From structures to heuristics to global solvers

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Mathematics for key technologies

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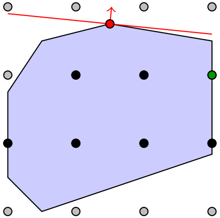
1. Undercover: the largest sub-MIP [B., Gleixner. Math. Prog., 2013]
2. RENS: the optimal rounding [B. Math. Prog. C, under review]
3. Measuring the impact of primal heuristics [B. OR Letters, accepted]

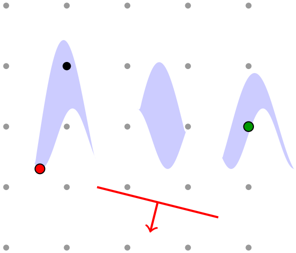
Mixed-Integer Linear Programming (MIP):

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x_i \in \mathbb{Z} \text{ for all } i \in \mathcal{I} \end{aligned}$$



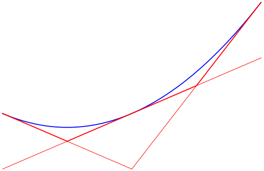
Mixed-Integer Nonlinear Programming (MINLP):

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & g(x) \leqslant 0, \quad g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m) \\ & x_i \in \mathbb{Z} \text{ for all } i \in \mathcal{I} \end{aligned}$$

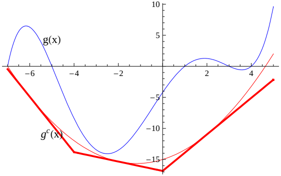


MINLP solving techniques

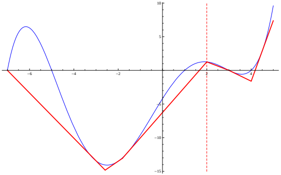
Gradient cuts



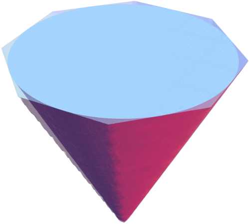
Underestimators



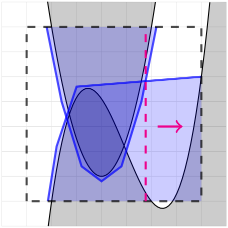
Spatial branching



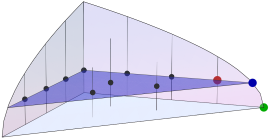
Reformulation



Bound tightening



Primal heuristics



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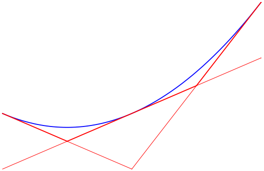

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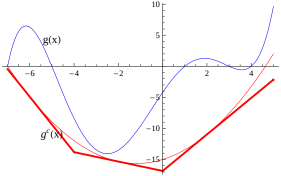

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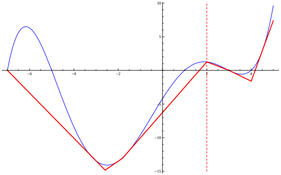

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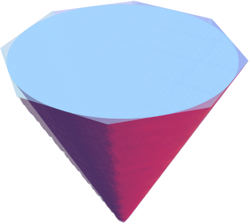

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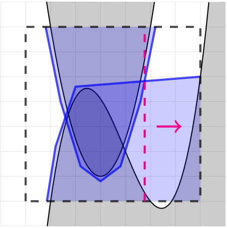
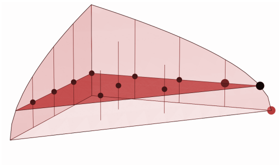


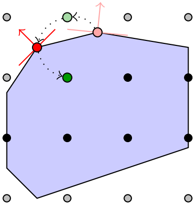
Primal heuristics



Primal heuristics inside a global solver

Primal heuristics . . .

- ▷ are incomplete methods which
- ▷ often find good solutions
- ▷ within a reasonable time
- ▷ without any warranty!



Why use primal heuristics inside a global solver?

- ▷ to prove feasibility of the model
- ▷ often nearly optimal solutions suffice in practice
- ▷ feasible solutions guide remaining search process

Improving the Feasibility Pump

DINS, a MIP Improvement Heuristic

OCTANE: A New Heuristic for Pure 0-1 Programs

RENS - the optimal rounding

An Evolutionary Algorithm for Polishing Mixed Integer Programming Solutions

Rapid Learning for Binary Programs

Proximity Search for 0-1 Mixed-Integer Convex Programming

A Generalized Wedelin Heuristic for Integer Programming

Shift-And-Propagate

Exploring relaxation induced neighborhoods to improve MIP solutions

Undercover: a primal MINLP heuristic exploring a largest sub-MIP

Local Branching

A Good Recipe for Solving MINLPs

ZI round, a MIP rounding heuristic

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- Large Neighborhood Search: common paradigm in MIP heuristics

  fix a subset of variables ⇝ easy subproblem ⇝ solve

  MIP: "easy" = few integralities
  MINLP: "easy" = few nonlinearities

- observation: any MINLP can be reduced to a MIP by fixing (sufficiently many) variables.

- idea: fix a small subset of variables to obtain a linear subproblem
- use solution of an LP or NLP relaxation to determine fixing values

Definition (cover of an MINLP)

Let

- a domain box $[L, U] = \times_i [L_i, U_i]$,
- a function $g_j : [L, U] \to \mathbb{R}$, $x \mapsto g_j(x)$ on $[L, U]$, and
- a set $\mathcal{C} \subseteq \mathcal{N} := \{1, \ldots, n\}$ of variable indices be given.

