Linear oscillations of a compressible hemispherical bubble on a solid substrate

Sergey Shklyaev[1] and Arthur V. Straube[2,a)]
[1]Department of Theoretical Physics, Perm State University, Bukirev 15, Perm 614990, Russia
[2]Department of Physics and Astronomy, University of Potsdam, Karl-Liebknecht-Str. 24/25, D-14476 Potsdam-Golm, Germany

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The linear natural and forced oscillations of a compressible hemispherical bubble on a solid substrate are under theoretical consideration. The contact line dynamics is taken into account by application of the Hocking condition, which eventually leads to nontrivial interaction of the shape and volume oscillations. Resonant phenomena, mostly pronounced for the bubble with the fixed contact line or with the fixed contact angle, are found out. A double resonance, where independent of the Hocking parameter, an unbounded growth of the amplitude occurs, is detected. The limiting case of weakly compressible bubble is studied. The general criterion identifying whether the compressibility of a bubble can be neglected is obtained.
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I. INTRODUCTION

The dynamics of bubbles and droplets is of great interest for their numerous applications. Bubbly fluids are widely used as displacing fluids in petroleum industry. Small bubbles and droplets allow to efficiently control heat and mass transfer in heat exchangers and reactors and to intensify mixing in microdevices.[1,2] Whereas oscillations of drops suspended in a fluid ambient away from the boundaries have been scrutinized for over a century,[3] oscillations of drops and bubbles in contact with solid surfaces have only received attention for the last few decades. Understanding fundamental aspects of drops and bubbles interaction with the solid surface is closely related to the problem of wetting.[4,5] This knowledge is of practical importance because many technological processes deal with spreading of a liquid (a paint, a lubricant, or a dye) over solid surfaces. From the theoretical point of view, the presence of a solid surface often meets another problem, the contact line dynamics, which is currently far from being fully understood.

Qualitatively, oscillations of a liquid drop of averaged radius R can be characterized by three time scales: The viscous relaxation time $\tau_v=R^2/\nu$, the capillary time scale $\tau_c=\sqrt{\rho R^3/\sigma}$, which is related to the period of the shape oscillations,[6] and the time scale of the acoustic oscillations $\tau_a=R/c$. Here, σ is the surface tension, ρ, ν, and c are the density, kinematic viscosity, and speed of sound, respectively. Quite often, these three time scales relate to each other as

$$\tau_a \ll \tau_c \ll \tau_v. \tag{1}$$

This hierarchy of the times allows for the description of the drop oscillations within the model of inviscid incompressible fluid, where the dynamics of the drop corresponds to the shape oscillations. For instance, these inequalities hold for water drops of $R>10^{-5}$ cm. For a liquid drop immersed in another liquid, the hierarchy of the time scales is the same. The only difference is in the meaning of ρ, ν, and c, which should now be considered as some effective quantities, e.g., the mean values for the two liquids.

For a gaseous bubble in an ambient liquid, the situation is similar to the case of the drop, but an additional, the so-called breathing mode, appears.[7] This mode corresponds to the radial (volume) oscillations of the bubble and is caused by the bubble compressibility. Note that since the densities of the gas and liquid considerably differ, the gas compressibility is non-negligible even at subacoustic frequencies, $\omega_b\tau_a \ll 1$, where ω_b is the frequency of the breathing mode. In this case, the gas pressure in the bubble very quickly adjusts to the instant volume of the bubble. Although the pressure field in the bubble changes in time, its instant distribution can be considered as spatially homogeneous. If the dissipation is insignificant during a period of oscillation, the gas can be described by the adiabatic law. For this situation,[8] $\omega_b=\sqrt{3\gamma P_g/\rho R^2}$, where γ is the adiabatic exponent and P_g is the equilibrium pressure in the bubble (for a correction of ω_b caused by surface tension see Sec. II). Detailed analyses of damping effects can be found in Refs. 1 and 8. Particularly, these studies figure out the inequalities

$$\omega_b\tau_v \gg 1, \quad \omega_b\tau_t \gg 1, \quad \omega_b\tau_a \ll 1, \tag{2}$$

which can be applied to neglect the effects caused by viscosity, heat transfer, and acoustic irradiation, respectively. Here, we introduce the characteristic time of thermal diffusion $\tau_t=R^2/\chi_m$, with χ_m being the maximal from heat diffusivities of the fluid χ and the gas χ_g. As it follows from Eq. (2), the predominant damping mechanism for an air bubble in water is heat dissipation. This mechanism can be neglected provided that $R>10^{-3}$ cm.

If the frequencies of the shape and volume oscillations become comparable, the oscillations of these types start to

a)Author to whom correspondence should be addressed. Electronic mail: arthur.straube@gmail.com.

interact. For instance, parametric excitation of the shape oscillations on top of the forced breathing mode has been addressed by Mei and Zhou,[9] who have found an instability of the radial oscillations and performed a weakly nonlinear analysis. Since then, the problem of the parametric instability has received much attention; see, e.g., a recent review by Feng and Leal.[10] Another example of the interaction has been provided by Longuet-Higgins.[11] He focuses on bubble oscillations in a liquid and shows that the nonlinear coupling of the shape oscillations can lead to the excitation of the volume mode. Nonlinear oscillations of an incompressible drop with the accurate account for the dynamics of the ambient gas have been studied in Ref. 12. The nonlinear coupling of the shape oscillations results in generation of sound. It is worth noting that in the previous studies, the interaction of the shape and volume oscillations arises as a nonlinear effect. In our paper, we report on another, pure linear, mechanism of coupling of the oscillations of the different kinds. This mechanism of linear coupling is caused by the contact line dynamics.

Last years have witnessed growing interest in understanding the contact line dynamics. Although the steady motion of the contact line has been well studied,[5,13] there has been no rigorous theory for unsteady motion yet. What is typically applied in this situation is a simplified approach. The thin viscous boundary layer is neglected and a phenomenological boundary condition imposed on the apparent contact angle is applied instead. Such condition has been proposed by Hocking for small oscillations of the contact line.[14] The velocity of the contact line is assumed to be proportional to the deviation of the contact angle from its equilibrium value (for simplicity, the equilibrium contact angle is considered to be $\pi/2$),

$$\frac{\partial \zeta}{\partial t} = \Lambda \mathbf{n} \cdot \nabla \zeta. \tag{3}$$

Here, ζ is the deviation of the interface from its equilibrium position and $\mathbf{n}$ is the external normal to the solid surface. The coefficient Λ has the dimension of velocity and is referred to as the wetting or the Hocking parameter. The mostly studied cases correspond to the simplest limiting situations of either the fixed contact line ($\zeta=0$, the pinned-end edge condition) or the fixed contact angle ($\mathbf{n}\cdot\nabla\zeta=0$, the free-end edge condition). Except for these two particular cases, the Hocking condition (3) leads to energy dissipation at the contact line. A simple generalization of Eq. (3) that is based on experimental observations[4] and accounts for hysteresis of the contact line has been provided in Ref. 15.

A consideration of natural oscillations of a hemispherical drop for the fixed contact angle has shown[16] that the eigenfunctions coincide with the even modes of the natural oscillations of a spherical drop. The problem of axisymmetric oscillations of a drop with the fixed contact line has been addressed both experimentally[17,18] and numerically.[19–22] Experimental studies[17,18] focus on the forced oscillations. In particular, eigenfrequencies are determined for different equilibrium contact angles. Numerical investigations deal with natural oscillations of an inviscid drop[19] and with

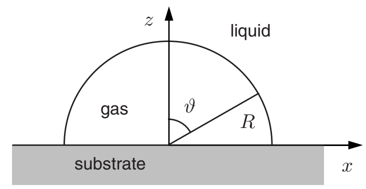

FIG. 1. The geometry of the natural oscillations problem.

natural[20] and forced[21,22] nonlinear oscillations of a viscous drop. Longitudinal vibrations have recently been studied theoretically[23] and experimentally.[24] In Ref. 23, the horizontal and vertical orientations of the substrate are considered. For the vertical orientation, gravity leads to asymmetry of oscillations. The study by Noblin *et al.*[24] has manifested the nontrivial dynamics of the contact line: At a relatively small amplitude of vibration, the contact line remains pinned, while at higher amplitudes, it starts to move in the stick-slip regime. Asymmetric vibrations as well as possible microfluidic applications have been discussed in Ref. 25.

