An Auctioning Approach to Railway Track Allocation

Thomas Schlechte

joint work with

Ralf Borndörfer and Martin Grötschel

16.07.2008

IAROR Summer Course on Railway Timetable Optimization
2008 Delft (Netherlands)
Konrad K. Zuse is the creator of the first fully automatic, programm controlled and freely programmable computer working in binary floating point arithmetic. The Z3 was finished in 1941.

Zuse-Institute-Berlin (ZIB)
Overview

1. Idea & Motivation
2. Train Timetabling Problem
3. Model and Algorithms
4. Computational Studies
Overview

1. Idea & Motivation
   1. Auction Setting
   2. Planning Process
2. Train Timetabling Problem
3. Model and Algorithms
4. Computational Studies
Examples

- In ancient times ...
  - Auctions are known since 500 b.c.
  - March 28, 193 a.d.: The pretorians auction the Roman Emperor’s throne to Marcus Didius Severus Iulianus, who ruled as Iulianus I. for 66 days
[193 A.D., March 28] When the emperor Pertinax was killed trying to quell a mutiny, no accepted successor was at hand. Pertinax's father-in-law and urban prefect, Flavius Sulpicianus, entered the praetorian camp and tried to get the troops to proclaim him emperor, but he met with little enthusiasm. Other soldiers scoured the city seeking an alternative, but most senators shut themselves in their homes to wait out the crisis. Didius Julianus, however, allowed himself to be taken to the camp, where one of the most notorious events in Roman history was about to take place. Didius Julianus was prevented from entering the camp, but he began to make promises to the soldiers from outside the wall. Soon the scene became that of an auction, with Flavius Sulpicianus and Didius Julianus outbidding each other in the size of their donatives to the troops. The Roman empire was for sale to the highest bidder. When Flavius Sulpicianus reached the figure of 20,000 sesterces per soldier, Didius Julianus upped the bid by a whopping 5,000 sesterces, displaying his outstretched hand to indicate the amount. The empire was sold, Didius Julianus was allowed into the camp and proclaimed emperor.
Arguments for Auctions

- Auctions can ...
  - resolve user conflicts in such a way that the bidder with the highest willingness to pay receives the commodity (efficient allocation, wellfare maximization)
  - maximize the auctioneer’s earnings
  - reveal the bidders’ willingness to pay
  - reveal bottlenecks and the added value if they are removed

- Economists argue ...
  - that a “working auctioning system” is usually superior to alternative methods such as bargaining, fixed prices, etc.
2 Tickets EM 2008 Deutschland-Österreich 16.06.08 KAT.3

Bis heute wurden 060 Anfragen erhoben

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Angemeldet als privater Verkäufer

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Angaben: seit 07.05.02 in Deutschland
Angemeldet als privater Verkäufer

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William Vickrey (1914-96)

William Vickrey invented this auction design. As in any auction the bidder with the highest value gets the surcharge, but he has only to pay the value of the second best (second-price-auction).

The advantage of this auction is that truthfull bidding is a dominant strategy for all bidders.

That means gambling makes no sense in contrast to a first price auction and therefore all bidders would prefer to bid equal to their willingness to pay.

The submission of the bids is often sealed and performed as a one shot auction.
Game Theory

- **Game (N,S,a)**
  - N={1,...,n} player
  - S={(s₁,...,sₙ)} strategies
  - a:S→Rⁿ payoff

- **Non-cooperative games**
  - Dominance
  - (Nash-)Equilibrium ŝ
    - \( aᵢ(ŝ₁,...,sᵢ,...,ŝₙ) ≤ aᵢ(ŝ₁,..., ŝₙ) \) ∀i
    - (i.g. no existence/uniqueness)
  - Matrix games: saddle point, minimax

- **Theorem (Nash):** Every finite non-cooperative n-person game has at least one equilibrium of mixed strategies.

- **Theorem (Nikaido, Isoda):** Generalization to auction frameworks.

