Fractal Power Spectrum in a Steady Plane Stokes Flow Past the Lattice of Cylinders

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We study spectral and correlation properties for tracer particles in steady two-dimensional flows of incompressible viscous fluids past doubly periodic arrays of solid circular cylinders. It is demonstrated that in a class of such flows, the Fourier spectrum is neither discrete nor absolutely continuous, and the autocorrelation decays in accordance with the power law.

§1. Introduction

It has been long recognized that irregular transport of passive tracers by viscous flows can have well-ordered flow patterns as its background.[1)] The origin of this phenomenon is rooted in the distinction between the Eulerian description of the flow and the Lagrangian description of individual tracer particles: the former characterizes the fluid motion in a fixed point of the physical space, whereas the latter is bound to the moving particle which is advected by the fluid and explores different regions of the physical space. Chaotic advection (known also as "Lagrangian chaos") has been detected in steady or time-periodic flows on micro-, meso- and macroscopic scales, from nanotechnology to astrophysics.[2)] If particle paths are viewed as phase trajectories, the volume filled with fluid turns into the phase space; from this point of view it is clear that in order to exhibit Lagrangian chaos, a flow pattern must be either three-dimensional, or time-dependent, or both.

Since the geometry of a two-dimensional phase space precludes chaos, the conventional examples of advection in time-independent two-dimensional flows have simple dynamical characteristics: motions of typical tracer particles are stationary or (quasi)-periodic in time; they are well correlated, and the power spectra of such motions are discrete. Here, we present a class of time-independent two-dimensional flows for which the Fourier spectrum sits on the fractal set, and the correlation decays in accordance to the power law.

§2. Problem formulation

We consider a steady flow of a viscous incompressible fluid through the square lattice of parallel solid circular cylinders. Let the flow be perpendicular to the axes of cylinders. Assuming that the Reynolds number is small, we neglect nonlinear terms in the Navier-Stokes equation and reduce it to the Stokes equation for the velocity

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$\boldsymbol{V}$ and pressure p:

$$-\nabla p + \eta \Delta \boldsymbol{V} = 0, \quad \operatorname{div} \boldsymbol{V} = 0, \tag{1}$$

where η is the viscosity of the fluid. As boundary conditions, we impose periodicity on the borders of the elemental cell of the lattice. Choosing the size of this cell as the length unit and directing the axes x and y along the axes of the lattice, we obtain

$$\boldsymbol{V}(0,y) = \boldsymbol{V}(1,y), \quad \boldsymbol{V}(x,0) = \boldsymbol{V}(x,1). \tag{2}$$

Along the boundary $\partial\Gamma$ of the solid cylinder Γ the velocity vanishes, ensuring the no-slip condition:

$$\boldsymbol{V}\big|_{\partial\Gamma} = 0. \tag{3}$$

To compensate the viscous dissipation, external factors (e.g. in the form of mean gradients of pressure) must be present; we ensure this balance by prescribing the components of mean flow along both axes:

$$\int_0^1 V_x dy = \alpha, \; \int_0^1 V_y dx = \beta. \tag{4}$$

Incompressibility allows to introduce in the standard way the stream function $\Psi(x,y)$: $V_x = \partial\Psi/\partial y$ and $V_y = -\partial\Psi/\partial x$. In this terms, Eq. (1) is reduced to the biharmonic equation $\Delta^2\Psi = 0$. Vanishing of velocity on the border of the cylinder turns into $\partial\Psi/\partial x\big|_{\partial\Gamma} = \partial\Psi/\partial y\big|_{\partial\Gamma} = 0$. Conditions for mean flow and periodicity can be satisfied simultaneously by putting $\Psi(x,y) = \alpha y - \beta x + \Phi(x,y)$ where Φ has period 1 with respect to both of its arguments. Since the flow is steady, the fluid particles move along the streamlines. Notably, description in terms of Ψ is tantamount to the Hamiltonian formalism; therefore the transport of particles is governed by the integrable Hamiltonian system with one degree of freedom.

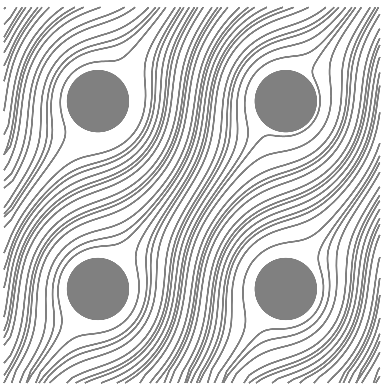

Fig. 1. Flow pattern with inclination $\alpha/\beta = (\sqrt{5}-1)/2$.

A solution of Eqs. (1) - (4) as series in elliptic functions of coordinates was obtained for the arbitrary inclination α/β of the flow by H. Hasimoto.[3] This allowed to estimate the drag on an individual cylinder and, further, to calculate the permeability of the array of cylinders.[4] For our studies of spectral and correlation properties of passive tracers carried in the velocity field, we used the numerical solution of the biharmonic equation which was computed on the polar grid with 600×1200 nodes; between the nodes, the field was interpolated by the scheme of the 2nd order. Figure 1 presents four adjacent cells of the flow pattern with inclination $\alpha/\beta = (\sqrt{5}-1)/2$ and radius of the cylinder 1/6.