Recently, the focus of attention has been on the impact of the contact line dynamics on the natural and forced oscillations of an *incompressible* hemispherical drop on a solid substrate.[26,27] Axisymmetrical modes of the natural oscillations caused by transversal vibrations of the substrate are studied in Ref. 26. Another paper[27] addresses the nonaxisymmetrical modes of the natural oscillations and the forced oscillations for the longitudinal vibrations of the substrate, where inertia of the ambient fluid is taken into account. To some extent, these analyses can be applied to describe oscillations of an incompressible bubble in a liquid. However, the bubble compressibility, a principal feature of the present paper, can become of crucial importance, which has been beyond the scope of the previous research.[26,27] A recent experimental study[28] has indicated an interesting crossover, where beyond a certain threshold of vibration acceleration, the bubble can split into smaller parts. Despite noticeable progress, the dynamics of a *compressible* bubble on a vibrated substrate has not been fully understood.

In the present paper, we address the behavior of a compressible hemispherical bubble on a solid substrate. The paper is outlined as follows. In Sec. II, we analyze natural oscillations. Section III deals with the forced oscillations for the normally vibrated substrate. A transition to the case of a weakly compressible bubble is performed in Sec. IV. This is the situation when the frequency of the volume oscillations is high compared to that of the shape oscillations. Particularly, we obtain a general criterion identifying whether compressibility of a bubble can be neglected. In Sec. V, we discuss the results and summarize the most important conclusions.

II. NATURAL OSCILLATIONS

Consider natural oscillations of a compressible gaseous bubble that sits on a solid substrate and is surrounded by a liquid, Fig. 1. Suppose that in equilibrium, the bubble is hemispherical, i.e., the equilibrium contact angle equals $\pi/2$ (although this assumption is of no crucial importance, it considerably simplifies the forthcoming analysis). We assume that the gas density is small enough so that the frequency of the volume oscillations is comparable to that of the shape oscillations. Next we impose the frequency restrictions (2), which ensure insignificance of the damping effects caused by acoustic irradiation, viscous, and heat dissipation. Finally, we admit that the bubble is sufficiently small so that the hydrostatic difference of the pressure is negligible compared to the pressure contribution caused by surface tension. This assumption allows us to neglect gravity and ensures that the averaged bubble surface is hemispherical to very high accuracy. Formally, this situation implies that the Bond number, $\mathrm{Bo}=\rho g R^2/\sigma$, is small. Here g is the acceleration due to gravity. For instance, even under terrestrial conditions, an air bubble of radius $R=0.1$ cm suspended in water is characterized by $\mathrm{Bo}\approx 0.1$. Thus, the impact of gravity on the bubble results in a relative surface distortion on the order of only 10%. We are interested in smaller bubbles so that the distortion effects are insignificant.

Because of symmetry, we use the spherical coordinates r, ϑ, α with the origin in the center of the bubble. As we have announced in Sec. I, we will demonstrate that the shape oscillations are able to interact with the breathing mode even within the linear approximation. Because in this approximation, the nonaxisymmetric modes of the shape oscillations do not interact with the breathing mode (see argumentation below in this section), we eventually restrict our consideration to the axisymmetric problem.

We now formulate the dimensionless governing equations and boundary conditions by measuring the distance, time, velocity potential, deviations of pressure, and the bubble surface in the scales of R, $\sqrt{\rho R^3/\sigma}$, $\epsilon\sqrt{\sigma R/\rho}$, $\epsilon\sigma/R$, and ϵR, respectively. As one sees, the small parameter ϵ has the meaning of the ratio of the amplitude of the surface oscillation to the equilibrium radius R of the bubble. Small oscillations of the inviscid incompressible ambient are governed by the Bernoulli equation and the condition of incompressibility,

$$p=-\frac{\partial\varphi}{\partial t},\quad \nabla^2\varphi=0. \tag{4}$$

The solid surface $\vartheta=\pi/2$ is impermeable for the liquid,

$$\frac{\partial\varphi}{\partial\vartheta}=0. \tag{5}$$

At the free surface, governed by the equation $r=1+\epsilon\zeta(\vartheta,t)$, we prescribe the kinematic and the dynamic conditions

$$\frac{\partial\zeta}{\partial t}=\frac{\partial\varphi}{\partial r},\quad p-p_g=(\nabla_\vartheta^2+2)\zeta, \tag{6}$$

$$\nabla_\vartheta^2=\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right),$$

where φ is the velocity potential and p and p_g are the pulsation parts of the pressure in the liquid and gas phases, respectively. These pulsation fields describe the pressure deviation from their equilibrium values: $\epsilon^{-1}p_{g0}$, $p_{g0}\equiv P_g R/\sigma$ in the gas, and $\epsilon^{-1}(p_{g0}-2)$ in the liquid.

Next, we define the spatially uniform oscillations of the gas pressure p_g. Because the pulsations of the gas pressure in the bubble are spatially homogeneous and the dissipative processes are assumed to be negligible during a period of oscillation [recall the accepted restrictions (2)], we can apply the adiabatic law for perfect gas. As a result, we arrive at the condition

$$[p_{g0}+p_g(t)]V^\gamma(t)=p_{g0}V_0^\gamma.$$

Here, γ is the adiabatic exponent, $V_0=\frac{2}{3}\pi$ is the dimensionless volume of motionless bubble, and

$$V(t)=V_0(1+3\epsilon\langle\zeta\rangle),\quad \langle\zeta\rangle=\frac{1}{2\pi}\int_S\zeta dS,$$

where the angle brackets denote the space averaging over equilibrium surface $S=S(\vartheta)$ of the bubble. Thus, we obtain for the pressure pulsation in the gas phase

$$p_g=-3\gamma p_{g0}\langle\zeta\rangle\equiv-\Pi_0\langle\zeta\rangle, \tag{7}$$

where we introduce parameter $\Pi_0=3\gamma P_g R/\sigma$. Because $\Pi_0\propto p_{g0}$, hereafter Π_0 is referred to as the dimensionless equilibrium pressure inside the bubble. Note that this parameter can be presented as a ratio of frequencies of the volume and shape oscillations squared, $\Pi_0=(\omega_b\tau_c)^2$. Thus, one can clearly see that for finite values of Π_0 inequalities (1) directly follow from requirements (2).

We emphasize that for nonaxisymmetric modes $\langle\zeta\rangle=0$. As it follows from Eq. (7), the pressure inside the bubble for these modes does not change in time, $p_g(t)=0$, so that the breathing mode cannot emerge. In other words, because in the linear problem, the modes with different azimuthal numbers are noninteracting, no excitation of the volume oscillations is possible. Hence, to be able to focus on the announced interaction between the breathing mode and the shape oscillations, hereafter we deal with the axisymmetric problem, $\langle\zeta\rangle\neq 0$.

The formulation of the boundary value problem is completed by prescribing the dynamics of the contact line, where we impose the Hocking condition (3),

$$\frac{\partial\zeta}{\partial t}=-\lambda\frac{\partial\zeta}{\partial\vartheta}, \tag{8}$$

where the dimensionless number $\lambda=\Lambda\sqrt{\rho R/\sigma}$ is the Hocking (also known as wetting) parameter.