- **Cooperative games**
  - Imputation (payoff to members of a coalition)
  - Concepts such as core, stable set, bargaining set, kernel, nucleolus, etc.
Combinatorial Auction - Easy Example

UMTS Frequencies?
# Frequency Auction

<table>
<thead>
<tr>
<th>Frequency / Bidder</th>
<th>A</th>
<th>B</th>
<th>A and B</th>
</tr>
</thead>
<tbody>
<tr>
<td>WodaPhon</td>
<td>10</td>
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<tr>
<td>Thelekom</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
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Vickrey-price for WodaPhon is $24 - (30 - 10) = 4$
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Vickrey-price for Thelekom is $24 - (30 - 20) = 14$
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"Collusion"

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<td>0</td>
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All vickrey-prices are 0

\( (= 24 - (48 -24)) \)
### "Shill-bidding"

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<td>30</td>
</tr>
</tbody>
</table>
```

Vickrey-prices are maximal
(WodaPhon 10 and Thelekom 20)
Combinatorial Auction

- Combinatorial Auction Problem (CAP)
  - $M$ objects, $N$ bidders, $b^j(S)$ bid by $j$ for $S \subseteq M$
  - $y(S,j)$ 0/1-variable for giving $S$ to $j$

$$\max \sum_{S \subseteq M} \sum_{j \in N} b^j(S) y(S,j)$$

$$\sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \quad \forall i \in M$$

$$y(S,j) \in \{0,1\} \quad \forall S \subseteq M, j \in N$$

- Set Packing Problem
- Auction framework
Vickrey Auction
(Nobel price in Economics 1996)

- Combinatorial auction

\[ E(N,b) := \max \sum_{S \subseteq M} \sum_{j \in N} b^j(S) y(S,j) \]

\[ \sum_{S \ni i} \sum_{j \in N} y(S,j) \leq 1 \quad \forall i \in M \]

\[ y(S,j) \in \{0,1\} \quad \forall S \subseteq M, j \in N \]

- Private values \( v_j \)

- Mechanism
  - Bids \( b_j = v_j \)
  - Payments
    \[ z_j = E(N \setminus j,v) - E(N,v) \mid N \setminus j \]
Auctions

- **Commodities/Bids**
  - Independent commodities (classical auction)/commodity bundles (combinatorial auction)
  - Combinatorial bids (and/or/xor)

- **Bidders**
  - Cooperation forbidden/cooperation allowed

- **Payment**
  - First price/second price (Vickrey) auction

- **Information**
  - Private Values/Common Values (winner's curse)
  - Sealed Bid/Open Bid

- **Mechanism**
  - English auction/dutch auction
  - Increment/number of rounds
  - Activity rules/taking bids back
  - Direct bidding/clock/proxy auction
Rail Track Auctioning

- Goals
  - More traffic at lower cost
  - Better service

- How do you measure?
  - Possible answer: in terms of willingness to pay

- What is the "commodity" of this market?
  - Possible answer: timetabled track
    = dedicated, timetabled track section

- How does the market work?
  - Possible answer: by auctioning timetabled tracks

- Auctions can be in-company auctions
Rail Track Auction

BEGIN
Minimum Bid = Basic Price

Bids are increased by a **minimum increment**

TOCs decide on bids for bundles of timetabled tracks

Bid is assigned

Bids is unchanged

Winner Determination: Find optimal track allocation with maximum earnings

Bid is not assigned

All bids assigned: **END**
Sears, Roebuck & Co.

- 3-year contracts for transports on dedicated routes
- First auction in 1994 with 854 contracts
- Combinatorial auction
  - "And-" and "or-" bids allowed
  - $2^{854} \approx 10^{257}$ theoretically possible combinations
  - Sequential auction (5 rounds, 1 month between rounds)
- Results
  - 13% cost reduction
  - Extension to 1.400 contracts (14% cost reduction)
**Frequency Auctions**


- Prices for mobile telecommunication frequencies (2x10 MHz+5MHz)