§3. Fractal nature of the Fourier spectrum

Viewed as a dynamical system, the velocity field of Eqs. (1) - (4) is the area preserving vector field with rotation number α/β on the 2-torus. Spectral properties of flows on 2-tori were first studied by Kolmogorov;[5] for nowhere vanishing fields and typical irrational rotation numbers he found the discrete spectrum. In the absence of fixed points the return time (duration of one turn around the torus) is bounded. If isolated non-degenerate fixed points (stagnation points for the fluid) are present, return time has logarithmic singularities; they are too weak to ensure mixing.[6]

A peculiarity of the vector field in the considered problem is its identical vanishing on the cylinder border where every point is the fixed point of the flow. The collective effect of this continuum of equilibria produces a strong, power-like singularity of return time: close to the border, the return time diverges as $\xi^{-1/2}$ where ξ is the minimal distance between the streamline and the cylinder. Since under irrational rotation numbers every streamline is dense on the torus, each fluid particle repeatedly passes arbitrarily close to the cylinder and exhibits a strong slowdown. This has a noticeable effect on the observables associated to such particles. A numerical estimate of the power spectrum of the velocity of the moving fluid particle is shown in Fig. 2. The discrete set of delta-peaks, typical for quasiperiodic dynamics, is apparently absent; the spectrum looks rather like a continuous one or a mixed one. Logarithmic presentation of abscissa is more illuminating (Fig. 3): in the low-frequency part a distinct self-similar pattern is shaped.

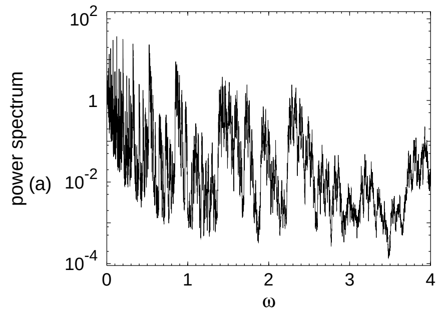

Fig. 2. Power spectrum for the tracer velocity; sample length 2^{13}, $\alpha/\beta = (\sqrt{5}-1)/2$.

A comparison of Figs. 3(a) and (b) shows that increase of the length of sample over which the spectrum is evaluated, does not ensure convergence to the limit spectral curve. Instead, the peaks become higher, the valleys lower and narrower, and

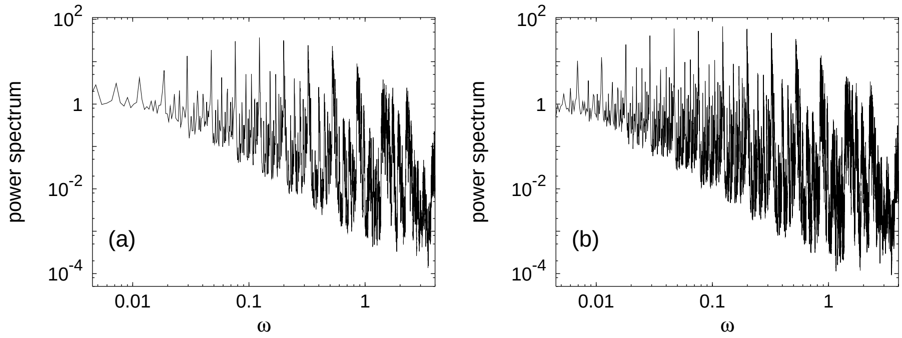

Fig. 3. Estimates of power spectrum. (a) Sample length 2^{13}; (b) sample length 2^{15}.

the whole plot is getting more dense. This process of "fractalization" of spectral curve reminds of "structure intermediate between quasi-periodic and random".[7] The power spectrum is singular continuous: neither discrete nor absolutely continuous with respect to the Lebesgue measure; the spectral measure is supported by the dense fractal set. It corresponds to dynamics intermediate between chaos and order.[8]

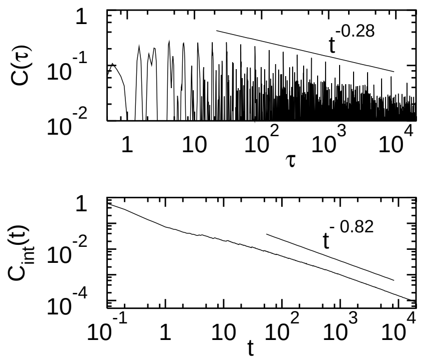

Fig. 4. Autocorrelation and integrated autocorrelation for the velocity of the tracer.

The unusual nature of the Fourier spectrum is reflected by the behavior of the autocorrelation $C(\tau)$: in contrast to conventional examples of two-dimensional dynamics, autocorrelation decays. This decay is slower than exponential: the highest peaks of $C(\tau)$, which form a log-periodic lattice, decrease in accordance with the power law $C(\tau) \sim \tau^{-0.28}$. Such "long-range correlations" are widespread in physics of critical phenomena, but here we encounter them in an integrable two-dimensional dynamical system where the correlations are usually supposed not to decay at all. Quantitative information on the fractal characteristics of the spectrum is delivered by the "integrated autocorrelation" $C_{\rm int}(t) = \frac{1}{t}\int_0^t C(\tau)^2 d\tau$: the decay of $C_{\rm int}(t)$ indicates to the absence of the discrete component in the Fourier spectrum. Further, the rate of this decay characterizes the fractal component: $C_{\rm int}(t) \sim t^{-D_2}$ where D_2 is the correlation dimension of the fractal set which supports the spectral measure.[9] According to our estimations, $D_2 \approx 0.82$.

Our result shows that viscous steady flows past periodic arrays of solid obstacles possess the fractal spectral component. As a consequence, these flows should display mixing. Of course, such mixing is slower and less intensive than in the case of chaotic advection; however, on the large timescale it leads to the same effect.

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