The boundary value problem (4)–(8) has been formulated in the linear approximation with respect to small ϵ. Particularly, this approximation allows us to pose the boundary conditions (6) at the time averaged, i.e., at the equilibrium, position of the surface, $r=1$. The formulated governing equations and boundary conditions involve the two dimensionless parameters Π_0 and λ. The limiting case of high pressure, $\Pi_0\to\infty$, refers to the consideration of incompressible gas. In this situation, the problem (4)–(8) transforms to the one describing the natural oscillations of an incompressible bubble immersed in a liquid[27] (see a detailed discussion in Sec. IV). Conversely, for small values of Π_0, one expects that the bubble collapses. Note that hereafter we assume the pressure in the ambient fluid to be positive and are not interested in consideration of cavitation. The limiting situations with respect to parameter λ correspond either to the fixed contact line ($\lambda \to 0$, the contact angle can change) or to the fixed contact angle ($\lambda \to \infty$, the motion of the contact line is allowed). As it has been mentioned before, apart from these particular situations, the Hocking condition leads to energy dissipation near the contact line. For this reason, the natural oscillations are generally damped.

Let us represent the decaying at the infinity solution to the Laplace equation (4), which satisfies the impermeability condition (5), in the form

$$\varphi = \mathrm{Re}\left[i\omega \sum_{n=0}^{\infty} \frac{A_n P_{2n}(\theta)}{r^{2n+1}} e^{i\omega t} \right]. \tag{9}$$

Here, we introduce the variable $\theta = \cos \vartheta$ and the Legendre polynomials $P_n(\theta)$.

Substitution of ansatz (9) into the Bernoulli equation (4) and kinematic condition [the first relation in Eq. (6)] leads to expressions

$$p = \mathrm{Re}\left[\omega^2 \sum_{n=0}^{\infty} \frac{A_n P_{2n}(\theta)}{r^{2n+1}} e^{i\omega t} \right], \tag{10a}$$

$$\zeta = -\mathrm{Re}\left[\sum_{n=0}^{\infty} (2n+1) A_n P_{2n}(\theta) e^{i\omega t} \right]. \tag{10b}$$

Next, we determine ζ from the dynamic condition [the second relation in Eq. (6)]

$$\zeta = \mathrm{Re}\left[\left\{ \frac{1}{2}\Pi_0 \langle \zeta \rangle - \sum_{n=0}^{\infty} \frac{\omega^2 A_n P_{2n}(\theta)}{(2n-1)(2n+2)} + C\theta \right\} e^{i\omega t} \right]$$

and equate it with Eq. (10b). Accounting for a relation $\langle \zeta \rangle = -A_0$, we obtain the coefficients introduced in. Eq. (9),

$$A_0 = \frac{C}{\Omega_0^2 - \omega^2}, \quad A_n = \frac{(4n+1) P_{2n}(0) C}{\Omega_n^2 - \omega^2} \quad (n > 0). \tag{11}$$

Here, $\Omega_n^2 = (2n-1)(2n+1)(2n+2)$ are the eigenfrequencies of the shape oscillations of a spherical bubble. These frequencies refer to the even modes; $\Omega_0^2 = \Pi_0 - 2$ is the frequency of the volume oscillations of a spherical bubble in liquid. Here, the term -2 presents the well-known correction to the frequency of breathing mode caused by surface tension.[29]

After the substitution of Eqs. (10) and (11) into condition (8), we arrive at the dispersion relation defining the spectrum of the eigenfrequencies of a bubble,

$$i\omega \left(\omega^2 \sum_{n=1}^{\infty} \frac{\alpha_n P_{2n}(0)}{\Omega_n^2 - \omega^2} - \frac{1}{2} - \frac{1}{\Omega_0^2 - \omega^2} \right) = \lambda$$
$$\alpha_n = -\frac{(4n+1) P_{2n}(0)}{(2n-1)(2n+2)}, \tag{12}$$

where α_n are the coefficients in expansion of θ in the series of the even Legendre polynomials at $\theta \in [0,1]$.

Note that except for some particular situations (e.g., small and high values of λ), the eigenfrequencies defined by Eq. (12) are complex. Separating the real and imaginary parts, one can demonstrate that the imaginary part of the eigenfrequency is non-negative for any $\Pi_0 > 2$, i.e., the oscillations decay. At $\Pi_0 \approx 2$ and a finite value of the wetting parameter, one of the eigenfrequencies is determined by the relation

$$\omega = i\lambda(\Pi_0 - 2), \tag{13}$$

which means that too small pressure inside the bubble results in the monotonic instability of the bubble with respect to collapse. Thus, relation (13) defines the stability threshold against the fast (adiabatic) compression. Clearly, at positive external pressure, the bubble becomes unstable for higher values of the gas pressure: $\Pi_0 < 6\gamma$ (or, in dimensional units, $P_g < 2\sigma/R$).

Consider now the dispersion relation in the limiting cases. For the fixed contact angle, $\lambda \to \infty$, the eigenmodes of oscillation of a hemispherical bubble coincide with the corresponding even modes of the spherical bubble. In this case, the oscillation frequencies equal Ω_k $(k=0,1,\ldots)$. For high but finite values of the wetting parameter, the eigenfrequencies obey the relation

$$\omega^{(k)} = \Omega_k + \frac{i\gamma_k}{\lambda} + \frac{\gamma_k \Omega_k}{2\lambda^2} \left(\sum_{\substack{n=0,\\ n\neq k}}^{\infty} \frac{4\gamma_n}{\Omega_n^2 - \Omega_k^2} - \frac{\gamma_k}{\Omega_k^2} \right), \tag{14}$$

defined for $k=0,1,\ldots$. Here, $\gamma_k = -\Omega_k^2 \alpha_k P_{2k}(0)/2$ $(k>0)$, $\gamma_0 = 1/2$. As it can be seen, the oscillations are weakly damped, a correction to the eigenfrequency is proportional to λ^{-2}. Formula (14) is identical to that for the incompressible bubble for positive k, if the first term in the sum is vanishing, which corresponds to the limit $\Omega_0^2 \to \infty$. We indicate that result (14) holds even for $\Pi_0 < 2$. In this case, all the terms for $k=0$ are imaginary, and relation (14) describes two branches in the spectrum (the stable and unstable): $\omega^{(0)} = \pm i\sqrt{2-\Pi_0} + O(\lambda^{-1})$.

The frequency of oscillation of the bubble with the fixed contact line ω_p is determined from the following real expression:

$$\omega_p^2 \sum_{n=1}^{\infty} \frac{\alpha_n P_{2n}(0)}{\Omega_n^2 - \omega_p^2} - \frac{1}{2} - \frac{1}{\Omega_0^2 - \omega_p^2} = 0. \tag{15}$$

In the case of $\lambda \ll 1$ (slight slip of the contact line), the results are qualitatively similar to those of incompressible liquid:[26,27] the decay rate is proportional to the small wetting parameter and a correction to the frequency ω_p is proportional to λ^2.

However, despite the qualitative similarity with the oscillations of the hemispherical drop studied in Refs. 26 and 27, there arises a number of peculiar effects for the compressible bubble. The appearance of an additional branch in the spectrum of natural oscillation, governed by the parameter Π_0, leads to a nontrivial effect: The volume and the shape oscillations start to interact. Thus, if for a spherical bubble of the same radius, the frequency Ω_0 of the volume oscillations is close to any of the frequencies Ω_k of the shape oscillations (for an even mode),

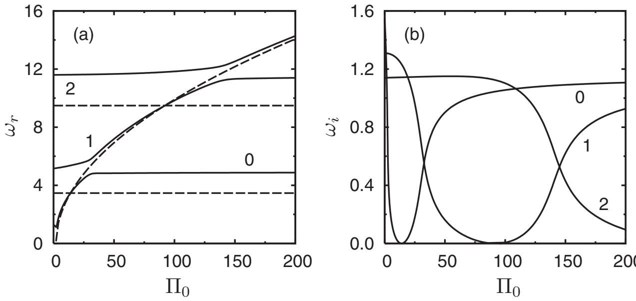

FIG. 2. Eigenfrequencies ω of the bubble oscillations as functions of pressure Π_0 for modes 0–2. The real (a) and imaginary (b) parts of ω are plotted for $\lambda=1$ (full line) and $\lambda=10$ (dashed line).

