Exercise 6.1 10 Points

Given a polyhedron $P(A,b)$ defined by

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad b = \begin{pmatrix}
2 \\
1 \\
1 \\
2 \\
2 \\
0 \\
0 \\
0 \\
0 \\
0 \\
1 \\
1 \\
1 \\
1 \\
0 \\
1 \\
1 \\
1 \\
\end{pmatrix}
\]

Show the following statements:

a) $\dim(P(A,b)) = 5$.

b) $-x_i \leq 0$ defines a facet of $P(A,b)$ for all $i = 1, \ldots, 5$.

c) $x_5 \leq 1$ defines a facet of $P(A,b)$.

d) Which of the inequalities are redundant?

Exercise 6.2 10 Points

a) Proof:

Let $\mathcal{P} = \mathcal{P} = (A,b) \subseteq \mathbb{R}^n$ be a polyhedron, $x \in \mathcal{P}$. The following statements are equivalent:

i) $x$ is a vertex of $\mathcal{P}$
ii) $\text{rank } A_{\text{supp}(x)} = |\text{supp}(x)|$

iii) $\{A_j\}_{j \in \text{supp}(x)}$ is linear independent

b) A matrix $A \in \mathbb{R}^{m \times n}$ is called totally unimodular if each square submatrix of $A$ has determinant equal to 0, $+1$, or $-1$. In particular, each entry of a totally unimodular matrix is 0, $+1$, or $-1$.

Let $A \in \mathbb{R}^{m \times n}$ totally unimodular and $b \in \mathbb{Z}^m$. Proof the following statement:

If $A$ is totally unimodular, then the polyhedron $P=(A, b)$ has only integer vertices.

*Hint: Use a) and Cramer’s Rule.*

**Exercise 6.3**

10 Points

a) Show that each non-trivial face of a polyhedron is the intersection of facets of the polyhedron.

b) Let $P$ be a polyhedron of dimension $d$, and let $F$ be a non-trivial face of $P$ whose dimension $k$ is less than $d$. Prove that there exist faces $F_{k+1}, F_{k+2}, \ldots, F_{d-1}$ of $P$ such that

$(i) \; F \subseteq F_{k+1} \subseteq F_{k+2} \subseteq \ldots \subseteq F_{d-1} \subseteq P,$

$(ii) \; \dim(F_{k+i}) = k + i, \; \text{für } i = 1, \ldots, d-k-1.$

*Hint: induction over $d-k$*

Homepage of the Lecture: [http://www.zib.de/ss17_Optimierung_I](http://www.zib.de/ss17_Optimierung_I)

Questions?: klug@zib.de