- Low earnings are not per se inefficient
- Only min. prices => insufficient cost recovery
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Planning in Public Transport

Strategic Stage
- Tracks
- Lines/Freq.
- Timetables
- Stops

Tactical Stage
- Timetables
- Prices
- Connections
- Rotations

Operational Stage
- Timetables
- Vehicles
- Crews
- Duties
Traffic Projects @ ZIB

92-94
94-97
97-00
00-03
03-07

VS-OPT
DS-OPT
IS-OPT
TS-OPT

VS: BVG
DS: BVG

Line+Price Planning

MCF
Telebus

CS-OPT
Planning in Public Transport

Tracks - B1 - B15

Lines/Freq.

Timetables

Vehicles

Crews

Strategic Stage

Tactical Stage

Operational Stage

Stops

Prices

Connections

Rotations

Duties
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Overview

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   1. Problem Definition
   2. Complexity Issues
   3. Standardization
3. Models and Algorithms
4. Computational Studies
The Problem (TraVis by M. Kinder)
Schedule in 3d
Conflict-Free-Allocation
Railway Timetabling – State of the Art

- Charnes and Miller (1956), Szpigel (1973), Jovanovic and Harker (1991),
- **Caprara, Fischetti and Toth (2002)**, Peeters (2003)
- Semet and Schoenauer (2005),
- **Caprara, Monaci, Toth and Guida (2005)**
- Kroon, Dekker and Vromans (2005),
- Vansteenwegen and Van Oudheusden (2006),
- Caprara, Kroon, Monaci, Peeters, Toth (2006)
- **Borndoerfer, Schlechte (2007)**

**non-cyclic timetabling literature**
Track Allocation Problem

Train Requests ➔ Scheduling Digraph ➔ Timetable
**Complexity**

**Proposition** [Caprara, Fischetti, Toth (02)]:

OPTRA/TTP is \( \mathcal{NP} \)-hard.

**Proof:**

Reduction from Independent-Set.
Independent/Stable Set Problem

\[ S \subseteq 2^V \]

\[ s \in S \iff \forall u, v \in s : (u, v) \notin E \]

\[ G = (V, E) \]

\[ \max_{s \in S} |s| \]
Polynomial Reduction

\[ s \quad (1,2) \quad (2,3) \quad (2,4) \quad (3,4) \quad (4,5) \quad t \]
Conflict (1,2)
Conflict (2,3)
Conflict (2,4)
Conflict (3,4)
Conflict (4,5)
Feasible Train Set $\Leftrightarrow$ Stable Set
Feasible Train Set $\Leftrightarrow$ Stable Set
Feasible Train Set $\Leftrightarrow$ Stable Set

$s \quad (1,2) \quad (2,3) \quad (2,4) \quad (3,4) \quad (4,5) \quad t$
Standardization http://ttplib.zib.de
MacroInfra.xsd
RequestSet.xsd
MacroTimetable.xsd
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   3. IP Models for TTP
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   5. Branch & Bound & Price
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The structure shown in Table 1 can be translated into equation form by moving a row of \( \lambda \)'s, one for each column, up through the rows and inserting the equal sign to the right of the \( P \) column. The first two equations, for example, would be:

\[
4 = \lambda_1 + \lambda_2 - \lambda_6 + \lambda_{10} \\
1 = \lambda_1 + \lambda_6 - \lambda_2 + \lambda_{10}
\]

With the addition of the variables, the problem has been reduced to a standard simplex problem of the form:

\[
\text{Min} \sum_{i=1}^{k} \lambda c_i
\]

subject to:

\[
\sum_{i=1}^{k} \lambda_i P_i = P_k \\
\lambda_i \geq 0
\]

and can be solved by the simplex technique.

Abraham Charnes
Finalist for Nobel Prize in Economic Sciences 1975

Merton H. Miller
Nobel Prize Winner in Economic Sciences 1990 with Markowitz and Sharpe

**TABLE 1**

<table>
<thead>
<tr>
<th>( c_i )</th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
<th>( P_5 )</th>
<th>( P_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