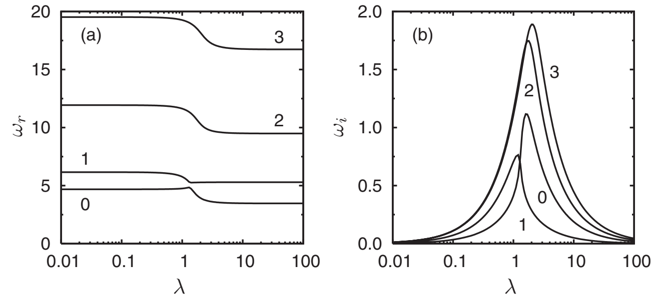

FIG. 3. Eigenfrequencies ω of the bubble oscillations as functions of wetting parameter λ for modes 0–3. The real (a) and imaginary (b) parts of ω are plotted for $\Pi_0=30$.

$$\Omega_0^2 = \Omega_k^2(1+\delta), \quad \delta \ll 1,$$

then one of the eigenfrequencies ω_{0k} for the hemispherical bubble is in between these frequencies,

$$\omega_{0k} = \frac{\gamma_0 \Omega_k}{\gamma_0+\gamma_k} + \frac{\gamma_k \Omega_0}{\gamma_0+\gamma_k}. \tag{16}$$

A complex correction to the frequency defined by Eq. (16) is proportional to δ^2. For this reason, the decay is weak for this mode irrespective of the wetting parameter and in the "resonant" situation ($\delta=0$) is absent at all. With the accuracy up to the terms of order δ^2, only two coefficients, A_0 and A_k, are nonvanishing in sums (9) and (10). Hence, this mode is a superposition of two kinds of motion: The radial pulsations and the kth mode of the shape oscillations of the bubble with the fixed contact angle. These oscillations occur in the antiphase, and their relative amplitudes are such that the contact line remains motionless.

For arbitrary values of the governing parameters, Eq. (14) was numerically solved. By using the secant method, we retained up to 200 terms. Careful analysis has shown that for most situations, retention of only 20 terms is enough to ensure convergence.

In Fig. 2, we present the frequency of natural oscillations as a function of the pressure in the bubble. At $\lambda=10$, the real part of the frequencies of the first modes almost coincide with the values Ω_0, Ω_1, Ω_2. The imaginary part of the frequency is small and is described by formula (14). A distinction exists only in the case $\Omega_0 \approx \Omega_k$: The imaginary part of the frequency drastically decreases for the volume oscillations and increases for the shape oscillations. At moderate values of the wetting parameter ($\lambda=1$), with the growth of Π_0 the spectrum of the eigenfrequencies rearranges. In this case, the damping time of oscillations is comparable to their period; the imaginary part of the frequency as a function of Π_0 is given in Fig. 2(b). As has been indicated before, near the "resonance" $\Omega_0 \approx \Omega_k$, the imaginary part of the frequency for one of the modes tends to zero according to the law $\omega_i \simeq \delta^2 \propto (\Omega_0 - \Omega_k)^2$.

The dependence of eigenfrequencies on the Hocking parameter at a fixed gas pressure in the bubble can be found in Fig. 3. For modes 2 and 3, the behavior resembles the case of the incompressible liquid:[26,27] The real part decreases with the growth of λ, the imaginary part has a maximum at a finite value of the wetting parameter, turning to zero at small and high λ. An interesting behavior of the eigenfrequencies can be observed for the two lowest modes, when the complex frequencies become close, see Fig. 4. Note that an insignificant change of Π_0 is able to qualitatively rearrange the dependence $\omega_r(\lambda)$. We indicate that the exact coincidence of complex decay rates of the two modes resulting in such a rearrangement of the spectrum is possible only in a discrete number of points (λ, Π_0), which obey the complex-valued equation $\omega^{(k)}(\lambda,\Pi_0)=\omega^{(n)}(\lambda,\Pi_0)$.

As the final point of this section, we point out to what extent the presence of the substrate influences damping of a bubble oscillation. Generally (see, e.g., Refs. 1 and 8), oscillation damping of a spherical bubble may be caused by viscosity, thermal diffusion, and radiation of acoustic wave. Although all these phenomena are relevant for the hemispherical bubble, they can be remarkably changed by the substrate.

As we stressed in Sec. I, understanding the impact of viscosity on the moving contact line is a notoriously complicated problem. However, an important estimation can be drawn from the problem with the contact line pinned,[26] $\lambda=0$. According to this analysis, the dimensionless decay rate is $\sqrt{\tau_c/\tau_v}$ against conventional τ_c/τ_v for a spherical bubble away from the solid surface.

A similar situation is expected for the damping caused by heat diffusion. Note that this mechanism may be important for both gas and fluid. Moreover, the ratios of thermal conductivities of the three media (gas, liquid, and solid) become important. If the conductivity of the solid is not negli-

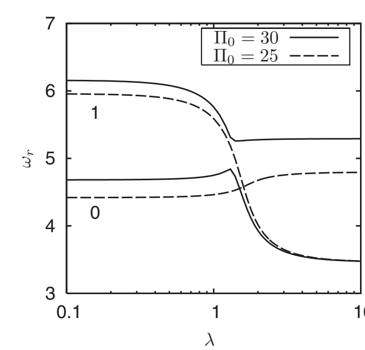

FIG. 4. Rearrangement of the two lowest modes 0 and 1 under small variation of pressure Π_0. Eigenfrequencies ω_r as functions of wetting parameter λ, plotted for $\Pi_0=30$ and $\Pi_0=25$, cf. Fig. 3(a).

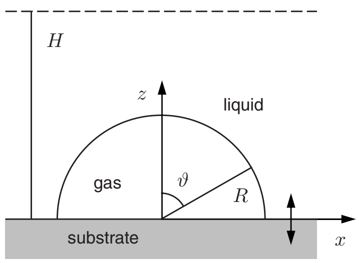

FIG. 5. The geometry of the forced oscillations problem.

gibly small, the thermal boundary layer near the solid surface is developed. The decay rate is given by $\sqrt{\omega_b \tau_t}$, whereas for a bubble in the absence of the solid surface one ends up with the usual value of $\omega_b \tau_t$.

Acoustic irradiation has nondissipative origin, and therefore the presence of the substrate is insignificant. Hence, the conventional condition $\omega_b \tau_a \ll 1$ as in Eq. (2) is sufficient to neglect this phenomenon.

As we see, the presence of the solid surface leads to the development of the boundary layers, which results in the faster oscillation damping compared to the case of a bubble away from the substrate. These changes, however, do not change restrictions (2). Thus, the criteria that allow us to neglect viscous and heat dissipation and acoustic irradiation loss remain conventional.

