{P}}(D + kE)|_X)$, which are Weil divisors on X for f in a set of density 1 by Lemma 2.1. The restriction of X_f to the complement of $X \cap \mathbb{P}_{\mathrm{sing}}$ is a Cartier divisor. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X. Taking the pullback under π and taking the closure gives a Cartier divisor on $\widetilde{X}$.

Step 1. For f in as set of density 1, the divisor X_f is irreducible.

Similar to [1, Proposition 4.1], one computes that for any positive constant k_0, the number of reducible X_f is at most

$$q^{\frac{k^2 E.E}{2} - \frac{k_0 k}{2} + O(k)}.$$

It remains to determine $\#H^0(X, \mathcal{O}_{\mathbb{P}}(D + kE)|_X)$. Let C be an effective Cartier divisor on X. Then there is an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0.$$

In particular, by tensoring with the k-th tensor power of the invertible sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}}(E)|_X$, for the Euler characteristic χ holds

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{L}^{\otimes k}) = \chi(\mathcal{L}^{\otimes k}) + \chi(\mathcal{O}_C(C) \otimes \mathcal{L}^{\otimes k}).$$

Since $\mathcal{O}_C(C) \otimes \mathcal{L}^{\otimes k}$ is supported on a codimension 1 subscheme of X, the leading terms of the Hilbert polynomials $\chi(\mathcal{O}_X(C) \otimes \mathcal{L}^{\otimes k})$ and $\chi(\mathcal{L}^{\otimes k})$ coincide.

Pick now ℓ large enough such that $\mathcal{O}_{\mathbb{P}}(D + \ell E)$ is globally generated. This allows to choose a section $g \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + \ell E))$ which does not vanish on X. Further choose a positive integer m such that mD is Cartier. Then there is a chain of injective maps

$$H^0(X, \mathcal{O}_{\mathbb{P}}(kE)|_X) \to H^0(X, \mathcal{O}_{\mathbb{P}}(D + (k + \ell)E)|_X) \to \cdots \to H^0(X, \mathcal{O}_{\mathbb{P}}(mD + (k + \ell m)E)|_X)$$

induced by multiplication with g. As a consequence, Serre vanishing yields for $k \gg 0$

$$\chi(\mathcal{L}^{\otimes k}) \leq \chi(\mathcal{O}_{\mathbb{P}}(D + \ell E)|_X \otimes \mathcal{L}^{\otimes k}) \leq \chi(\mathcal{O}_{\mathbb{P}}(mD + \ell m E)|_X \otimes \mathcal{L}^{\otimes k}).$$

But $\mathcal{O}_{\mathbb{P}}(mD + \ell mE)|_X$ is the sheaf of an effective Cartier divisor, so by the previous, the leading terms of these three Hilbert polynomials agree. Thus

$$\# H^0(X, \mathcal{O}_{\mathbb{P}}(D+kE)|_X) = q^{\chi(\mathcal{L}^{\otimes k})+O(k)} = q^{\frac{k^2 E.E}{2}+O(k)}, \quad k \gg 0.$$

Choosing k_0 large enough, we obtain that the density of reducible X_f is 0.

Step 2. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\mathbb{F}} \to \operatorname{Irr}(X_f)_{\mathbb{F}}$, $C \mapsto C \cap X_f$.

Due to the assumption that $X \cap \mathbb{P}_{\text{sing}}$ be finite, Lemma 2.2, 2.3 and 2.4 can be applied to show that the density of f for which the claim fails is 0 as in [1, Proposition 4.1]. $\square$

4. Induction

Lemma 4.1. *Let $X \subseteq \mathbb{P}_{\mathbb{F}}$ be a smooth irreducible subscheme of dimension $m \geq 3$. Suppose that $\dim \overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq m-2$. Then:*

(1) *There exists a hypersurface $J \subseteq \mathbb{P}$ defined over k such that*
- *$J \cap X$ is irreducible,*
- *$\dim J \cap X = m-1$,*
- *$\dim J \cap (\overline{X} \setminus X) \leq m-2$,*
- *$\dim J \cap \overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq m-3$.*

(2) *For any J as in (1), there is a density 1 set of f for which the implication*

$$(J \cap X)_f \textit{ irreducible } \Rightarrow X_f \textit{ irreducible}$$

holds.