**Chart 1.** Simplified map of terminal switching railroad, showing connections with tracks, major interchange and customer yard areas, and traffic requirements (in train-loads) between major points.

postponed until the description of the model and the computational routine has been completed.

Above the routes, in the row labeled \( c_i \), are entered the costs of assigning a single crew and engine package to the route in question. These costs may be stated either as the standard crew and engine expenses, or as the expected costs reflecting the fact that on longer runs there is a greater probability of running into overtime. We constructed working models both ways and found, that optimal programs were not particularly sensitive to variations in the cost of crews. In fact, it was usually possible to simplify the calculation by minimizing the number of crews, that is treating the cost of each crew as 1.

\( P_1 \) to \( P_6 \) in the tableau are overfulfillment slack vectors. In the train scheduling context they correspond to "light moves", or trips by a crew and engine without cars. If, for example, four crews should be assigned to the route \( P_1 \)—which runs
Integer Program

- Graph Model
- Algebraic Model

Min \( x_1 + x_2 + x_3 + 1.2x_4 + 1.2x_5 \)

\[
\begin{align*}
4 & \leq x_1 + x_4 \\
1 & \leq x_1 + x_5 \\
9 & \leq x_3 + x_5 \\
5 & \leq x_3 + x_4 \\
6 & \leq x_2 + x_4 \\
3 & \leq x_2 + x_5 \\
x & \geq 0
\end{align*}
\]

\( x \) integer
Linear Optimization

- **Algebraic Model**
  - (LP, Simplex algorithm)
  
  \[
  \text{Min } x_1 + x_2 + x_3 + 1.2x_4 + 1.2x_5 \\
  4 \leq x_1 + x_4 \\
  1 \leq x_1 + x_5 \\
  9 \leq x_3 + x_5 \\
  5 \leq x_3 + x_4 \\
  6 \leq x_2 + x_4 \\
  3 \leq x_2 + x_5 \\
  x \geq 0
  \]

Leonid V. Kantorovich
Nobel Prize in Economic Sciences 1975

Tjalling C. Koopmans
Nobel Prize in Economic Sciences 1975

George W. Dantzig
Father of Linear Programming
LP Progress: 1988-2004


- Algorithms (*machine independent*):
  - Primal *versus* best of Primal/Dual/Barrier: 3300x
- Machines (workstations → PCs): 1600x
- NET: Algorithm × Machine: 5 300 000x
  - (2 months/5300000 = 1 second)
IP Solvers CPLEX, XPRESS, SCIP...
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General Framework - SCIP
Linear Relaxation

- solve linear program by primal, dual simplex or barrier (interior point method)
current solution has fractional variables
improve by valid cutting planes
Branching

- current solution is infeasible (at least one forbidden fractional value)
Branching on Variables

- split problems into sub problems to cut off current solution
Branching on Constraints

- split problems into subproblems to cut off current solution
- divide and conquer (split problem into smaller subproblems)
- use conflict analysis (constraint propagation)
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### Arc Packing Problem

\[(APP)\]

\[
\max \quad \sum_{i \in \mathcal{I}} \sum_{a \in A} p^i_a x^i_a
\]

s.t.

\[
\sum_{a \in \delta^\text{out}_i(v)} x^i_a - \sum_{a \in \delta^\text{in}_i(v)} x^i_a \leq \delta_i(v) \quad \forall v \in V, \forall i \in \mathcal{I} \quad (i)
\]

\[
\sum_{i \in \mathcal{I}} \sum_{a \in A} x^i_a \leq 1 \quad \forall c \in C \quad (ii)
\]

\[
x^i_a \in \{0, 1\} \quad \forall a \in A, \forall i \in \mathcal{I} \quad (iii)
\]

#### Variables
- Arc occupancy (request i uses arc a)

#### Constraints
- Flow conservation and
- Arc conflicts (pairwise )

#### Objective
- Maximize proceedings

(PPP) transformation from arc to path variables (see Cachhiani (2007))
TTP as APP

- macroscopic conflicts
  - headways
  - station capacities
- simplified
  - block occupation

Amsterdam    Delft
Packing Constraints

Strengthen Constraints

Instead of:

\[ + \quad + \quad \leq 1 \]