III. FORCED OSCILLATIONS

Consider the behavior of a gas bubble sitting on a plane solid substrate, as sketched in Fig. 5. We neglect gravity as before and now look at the problem of forced oscillations. Assume that the substrate performs transversal vibrations (with respect to its plane) with the amplitude a and the frequency ω; the substrate velocity in an inertial reference frame is $a\omega \sin \omega t$. We consider the frequency of vibrations to be comparable with both the frequencies of the shape and volume oscillations. We stress that spatially uniform pulsations are of crucial importance for the consideration of the bubble. This is in contrast to the problems addressing an incompressible drop,[26,27] where the system is dominantly governed by the pressure difference in the liquid. In our system, to define the uniform part of the pressure field, one has to specify an additional condition away from the bubble. From the experimental point of view, the most convenient way to overcome this difficulty is to attach the bubble to the bottom of a liquid layer of depth H (large compared to the size of bubble) with the free surface,[30] see Fig. 5. For a spherical bubble, a similar way of treating this peculiarity of the pressure field has been applied before,[1,31] where the bubble has been immersed in a vibrated column of liquid with a free surface and the impact of vibrations on the spherically symmetric oscillation mode has been studied.

In the reference frame moving together with the substrate, the small oscillations of the bubble are governed by the following dimensionless equations and boundary conditions:

$$p = -\frac{1}{\Omega^2}\frac{\partial \varphi}{\partial t} + \left(1 - \frac{z}{h}\right)\cos \Omega t, \quad \nabla^2 \varphi = 0, \tag{17a}$$

$$\vartheta = \frac{\pi}{2}: \quad \frac{\partial \varphi}{\partial \vartheta} = 0, \tag{17b}$$

$$r = 1: \quad \frac{\partial \zeta}{\partial t} = \frac{\partial \varphi}{\partial r}, \quad \Omega^2 p + \Pi_0 \langle \zeta \rangle = (\nabla_\vartheta^2 + 2)\zeta, \tag{17c}$$

$$z = h: \quad p = 0, \tag{17d}$$

$$r = 1, \quad \vartheta = \frac{\pi}{2}: \quad \frac{\partial \zeta}{\partial t} = -\lambda \frac{\partial \zeta}{\partial \vartheta}. \tag{17e}$$

The problem (17a)–(17e) has been nondimensionalized by using the scales aH/R, $\rho a \omega^2 H$, $aH\sqrt{\sigma/\rho R^3}$ for the deviation of the bubble surface from its equilibrium form, the pressure, and the velocity potential, respectively; for the distance and time, we use the same scales as before, in problem (4)–(8). We note that the oscillations of the bubble surface can be considered to be small only provided that

$$a \ll \frac{R^2}{H}. \tag{18}$$

As we see, the restriction imposed on the amplitude a is much stricter than in the situation of an incompressible drop $(a \ll R)$. This restriction, however, can be weakened for the weakly compressible bubble, see Sec. IV.

Among the dimensionless parameters Π_0 and λ, defined earlier, there appear two more parameters in problem (17a)–(17e): The dimensionless frequency Ω of the substrate oscillation, which is related to the Weber number $\Omega^2 = \rho \omega^2 R^3/\sigma$, and the relative width of the layer $h = H/R$. As we have assumed above, the last parameter is high,

$$h \gg 1. \tag{19}$$

This inequality allows us to neglect the term proportional to z/h in the Bernoulli equation (17a) and apply an approximation. We replace the exact condition (17d) for the pressure at the free surface with a requirement

$$r \to \infty, \quad \varphi \to 0, \tag{20}$$

which ensures the disturbance decay far from the bubble.

Indeed, in the absence of the bubble, the vibrations of the layer cause pulsations of the pressure in the liquid: $p_0 = (1 - z/h)\cos \omega t$. We note that near the bottom, where $z \ll h$, the inertial force is negligible, therefore these pulsations can be considered as spatially uniform. Thus, far away from the bubble but close to the solid surface, the time oscillations of the pressure are spatially uniform, which causes the volume (and hence also the shape) oscillations of the bubble. Surely, the wave scattered by the bubble does not satisfy condition (17d) precisely. This solution, however, can be easily corrected by the method of images. Such a procedure introduces an error of order h^{-1} in the condition for the pressure at the bubble surface (17c), which is of the same order compared to the neglected inertial term.

The solution to the Laplace equation for the velocity potential that decays at infinity and satisfies the impermeability condition (17b) along with the fields of pressure and surface deviation can be represented as follows:

$$\varphi = \mathrm{Re}\left[i\Omega \sum_{n=0}^{\infty} \frac{A_n P_{2n}(\theta)}{r^{2n+1}} e^{i\Omega t} \right], \tag{21a}$$

$$p = \mathrm{Re}\left\{ \left[\sum_{n=0}^{\infty} \frac{A_n P_{2n}(\theta)}{r^{2n+1}} + 1 \right] e^{i\Omega t} \right\}, \tag{21b}$$

$$\zeta = -\mathrm{Re}\left[\sum_{n=0}^{\infty} (2n+1) A_n P_{2n}(\theta) e^{i\Omega t} \right]. \tag{21c}$$

Applying the dynamic boundary condition, we obtain the expansion coefficients

$$A_0 = \frac{\Omega^2 + C}{\Omega_0^2 - \Omega^2}, \quad A_n = \frac{(4n+1)P_{2n}(0)C}{\Omega_n^2 - \Omega^2} \quad (n>0). \tag{22}$$

Here, the complex constant C is found from the Hocking condition (17e),

$$C = \Omega^2 \left\{ (\Omega_0^2 - \Omega^2)\left[\Omega^2 \sum_{n=1}^{\infty} \frac{\alpha_n P_{2n}(0)}{\Omega_n^2 - \Omega^2} - \frac{1}{2} + \frac{i\lambda}{\Omega} \right] - 1 \right\}^{-1}. \tag{23}$$

As it can be seen, $\arg(A_n)$ is nonzero and identical for any $n>0$, but different from $\arg(A_0)$. This indicates that the bubble oscillations are a superposition of the standing wave (shape oscillations) and the volume oscillations, which have phase shifts relative to each other and relative to the substrate vibrations.

Further, one can show[26,27] that the frequencies Ω_k are not resonant at any value of λ. At $\Omega=\Omega_k$, the motion of the bubble represents a combination of the radial (volume) oscillation and the kth mode of the shape oscillations (both these oscillations are in phase with the vibration motion of the substrate),

$$\zeta = \frac{\Omega_k^2}{\Omega_0^2 - \Omega_k^2}\left[\frac{P_{2k}(\theta)}{P_{2k}(0)} - 1 \right] \cos \Omega_k t \quad (k>0), \tag{24}$$

which means that the contact line remains motionless.

At a frequency close to Ω_0, we put $\Omega=\Omega_0(1+\delta_0)$, where the frequency mismatch $\delta_0 \ll 1$, and obtain from Eqs. (22) and (23)

$$A_0 \approx -\Omega_0^2 \left[\Omega_0^2 \sum_{n=1}^{\infty} \frac{\alpha_n P_{2n}(0)}{\Omega_n^2 - \Omega_0^2} - \frac{1}{2} + \frac{i\lambda}{\Omega_0} \right],$$

$$A_n \approx -\frac{(4n+1)P_{2n}(0)\Omega_0^2}{\Omega_n^2 - \Omega_0^2} \quad (n>0),$$

$$C \approx -\Omega^2 \left\{ 1 - \delta_0 \Omega_0^2 \left[\Omega_0^2 \sum_{n=1}^{\infty} \frac{\alpha_n P_{2n}(0)}{\Omega_n^2 - \Omega_0^2} - \frac{1}{2} + \frac{i\lambda}{\Omega_0} \right] \right\}.$$

By performing evaluation for $\zeta_0=\zeta(\theta=0)$ and accounting for a relation $\sum_{n=0}^{\infty} \alpha_n P_{2n}(0)=0$, we end up with

$$\zeta_0 \approx \mathrm{Re}[i\lambda\Omega_0 e^{i\Omega_0 t}] = -\lambda\Omega_0 \sin \Omega_0 t. \tag{25}$$

This result clearly indicates that at a finite value of λ and a vibration frequency close to the frequency of the breathing mode, the bubble oscillates with a finite amplitude.