Proof. (1) Pick a positive integer k and choose sections $h_i \in H^0(\mathbb{P}, kE)$, $i = 0, \dots, m$, such that $\dim V(h_0, \dots, h_r) \cap \overline{X} = m - r - 1$ for $r = 0, \dots, m-1$ and $V(h_0, \dots, h_m) \cap \overline{X} = \emptyset$. This is possible since kE has no base points for $k \gg 0$. The sections $h_0, \dots, h_m$ give rise to a map

$$\pi : \overline{X} \to \mathbb{P}^m_{\mathbb{F}}, \quad P \mapsto (h_0(P) : \dots : h_m(P)).$$

The fiber over $(0 : \dots : 0 : 1)$ is zero-dimensional, therefore π is a generically finite dominant morphism. Define

$$\begin{aligned} Z := &\{P \in \mathbb{P}^m_{\mathbb{F}_q} \mid \operatorname{codim}_{\overline{X}} \pi^{-1}(P) = 1\} \\ &\cup \{\pi(C) \subseteq \mathbb{P}^m_{\mathbb{F}_q} \mid C \in \operatorname{Irr}(\overline{X} \setminus X) \cup \operatorname{Irr}(\overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}}), \dim \pi(C) = 0\}. \end{aligned}$$

Then Z is finite. By [4, Theorem 1.2, Proposition 2.7], Lemma 2.1 and [1, Lemma 5.2], there is a positive density of homogeneous polynomials $g \in k[x_1, \dots, x_m]$ such that
- $\{g = 0\}$ is geometrically integral,
- $\{g = 0\} \cap Z = \emptyset$,
- $\pi(C) \not\subseteq \{g = 0\}$ for any $C \in \operatorname{Irr}(\overline{X} \setminus X) \cup \operatorname{Irr}(\overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}})$ with $\dim \pi(C) \geq 1$,
- $X \cap \pi^{-1}(\{g = 0\})$ is irreducible of dimension $m-1$.

Pick such a g and set $J := g(h_0, \dots, h_m) \in H^0(\mathbb{P}, k \deg g \cdot E)$. Then:
- $J \cap X = X \cap \pi^{-1}(\{g = 0\})$,
- J contains no irreducible component of $\overline{X} \setminus X$ or $\overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}}$, whence

$$\dim J \cap (\overline{X} \setminus X) \leq m-2 \quad \text{and} \quad \dim J \cap X \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq m-3.$$

(2) Similar to [1, Lemma 5.3], if $(J \cap X)_f$ is irreducible and X_f is reducible, then $X_f = V_1 \cup V_2$ for subschemes V_1, V_2 such that $V_1 \not\subseteq V_2$, $V_2 \not\subseteq V_1$ and $\dim V_1, \dim V_2 \geq m-1$. Moreover, for $i = 1, 2$, $J \cap \overline{V_i}$ is nonempty of dimension $\geq m-2$. For f in a set of density 1, Lemma 2.1 implies that

$$\dim J \cap (\overline{V}_i \setminus V_i) \leq \dim J \cap (\overline{X} \setminus X) \cap X_f \leq m-3.$$

This implies that $J \cap V_i$ is of dimension $\geq m-2$. Using that $(J \cap X)_f$ is irreducible, we can assume w. l. o. g. that $J \cap V_1 \subseteq J \cap V_2$. As a consequence,

$$m-2 \leq \dim J \cap V_1 \leq \dim J \cap V_1 \cap V_2 \leq \dim J \cap (X_f)^{\text{sing}}.$$

Let $U := X \cap \mathbb{P}^{\text{sm}}$. Clearly $(X_f)^{\text{sing}} \subseteq (U_f)^{\text{sing}} \cup (X \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}})$. By Lemma 2.5, $(U_f)^{\text{sing}}$ is finite for f in a set of density 1, as U is smooth and does not meet $(\mathbb{P}_{\mathbb{F}})^{\text{sing}}$. In particular, for these f,

$$\begin{aligned}\dim J \cap (X_f)^{\text{sing}} &\leq \max\{\dim J \cap (U_f)^{\text{sing}}, \dim J \cap X \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}}\} \\ &\leq \max\{0, \dim J \cap \overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}}\} \\ &\leq m-3.\end{aligned}$$

This leads to the contradiction

$$m-2 \leq \dim J \cap (X_f)^{\text{sing}} \leq m-3.$$

Thus for f in a set of density 1, $(J \cap X)_f$ irreducible implies X_f irreducible. □

Proposition 4.2. *Let X be an irreducible subscheme of $\mathbb{P}$ of dimension $m \geq 2$ such that $\dim \overline{X} \cap \mathbb{P}_{\text{sing}} \leq m-2$. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\mathbb{F}} \to \operatorname{Irr}(X_f)_{\mathbb{F}}$ sending C to C_f.*

Proof. We may assume that X is reduced. For surfaces note that $\overline{X} \cap \mathbb{P}^{\text{sing}}$ is finite, thus the assertion for $\overline{X}$ follows from Proposition 3.1. Now Lemma 2.3 allows to proceed to X.