\[ + \quad \leq 1 \]

\[ + \quad \leq 1 \]

\[ + \quad \leq 1 \]
Headway conflicts

Is it possible to get all maximal (exp. ?) packing constraints in polynomial time?

Block occupation

**Intervall Graph !**

[Helly 1912, Borndörfer, Schlechte 2007]

Quadrangle-linear-headway matrices

**Perfect Graph !**

[Lukac 2004]

Arbitrary Headway Matrices?
Packing Models

- **Proposition:**
  The LP-relaxation of APP can be solved in polynomial time.
- ... and in practice.
A(rc) C(oupling) P(roblem)
Path Coupling Problem

\[(PCP) \quad (i) \quad \max \sum_{p \in P} w_p x_p \]

\[ (ii) \quad \sum_{p \in P_i} x_p \leq 1, \quad \forall i \in I \]

\[ (iii) \quad \sum_{q \in Q_j} y_q \leq 1, \quad \forall j \in J \]

\[ (iv) \quad \sum_{a \in P} x_p - \sum_{a \in Q} y_q \leq 0, \quad \forall a \in A_{LR} \]

\[ (v) \quad x_p, y_q \in \{0, 1\} \quad \forall p \in P, \quad q \in Q \]

Variables
- Path und config usage (request i uses path p, track j uses config q)

Constraints
- Path and config choice
- Path-config-coupling (track capacity)

Objective Function
- Maximize proceedings
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Duality of Linear Programs

Julius Farkas [1847-1930]:
Duality theorem based
Farkas Lemma „a theorem of alternatives“

\[(P)\]
\[\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b, \ x \geq 0
\end{align*}\]

\[(D)\]
\[\begin{align*}
\text{maximize} & \quad b^T y \\
\text{subject to} & \quad A^T y \geq c, \ y \geq 0
\end{align*}\]

\text{PRICE}(x) \text{ is equivalent to } \text{SEPERATION}(y) !
LP with too many variables

- Set Partitioning Problem
  - minimize $c^T x$
  - row sum equal to 1
**Strong Duality:** If there exists an optimal solution $x^*$ for (P), then there exists an optimal solution $y^*$ for (D) and their objective values are equal.
## Column Generation Example

| nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| c  | 5 | 5 | 3 | 3 | 3 | 3 | 3 | 8 | 9 | 6 | 7 | 8 | 9 | 4 | 4 | 4 | 4 | 3 | 4 | 5 | 5 | 4 | 10 | 11 | 12 | 7 | 8 | 9 | 5 | 6 | 5 | 11 | 12 | 12 | 8 | 9 | 9 | 6 | 12 | 9 | y |
| 1  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6  | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

- \( x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1 \), primal objective \( 2 \cdot 5 + 4 \cdot 3 = 22 \)
- \( y_1 = 5 \)
- \( y_2 = 5 \)
- \( y_3 = 3 \)
- \( y_4 = 3 \)
- \( y_5 = 3 \)
- \( y_6 = 3 \)
- \( c_{19} = 9, y_1 + y_2 + y_3 = 5 + 5 + 3 = 13 > 9 \Rightarrow x_{19} = 1, x_1 = x_2 = x_3 = 0 \)
Column Generation Example

- $x_1 = x_2 = x_3 = 0$, $x_4 = x_5 = x_6 = x_{19} = 1$, primal objective $9 + 3 \times 3 = 18$ [22]
- $y_1 + y_2 + y_3 = 11 \Rightarrow y_1 = 1$, $y_2 = 5$, $y_3 = 3$
- $y_4 = 3$
- $y_5 = 3$
- $y_6 = 3$
- $c_{34} = 9$, $y_2 + y_4 + y_5 + y_6 = 5 + 9 = 14 > 9$
- $c_{36} = 12$, $y_1 + y_3 + y_4 + y_5 + y_6 = 1 + 12 = 13 > 12$
Column Generation Example

- $x_1 = x_2 = x_3 = 0$, $x_4 = x_5 = x_6 = x_{19} = 1$, primal objective $9 + 3 \times 3 = 18$ [22]
- $y_1 + y_2 + y_3 = 11 \Rightarrow y_1 = 1$, $y_2 = 5$, $y_3 = 3$ (solution not unique)
  - $y_4 = 3$
  - $y_5 = 3$
  - $y_6 = 3$
- $c_{34} = 9$, $y_2 + y_4 + y_5 + y_6 = 5 + 9 = 14 > 9$
- $c_{36} = 12$, $y_1 + y_3 + y_4 + y_5 + y_6 = 1 + 12 = 13 > 12$
  $\Rightarrow x_{34} = x_{36} = x_{19} = 1/2$, $x_4 = x_5 = x_6 = 0$
### Column Generation Example

| nr | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 |
|----|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| c  | 5 | 5 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   | 9 |   |
| y  | 1 | 1 |   |   |   |   |   |   |   | 1 |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| x  |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |

- \( x_{34} = x_{34} = x_{19} = 1/2 \), primal objective \((9+9+12)/2 = 15\) [18]
- \( y_1 + y_2 + y_3 = 9\)
  \( y_2 + y_4 + y_5 + y_6 = 9\)
  \( y_1 + y_3 + y_4 + y_5 + y_6 = 12 \Rightarrow y_1 = 5, y_2 = y_4 = y_5 = 3, y_3 = 1\)
- \( c_{28} = 5, y_4 + y_5 + y_6 = 6 \geq 5\)
  \( \Rightarrow x_{28} = x_{19} = 1, x_{34} = x_{36} = 0\)
Column Generation Finished

- \( x_{19} = x_{28} = 1 \), primal objective 9+5=14 [15]
- \( y_1 + y_2 + y_3 = 9 \)
  \( y_4 + y_5 + y_6 = 5 \Rightarrow y_1 = 5, y_2 = 3, y_3 = y_6 = 1, y_4 = y_5 = 2 \)
- No column exists with negative reduced cost
- \( \Rightarrow x^* \) and \( y^* \) are optimal solutions
# History of Column Generation

<table>
<thead>
<tr>
<th>Article</th>
<th>Constraints</th>
<th>Variables</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charnes &amp; Miller 1956</td>
<td>6</td>
<td>17</td>
<td>manual</td>
</tr>
<tr>
<td>Hoffman &amp; Padberg 1993</td>
<td>145</td>
<td>1.053.137</td>
<td>5 min</td>
</tr>
<tr>
<td>Bixby, Gregory, Lustig, Marsten, Shanno 1992</td>
<td>837</td>
<td>12.753.313</td>
<td>249 sec</td>
</tr>
<tr>
<td>Barnhart, Johnson, Nemhauser, Savelsbergh, Vance 1998</td>
<td>&gt;10.000</td>
<td>unknown</td>
<td>several days</td>
</tr>
</tbody>
</table>
Linear Relaxation of PCP

\[
\begin{align*}
  (MLP) \quad & \max \sum_{p \in \mathcal{P}} w_p x_p + \sum_{q \in \mathcal{Q}} r_q y_q \\
  \text{s.t.} \quad & \sum_{p \in \mathcal{P}_i} x_p \leq 1 \quad \forall i \in I \quad (i) \\
  & \sum_{q \in \mathcal{Q}_j} y_q \leq 1 \quad \forall j \in J \quad (ii) \\
  & \sum_{a \in \text{a} \in \mathcal{P}} x_p - \sum_{q \in \mathcal{Q}} y_q \leq 0 \quad \forall a \in A_{LR} \quad (iii) \\
  & 0 \leq y_q \leq 1 \quad \forall q \in \mathcal{Q} \quad (iii) \\
  & 0 \leq x_p \leq 1 \quad \forall p \in \mathcal{P} \quad (iv)
\end{align*}
\]

<table>
<thead>
<tr>
<th>dual variable</th>
<th>information about</th>
<th>useful to</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \gamma_i )</td>
<td>bundle price</td>
<td>analyse request</td>
</tr>
<tr>
<td>( \pi_j )</td>
<td>track price</td>
<td>analyse network</td>
</tr>
<tr>
<td>( \lambda_a )</td>
<td>arc price</td>
<td>-</td>
</tr>
</tbody>
</table>
Dualization

\[ \text{(DLP)} \]
\[ \begin{align*}
\min & \quad \sum_{j \in J} \pi_j + \sum_{i \in I} \gamma_i \\
\text{s.t.} & \quad \gamma_i + \sum_{a \in p} \lambda_a \geq w_p \quad \forall p \in \mathcal{P}_i, \forall i \in I \quad (i) \\
& \quad \pi_j - \sum_{a \in q} \lambda_a \geq r_q \quad \forall q \in \mathcal{Q}_j, \forall j \in J \quad (ii) \\
& \quad \gamma_i \geq 0 \quad \forall i \in I \quad (iii) \\
& \quad \lambda_a \geq 0 \quad \forall a \in A_{LR} \quad (iv) \\
& \quad \pi_j \geq 0 \quad \forall j \in J \quad (v)
\end{align*} \]
Pricing of x-variables

\[(\text{PRICE } (x)) \exists \bar{p} \in P_i : \gamma_i < \sum_{a \in \bar{p}} (p_a - \lambda_a)\]

\[c_a = -p_a + \lambda_a\]

Pricing Problem(x):
Acyclic shortest path problems for each slot request i with modified cost function c!
Pricing of \( y \)-variables

\[
(\text{PRICE } (y)) \quad \exists \, q \in Q_j : \quad \pi_j < r_q + \sum_{a \in q} \lambda_a
\]

\[
c_a = -r_a - \lambda_a
\]

Pricing Problem(\( y \)):
Acyclic shortest path problem for each track \( j \) with modified cost function \( c \)!
Observation

- **Lemma** [ZR-07-02]: The linear relaxation of PCP can be solved in polynomial time, due to the equivalence of optimization and separation (see Groetschel, Lovasz & Schrijver [88]).
(PRICE (x)) \( \exists \bar{p} \in \mathcal{P}_i : \gamma_i < \sum_{a \in \bar{p}} (p_a - \lambda_a) \)