Next, in the limiting case of the fixed contact angle ($\lambda \gg 1$), we arrive at an obvious conclusion: At any frequency the bubble performs radial oscillations with an amplitude,

$$A_0 = \frac{\Omega^2}{\Omega_0^2 - \Omega^2}. \tag{26}$$

Other coefficients in Eqs. (21a)–(21c) are small (of order λ^{-1}) because the contact line only weakly interacts with the substrate. If, however, the frequency Ω of the external force is close to the frequency Ω_k of the kth mode of the shape oscillations, then the resonant amplification of this mode occurs,

$$\zeta = -b_0\left[\cos \Omega_k t - a_k \frac{P_{2k}(\theta)}{P_{2k}(0)} \cos(\Omega_k t + \beta_k) \right],$$
$$a_k = \frac{\gamma_k}{\sqrt{(\Omega_k - \Omega)^2\lambda^2 + \gamma_k^2}}, \quad \tan \beta_k = \frac{(\Omega_k - \Omega)\lambda}{\gamma_k}, \tag{27}$$

where $b_0=\Omega_k^2/(\Omega_0^2-\Omega_k^2)$. Thus, close to the resonant frequency (for the fixed contact angle), the bubble motion consists of a superposition of the radial oscillation and a standing wave (one mode in the expansion), which have a relative phase shift. Exactly at the point of resonance, the solution is given by formula (24).

For large λ, in the situation where the frequency of oscillations is close to the eigenfrequency of the volume oscillations, $\Omega \approx \Omega_0$, a resonant amplification of the radial pulsations takes place,

$$\zeta = a_0 \cos(\Omega_0 t + \beta_0),$$
$$a_0 = \frac{\lambda\gamma_0\Omega_0}{\sqrt{(\Omega_0 - \Omega)^2\lambda^2 + \gamma_0^2}}, \quad \tan \beta_0 = \frac{\gamma_0}{(\Omega - \Omega_0)\lambda}. \tag{28}$$

Note that exactly at the point of resonance, $\Omega=\Omega_0$, the obtained result (28) is reduced to Eq. (25), which means that the two asymptotics match in the overlapping range of parameters: $\Omega \approx \Omega_0$ and $\lambda \gg 1$. On the other hand, the amplitudes in Eq. (28) and relation (26) become the same for $\Omega_0 \gg |\Omega - \Omega_0| \gg \lambda^{-1}$.

We also indicate that resonant solution (28) is characterized by oscillations with the amplitude $a_0=O(\lambda)$, which is much higher than the resonant amplitude a_k at a frequency of the shape oscillations. As it follows from the resonant solution (27), the amplitude $a_k \simeq O(1)$ and hence $a_0/a_k \sim \lambda \gg 1$.

We now consider the bubble dynamics when the three frequencies are close: $\Omega \approx \Omega_0 \approx \Omega_k$. For arbitrary values of λ, we obtain

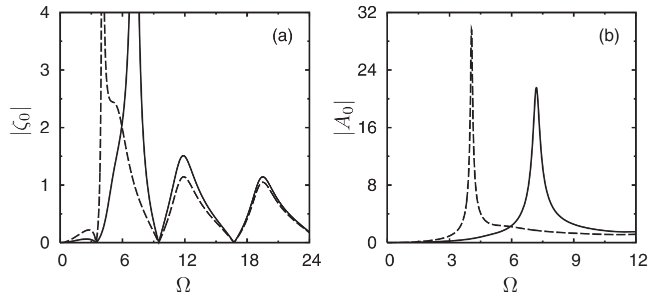

FIG. 6. Amplitude-frequency response for the contact line ζ_1 (a) and volume (b) oscillations evaluated for $\lambda=1$ and pressure values $\Pi_0=50$ (full line) and $\Pi_0=20$ (dashed line).

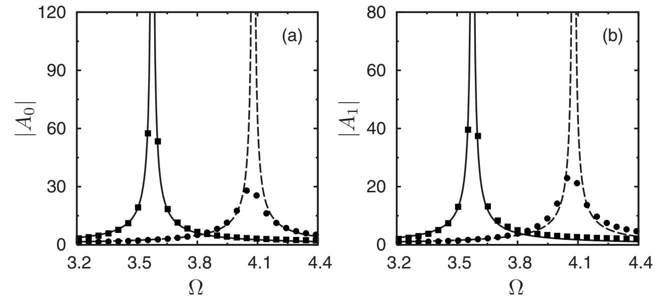

FIG. 7. Absolute values of coefficients A_0 (a) and A_1 (b) as functions of Ω near the double resonance, $\lambda=1$. Full and dashed lines correspond to Eqs. (29) and (30) at $\Pi_0=15$ ($\delta=1/12$) and $\Pi_0=20$ ($\delta=1/2$), respectively. Squares ($\Pi_0=15$) and circles ($\Pi_0=20$) present numerically obtained results according to Eqs.(22) and (23).

$$\zeta = A\left[1-\frac{P_{2k}(\theta)}{P_{2k}(0)}\right]\cos\omega_k t, \tag{29}$$

i.e., the bubble oscillations are in phase with the substrate vibrations and the contact line is motionless. The amplitude of oscillations reads

$$A=\frac{\gamma_0\gamma_k}{\gamma_0+\gamma_k}\frac{\Omega_k}{\Omega-\omega_{0k}}, \tag{30}$$

where ω_{0k} is the eigenfrequency of oscillations for the regime $\Omega_0\approx\Omega_k$, described by Eq. (16). A more rigorous analysis indicates that the phase of the resonant oscillations is shifted by $\pi/2$ with respect to the substrate motion, and their amplitude is

$$A_{\rm res}=\frac{\Omega_k}{\lambda}\left(\frac{\gamma_0+\gamma_k}{\gamma_0}\right)^2\delta^{-2}, \qquad \delta=\frac{\Omega_0^2-\Omega_k^2}{\Omega_k^2}. \tag{31}$$

The resonant amplitude remains bounded even at the eigenfrequency ω_{0k}, except for the case of precise coincidence of all the three frequencies. Note that the fact that the amplitude of the bubble oscillations is inversely proportional to the frequency mismatch squared δ^2 is in agreement with the result obtained in Sec. II: At $\Omega_0\approx\Omega_k$, energy dissipation is proportional to δ^2.

At arbitrary values of the governing parameters series (21a)–(21c) were numerically evaluated. In Fig. 6, we present the amplitudes of the contact line oscillations $\zeta_0\equiv\zeta(\theta=0)$ and the radial pulsation as functions of frequency Ω for $\lambda=1$. Because of the dissipative processes at the contact line, the amplitude of resonant oscillations remains bounded. However, the interaction of the volume and the shape oscillations leads to considerable increase of the amplitude near the frequency defined by relation (16).

As it follows from solution (24), the contact line is fixed at the frequencies Ω coinciding with Ω_k, the eigenfrequencies of the bubble oscillations with the fixed contact angle. At $\Omega=\omega_p$ (recall that ω_p is the eigenfrequency of a bubble with the fixed contact line), the amplitude of the contact line motion does not depend on the Hocking parameter,

$$\zeta_0(\Omega=\omega_p)=\frac{\omega_p^2}{\omega_p^2-\Omega_0^2}. \tag{32}$$

Note that for sufficiently small λ, the function $\zeta_0(\Omega)$ possesses a local maximum at $\Omega=\omega_p$. Moreover, the local maximum is rather close to $\Omega=\omega_p$ even at $\lambda=1$, see Fig. 6. For instance, the second maximum for $\Pi_0=20$ (the dashed line) takes place at $\Omega_{\rm max}=11.89$, whereas $\omega_p=11.94$.