For $m \geq 3$, we can assume that X is smooth by Lemma 2.3. Pick an irreducible component $C \in \operatorname{Irr} X_{\mathbb{F}}$. Then C is a smooth irreducible subscheme of $\mathbb{P}_{\mathbb{F}}$ of dimension $m \geq 3$ and

$$\dim \overline{C} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq \dim \overline{X} \cap \mathbb{P}^{\text{sing}} \leq m-2.$$

Lemma 4.1 applied to C produces a hypersurface $J \subseteq \mathbb{P}$ defined over k such that $J \cap C$ is irreducible of dimension $m-1$ and

$$\dim J \cap \overline{C} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq m-3.$$

Using the map $C \hookrightarrow X_{\mathbb{F}} \to X$, this means that $J \cap X$ is irreducible of dimension $m-1$ as well and

$$\dim J \cap \overline{X} \cap (\mathbb{P}_{\mathbb{F}})^{\text{sing}} \leq m-3.$$

Performing induction on $J \cap X$ shows that for f in a set of density 1, $(J \cap C)_f$ is irreducible for any $C \in \operatorname{Irr} X_{\mathbb{F}}$. For a possibly smaller set of density 1, this implies that C_f is irreducible by part 2 of Lemma 4.1. Moreover every C_f is horizontal, since $\dim C_f \geq m-1$, whereas

$$\dim C_f \cap (\mathbb{P}_{\mathbb{F}})_{\text{sing}} \leq \dim \overline{X} \cap (\mathbb{P}_{\mathbb{F}})_{\text{sing}} \leq m-2.$$

Finally Lemma 2.4 yields a bijection

$$\operatorname{Irr} X_{\mathbb{F}} \xrightarrow{\sim} \operatorname{Irr}_{\text{horiz}}(X_f)_{\mathbb{F}} \xrightarrow{\sim} \operatorname{Irr}(X_f)_{\mathbb{F}}, \quad C \mapsto C_f.$$ □

5. Finishing the proof

Lemma 5.1. *Let X and Y be irreducible finite type $\mathbb{F}$-schemes. Suppose that $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}_{\mathbb{F}}$ are morphisms such that π is finite and étale, ψ has relative dimension s at each point and $\dim \overline{\psi(Y)} \geq 2$. Then for f in a set of density 1, the implication*

$$Y_f \text{ irreducible } \Rightarrow X_f \text{ irreducible}$$

holds.

Proof. Following the proof of [1, Lemma 5.1], we only need to adjust the density estimate for f such that $V(f)$ misses at least $(c' + o(1))r^{me}/e$ points of $\overline{\psi(Y)}$ with residue field of size at most r^e, for fixed $c' > 0$, $e, m, r \in \mathbb{N}$, $m \geq 2$. As in the proof of Lemma 2.2, this density either equals zero or is bounded from above by

$$(1 - r^{-e})^{(c'+o(1))r^{me}/e}.$$

As $e \to \infty$, this quantity goes to zero due to $m \geq 2$. □

Lemma 5.2. *Let X and Y be irreducible finite type $\mathbb{F}$-schemes with morphisms $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}_{\mathbb{F}}$ such that π is dominant, $\dim \overline{\psi(Y)} \geq 2$ and $\dim \overline{\psi(Y)} \cap (\mathbb{P}_{\mathbb{F}})^{\mathrm{sing}} \leq \dim \overline{\psi(Y)} - 2$. Then for f in a set of density 1, the implication*

$$(Y_f)_{\mathrm{horiz}} \text{ irreducible } \Rightarrow (X_f)_{\mathrm{horiz}} \text{ irreducible}$$

holds.

Proof. As in [1, Lemma 5.2]. □

Proposition 5.3. *Let X be a $\mathbb{F}_q$-scheme of finite type. Let $\phi : X \to \mathbb{P}$ be a morphism such that $\dim \overline{\phi(C)} \geq 2$ and $\dim \overline{\phi(C)} \cap (\mathbb{P}_{\mathbb{F}})_{\mathrm{sing}} \leq \dim \overline{\phi(C)} - 2$ for each $C \in \operatorname{Irr} X$. Then for f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\mathbb{F}} \to \operatorname{Irr}_{\mathrm{horiz}}(X_f)_{\mathbb{F}}$ sending C to $(C_f)_{\mathrm{horiz}}$.*

Proof. By Lemma 2.3, we may again assume that X is reduced and smooth, so its irreducible components are disjoint. Without loss of generality, we can thus further suppose that X is irreducible. Let $C \in \operatorname{Irr} X_{\mathbb{F}}$, then $\overline{\phi(C)}$ is an irreducible component of $\overline{\phi(X)}_{\mathbb{F}}$. By Proposition 4.2, $\overline{\phi(C)}_f$ is irreducible for f in a set of density 1. Applying Lemma 5.2 to $C \to \overline{\phi(C)} \hookrightarrow \mathbb{P}_{\mathbb{F}}$ shows that $(C_f)_{\mathrm{horiz}}$ is irreducible. Together with Lemma 2.4, this implies the existence of a bijection $\operatorname{Irr} X_{\mathbb{F}} \to \operatorname{Irr}_{\mathrm{horiz}}(X_f)_{\mathbb{F}}$ sending C to $(C_f)_{\mathrm{horiz}}$. □

Proof of Theorem 1.1. The proof is as in [1, Theorem 1.6], adjusting the notion of horizontal components. □

References

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