\( \eta_i := \max_{p \in \mathcal{P}_i} \sum_{a \in p} (p_a - \lambda_a) - \gamma_i, \forall i \in I \)

\( \eta_i + \gamma_i \geq \sum_{a \in p} (p_a - \lambda_a) \forall i \in I, p \in \mathcal{P}_i \)
And analogously...

(PRICE (y)) \exists \overline{q} \in Q_j : \pi_j < \sum_{a \in \overline{q}} \lambda_a

\theta_j := \max_{\overline{q} \in Q_j} \sum_{a \in \overline{q}} \lambda_a - \pi_j, \ \forall j \in J

\theta_j + \pi_j \geq \sum_{a \in q} \lambda_a \ \forall j \in J, q \in Q_j
Pricing Upper Bound

$\max\{\eta+\gamma, 0\}, \max\{\theta+\pi, 0\}, \lambda$ is feasible for (DLP)

\[ \beta(\gamma, \pi, \lambda) := \sum_{i \in I} \max\{\gamma_i + \eta_i, 0\} + \sum_{j \in J} \max\{\pi_j + \theta_j, 0\} \]

- **Lemma** [ZR-07-02]: Given (infeasible) dual variables of PCP and let $v_{LP}(PCP)$ be the optimum objective value of the LP-Relaxation of PCP, then:

\[ v_{LP}(PCP) \leq \beta(\gamma, \pi, \lambda) \]
Overview

1. Idea & Motivation
2. Train Timetabling Problem
3. Models and Algorithms
   1. Historical Example
   2. IP Solving
   3. IP Models for TTP
   4. Column Generation
   5. Branch & Bound & Price
4. Computational Studies
Two Step Approach

- **TS-OPT**
  - 1. LP Solving
  - 2. IP Solving

- **Duals by Bundle Method**

- **Column Generation**

- **Pricing by Dijkstra’s Shortest Path**

- **Rapid Branching Heuristic**
Evaluation of only few highly different sub-problems at iteration $j$ to reach IP-Solutions fast.
Rapid Branching

Node selection of set of fixed to 1 variables by using perturbated cost function (bonus close to 1.0).

Update Upper Bound

$S^0_j$ $\rightarrow$ $S_{j+1}$

Go on if target was reached, otherwise pseudo-backtrack.

Column Generation

$S^l_j$
Overview

1. Idea & Motivation
2. Train Timetabling Problem
3. Models and Algorithms
4. Computational Studies
Results

- **Test Network**
  - 45 Tracks
  - 37 Stations
  - 6 Traintypes
  - 10 Trainsets
  - 146 Nodes
  - 1480 Arcs
  - 96 Station Capacities
  - 4320 Headway Times
Computational Results

- **Test Scenarios**
  - 146 Train Requests
  - 285 Train Requests
  - 570 Train Requests

- **Flexibility**
  - 0-30 Minutes
  - earlier departure penalties
  - late arrival penalties
  - train type depending profits
Run of TS-OPT/LP Stage

scenario 570 trains

objective value

\( x \times 10^5 \)

\( \beta(\gamma, \pi, \lambda) \)

\( v(\text{RPLP}) \)

column generation iterations
Model Comparison

For details see [ZR-07-02, ZR-07-20].
Scenario: Hannover-Kassel-Fulda

- **Situation**
  - ~ 8h time window
  - 250 trains (ICE,IC,RE,RB,S)
  - ~ original public timetable 2002
  - utility function 156468.0 (~track prices)
Optimization problem with thousands of possible paths

- additional demand of 35 cargo trains

- 265 trains scheduled
  - 237 „old“ passenger trains
  - 28 „new“ cargo trains
  - utility function 174628.0 (ca. +12%)
Don’t try this by hand!

- 237 “old” passenger trains
  - 13 deleted
  - more than 100 trains were shifted (at least one difference in departure or arrival at some station)
  - computation time: several minutes
Scheduling with guarantee!

- 28 “new” cargo trains
  - scheduling per hand is too complex, time-consuming, and not resistant against mistakes of planners
  - there is a need to have decision support for macroscopic timetabling to support crucial strategic decisions
  - there is a need to evaluate macroscopic timetables on “real-world” microscopic stage
Outlook

Future Plans

- Model refinement (robustness)
- Model refinement (connections)
- Solver speedup by bundle method
- Adaptive IP heuristics
- Transformation Micro->Macro->Micro

Simulation of results by

[Company Logos]

[Image]
Thank you for your attention!