As it becomes clear from Fig. 7, formulas (29) and (30) work remarkably well at significant deviations from the "double resonance," e.g., the amplitude of the radial pulsations at $\Pi_0=20$ ($\delta=-1/2$) is still in good agreement with Eq. (30).

IV. OSCILLATIONS OF A WEAKLY COMPRESSIBLE BUBBLE

We now turn to the consideration of the weakly compressible bubble, which implies high gas pressure, $\Pi_0\approx\Omega_0^2 \gg 1$. Compared to the situation analyzed in Sec. III, the high pressure in the bubble means that the radial pulsations as well as the induced shape oscillations become small. Mathematically, these facts are reflected by the smallness of ζ in Eq. (21c): As it can be seen from Eq. (22), all the coefficients $A_n\to 0$ as $\Omega_0\to\infty$. What is important is that the condition $\Pi_0\gg 1$ does not guarantee against the insignificance of the bubble compressibility. Indeed, the amplitude of the uniform part p_0 of pressure oscillations is based on the width of the layer: $\rho a\omega^2 H$. This pressure contribution causes surface deviations proportional to Π_0^{-1}; it is of crucial importance for the bubble as a compressible object. Another source of bubble oscillations comes from the inertial force and induces bubble distortion independent of Π_0. The corresponding contribution in the pressure $p_{\rm in}$ is important for any bubble, and does not matter compressible or not. This nonuniform part of pressure $p_{\rm in}\sim\rho a\omega^2 R$, produced by the inertial force, is much smaller than the uniform part because $p_{\rm in}/p_0\simeq h^{-1}\ll 1$. Thus, the dynamics of the weakly compressible bubble is determined by the competition of the two different factors: The weak compressibility itself ($\Pi_0^{-1}\ll 1$) and the smallness of the inertial part of pressure ($h^{-1}\ll 1$).

Formally, there appear two small parameters in the problem: h^{-1} and Π_0^{-1}. Their ratio, the parameter that describes the impact of compressibility on the bubble oscillations, is assumed to be finite. Note that restriction (18) imposed on the amplitude in Sec. III becomes much milder (see also Ref. 32),

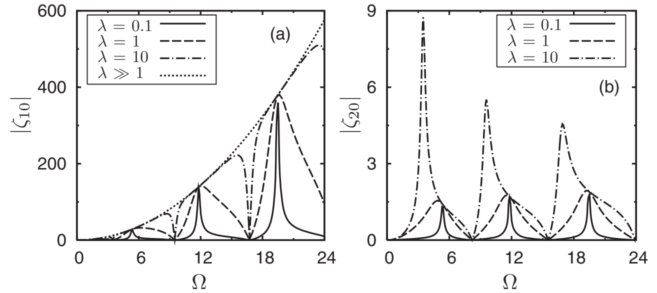

FIG. 8. Amplitude-frequency response for the contact line of the weakly compressible bubble, ζ_1 (a) and ζ_2 (b), evaluated for different values of λ. The asymptotics of large λ (left panel) is given by formula $\zeta_1=-\Omega^2$.

$$a \ll R, \quad a \ll \frac{\Pi_0}{h}R. \tag{33}$$

Thus, the solution to problem (17a)–(17e) can be presented as

$$\varphi = \Pi_0^{-1}\varphi_1 + h^{-1}\varphi_2, \tag{34a}$$

$$p = \cos\Omega t + \Pi_0^{-1}p_1 + h^{-1}p_2, \tag{34b}$$

$$\zeta = \Pi_0^{-1}\zeta_1 + h^{-1}\zeta_2. \tag{34c}$$

Here, the fields φ_1, p_1, and ζ_1 describe the motion caused by the spatially uniform part of the pressure and φ_2, p_2, and ζ_2 present the contribution induced by the inertial force. The latter contribution is related to the dynamics of an incompressible bubble. We also emphasize that the accepted in Sec. II condition $\mathrm{Bo} \ll 1$ is sufficient to neglect gravity here, for the case of weakly compressible bubble. The smallness of the Bond number ensures that the capillary effects produced by the inertial force dominate over the contributions caused by gravity.

The solution to the first problem, for the fields φ_1, p_1, and ζ_1, is obtained from the results of Sec. III in the limit $\Pi_0 \gg 1$ ($\Omega_0 \gg 1$). Writing down the fields of the velocity potential φ_1, pressure p_1, and surface deviation ζ_1 in terms of series (21a)–(21c), we obtain from Eq. (22) the coefficients

$$A_0 = \Omega^2, \quad A_n = \frac{(4n+1)P_{2n}(0)C_1}{\Omega_n^2 - \Omega^2} \quad (n > 0), \tag{35a}$$

$$C_1 = \Omega^2\left[\Omega^2\sum_{n=1}^{\infty}\frac{\alpha_n P_{2n}(0)}{\Omega_n^2-\Omega^2} - \frac{1}{2} + \frac{i\lambda}{\Omega}\right]^{-1}. \tag{35b}$$

In Fig. 8(a), we present the amplitude of the contact line oscillations $\zeta_{10} \equiv \zeta_1(\theta=0)$ as a function of frequency Ω for different λ. It is clearly seen that at a fixed value of Ω, this amplitude grows with the increase of λ. This solution is closely related to the mode discussed in Sec. III. Particularly, $\zeta_{10}=0$ at $\Omega=\Omega_k$ [see Eq. (24)]. At $\Omega=\omega_p^{(\infty)}$, where $\omega_p^{(\infty)}$ is ω_p evaluated at the limit $\Pi_0 \to \infty$, the real amplitude of the contact line oscillations is independent of λ and equals Ω^2 [cf. Eq. (32) at $\Omega_0^2 \approx \Pi_0 \gg 1$, recall the weight factor Π_0^{-1} in Eq. (34c)]. At $\lambda \gg 1$, we conclude from Eqs. (35a) and (35b) that C_1 is small and $\zeta_1 \approx -A_0 = -\Omega^2$. Note that this asymptotics works well even at $\lambda=10$ [see Fig. 8(a)], except for the close vicinity of "antiresonant" frequencies $\Omega=\Omega_k$, at which the contact line is motionless.

The second problem, for φ_2, p_2, and ζ_2, is governed by the following equations and boundary conditions:

$$p_2 = -\frac{1}{\Omega^2}\frac{\partial\varphi_2}{\partial t} - z\cos\Omega t, \quad \nabla^2\varphi_2 = 0, \tag{36a}$$

$$\vartheta = \frac{\pi}{2}: \quad \frac{\partial\varphi_2}{\partial\vartheta} = 0, \tag{36b}$$

$$r = 1: \quad \frac{\partial\zeta_2}{\partial t} = \frac{\partial\varphi_2}{\partial r}, \quad \Omega^2 p_2 = (\nabla_\vartheta^2 + 2)\zeta_2, \tag{36c}$$

$$r \to \infty: \quad \varphi_2 = 0, \tag{36d}$$

$$r = 1, \quad \vartheta = \frac{\pi}{2}: \quad \frac{\partial\zeta_2}{\partial t} = -\lambda\frac{\partial\zeta_2}{\partial\vartheta}. \tag{36e}$$

The problem (36a)–(36e) describes the forced oscillations of an incompressible bubble immersed in a liquid. Qualitatively, this problem is similar to that of the forced oscillations of a hemispherical drop.[26] Hence, the solution can be represented as

$$\varphi_2 = \mathrm{Re}\left[i\Omega\sum_{n=1}^{\infty}\frac{B_n P_{2n}(\theta)}{r^{2n+1}}e^{i\Omega t}\right], \tag{37a}$$

$$p_2 = \mathrm{Re}\left\{\left[\sum_{n=1}^{\infty}\frac{B_n P_{2n}(\theta)}{r^{2n+1}} - r\theta\right]e^{i\Omega t}\right\}, \tag{37b}$$

$$\zeta_2 = -\mathrm{Re}\left[\sum_{n=1}^{\infty}(2n+1)B_n P_{2n}(\theta)e^{i\Omega t}\right], \tag{37c}$$

where the expansion coefficients read

$$B_n = -\Omega^2\frac{(4n+1)P_{2n}(0)C_2 + \alpha_n}{\Omega_n^2 - \Omega^2} \quad (n > 0), \tag{38a}$$

$$C_2 = \frac{\displaystyle\sum_{n=1}^{\infty}\frac{(2n+1)\alpha_n P_{2n}(0)}{\Omega_n^2-\Omega^2}}{\displaystyle\Omega^2\sum_{n=1}^{\infty}\frac{\alpha_n P_{2n}(0)}{\Omega_n^2-\Omega^2} - \frac{1}{2} + \frac{i\lambda}{\Omega}}. \tag{38b}$$

The dependence of $\zeta_{20} \equiv \zeta_2(\theta=0)$ on Ω for solutions (37a)–(37c) and (38) is plotted in Fig. 8(b). As before, we observe a significant increase of the amplitude of the contact line oscillations with the growth of λ. Note that at its own antiresonant frequencies, which are the zeros of the numerator of C_2, we have $\zeta_{20}=0$ irrespective of λ. In particular, every standing wave, which is referred by index n, has its own phase shift, i.e., solution (37a)–(37c) with (38) describes traveling waves propagating along the bubble surface. Although not exactly the same, these phenomena related to the antiresonant frequencies and traveling waves are rather similar to those found in Ref. 26.

Having discussed separate contributions in Eqs. (34a)–(34c), we can make the final note about the full solution. The expansion coefficients of this solution in terms of series in spherical harmonics are a superposition $h^{-1}B_n + \Pi_0^{-1}A_n$, where both the contributions are small. The second term becomes negligible compared to the first one provided that

$$\rho\omega^2 RH \ll P_g.$$

This requirement ensures that the compressibility of a bubble is negligible and incompressible approximation is valid. We note that a similar inequality has been applied in Ref. 32 in the context of bubbly media.

V. CONCLUSIONS

We have investigated natural and forced oscillations of a compressible hemispherical bubble put upon a solid substrate. The contact line motion has been taken into account by applying the Hocking boundary condition. We have proven that the *linear* shape and volume oscillations demonstrate interaction, which is the main qualitative result of our study. Having performed detailed analysis, we have found out two important features accompanying this interaction. First, we have detected a double resonance, where independent of the Hocking parameter, an unbounded growth of the amplitude occurs. Second, we have figured out the general condition that can be used to neglect the bubble compressibility. This requirement turns out to be stricter than it might be expected. Finally, although our analysis bases on the assumption of adiabatic oscillations, we indicate below how the obtained results can be used for an arbitrary polytropic process in the gas.

We have focused on the natural oscillations and analyzed the eigenfrequency spectrum. Particularly, we have addressed the question as to how wetting at the contact line, which is governed by the Hocking parameter λ, influences the oscillation damping. We have neglected viscous and heat dissipation as well as acoustic irradiation loss. However, because the applied Hocking condition includes its own energy dissipation mechanism, the eigenoscillations are generally damped. The dependence of eigenfrequencies and the decay rate on λ are qualitatively similar to the case of a drop[26] or an incompressible bubble.[27] Here, the decay rate is maximal for $\lambda = O(1)$ and tends to zero at the limits of the fixed contact line ($\lambda \to 0$) and the fixed contact angle ($\lambda \to \infty$). In the latter case, the eigenfrequencies of the hemispherical bubble directly refer to those of the even eigenmodes for a spherical bubble of the same radius: The breathing mode frequency $\Omega_0^2 = \Pi_0^2 - 2$ and the frequencies of the shape oscillations $\Omega_k^2 = (2k-1)(2k+1)(2k+2)$, $k > 0$.

However, the compressibility of the bubble leads to a number of peculiar effects. Generally, the eigenfrequency spectrum becomes dependent on an additional parameter, dimensionless pressure in the bubble Π_0. This dependence results in nontrivial interaction of the volume and shape oscillation and newly found rearrangement of branches in the spectrum, see Fig. 4. As a particular consequence of the rearrangement, the eigenfrequencies of the compressible bubble are able to not only monotonically decrease with λ, as in the case of the drop or incompressible bubble, but also monotonically grow. Of special attention is the resonant case when $\Omega_0 \approx \Omega_k$, characterized by weak dissipation. At the exact coincidence of these frequencies, the contact line is motionless and there is no dissipation irrespective of λ. The bubble surface dynamics corresponds to a superposition of the kth mode of the shape oscillations and the antiphase radial pulsation.

We have considered normal vibrations of the substrate and studied the problem of the forced oscillations. In this situation the main role is played by the spatially uniform pulsations of the pressure field, which cause the volume oscillations of the bubble. Through the interaction with the substrate via the moving contact line, these volume oscillations induce the shape oscillations. The performed analysis of the forced oscillations has shown resonance phenomena to exist. Particularly, for weak dissipation, we have obtained analytical expressions for the oscillation amplitudes valid close to the resonance. We have also found out the double resonance, $\Omega \approx \Omega_0 \approx \Omega_k$, where Ω is the frequency of substrate vibration. As it follows from Eq. (31), in this oscillation regime the resonant amplitude $A_{\mathrm{res}} \propto (\Omega_0^2 - \Omega_k^2)^{-2}$. The divergence of A_{res} as $\Omega \to \Omega_0 = \Omega_k$ directly follows from the problem of natural oscillations, where the specific case $\Omega_0 = \Omega_k$ predicts no damping of oscillations.

We indicate that although we have applied the adiabatic law leading to Eq. (7), our analysis holds for an arbitrary polytropic process. In this case, the adiabatic exponent γ, which enters Eq. (7) and the parameter Π_0, should be now replaced with the polytropic exponent m. This generalization makes our theory applicable to a wider range of bubble sizes and allows for the description of smaller bubbles, for which heat conductivity becomes the dominant dissipative effect.[1,8] For instance, low frequency oscillations, $\omega \ll \chi_g R^{-2}$, are governed by the isothermal process, $m=1$. We note, however, that this generalization is appropriate only for the forced oscillations. For natural oscillations, any polytropic process is accompanied by intensive additional damping, which is different from the Hocking mechanism. Independent of λ, this damping results in the complete attenuation of oscillations within a few periods.

We have considered the special case of weakly compressible bubble and obtained the criterion identifying whether the bubble compressibility is insignificant. We have shown that the compressibility can be neglected only if the dimensionless pressure Π_0 in the bubble is large compared to the large h, which is the ratio of the layer depth H to the averaged bubble radius R (see Fig. 5). Our analysis allows us to draw a general conclusion about the impact of compressibility under the action of vibrations. It might be naively expected that compressibility effects become negligible at small frequencies ω in the sense that $\omega/\omega_c \ll 1$, where ω_c is a characteristic frequency of the volume oscillations. For bubble dynamics,[32] ω_c is the frequency of the breathing mode ω_b, for homogeneous fluid media,[33] it has the meaning of the acoustic frequency, $\omega_c \simeq c/H$. However, the correct condition that does guarantee that the compressibility effects can be neglected is significantly stricter and can be formulated as $(\omega/\omega_c)^2 \ll \mu$. Here μ is a small parameter typical for a concrete physical situation. For instance, as we saw for the bubble dynamics $\mu = h^{-1} \ll 1$, whereas for thermoacoustic convection $\mu = \beta\Theta \ll 1$, where β is the thermal expansion coefficient and Θ is the characteristic temperature difference.

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