We call $\mathcal{C}$ a *cover of g* if and only if for all $\bar{x} \in [L, U]$ the set

$$\{(x, g_j(x)) \mid x \in [L, U], x_k = \bar{x}_k \text{ for all } k \in \mathcal{C}\}$$

is an affine set intersected with $[L, U] \times \mathbb{R}$.

We call $\mathcal{C}$ a *cover of P* if and only if $\mathcal{C}$ is a cover for $g_1, \ldots, g_m$.

Input: MINLP P

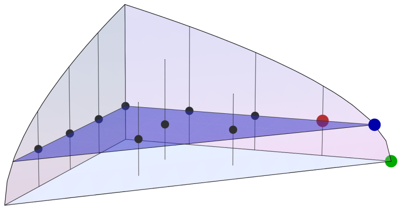
begin

compute a solution $\bar{x}$ of an approximation of P;

round $\bar{x}_k$ for all $k \in \mathcal{I}$;

determine a cover $\mathcal{C}$ of P;

solve the sub-MIP of P given by fixing $x_k = \bar{x}_k$ for all $k \in \mathcal{C}$;



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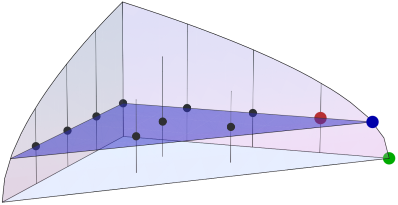

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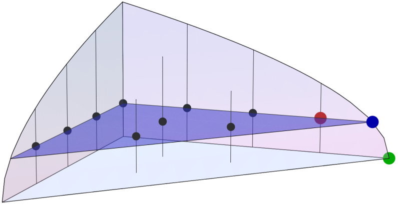

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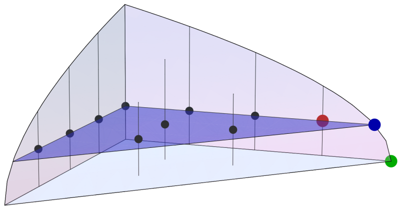

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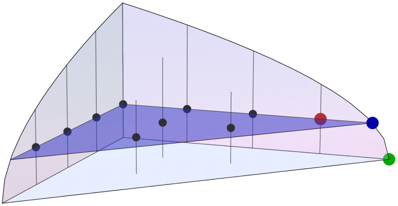

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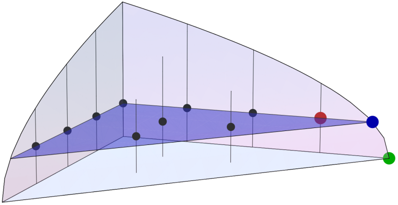

Remark:

- MIP heuristics: trade-off fixing many vs. few variables

  here: eliminate nonlinearities by fixing as few as possible variables

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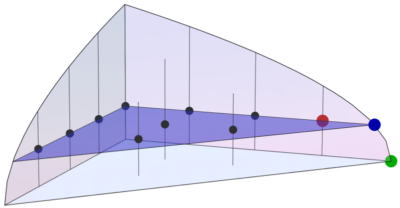

Remark:

- MIP heuristics: trade-off fixing many vs. few variables

  here: eliminate nonlinearities by fixing as few as possible variables
- How to find minimum cover?

Let $g_j : \mathbb{R}^n \to \mathbb{R}, x \mapsto x^T Qx + qx + c,\ Q \in \mathbb{R}^{n\times n}$ symmetric for all j.

Auxiliary binary variables: $\alpha_k = 1 :\Leftrightarrow\ x_k$ is fixed in P

Set covering constraints:

$\mathcal{C}(\alpha) := \{k \mid \alpha_k = 1\}$ is a cover of P if and only if

$$\alpha_k = 1 \qquad \text{for all square nonzeros: } Q^i_{kk} \neq 0, \qquad \text{(SN)}$$

$$\alpha_k + \alpha_j \geqslant 1 \qquad \text{for all bilinear nonzeros: } Q^i_{kj} \neq 0, k \neq j. \qquad \text{(BN)}$$

Solve the covering problem

$$\min \left\{ \sum_{k=1}^{n} \alpha_k : \text{(SN)}, \text{(BN)}, \alpha \in \{0,1\}^n \right\}.$$

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Solve the covering problem

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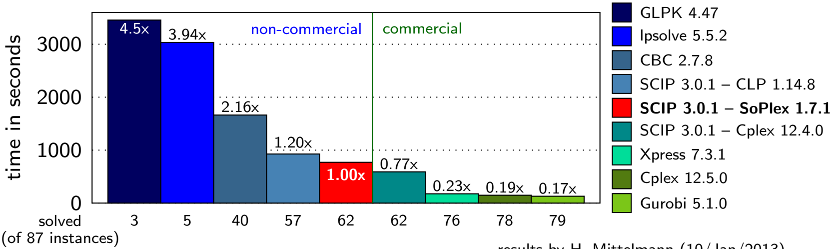
- Covering problem is $\mathcal{NP}$-hard, but branch-and-cut (empirically) very fast

- For general MINLPs, the covering problem becomes more difficult, e.g. for a global cover of a monomial $x_1^{p_1} \cdots x_n^{p_n}$, $p_1, \ldots, p_n \in \mathbb{N}_0$:

$$\alpha_k = 1 \qquad \text{for all } p_k \geqslant 2$$
$$\sum_{k:p_k=1} (1 - \alpha_k) \leqslant 1.$$

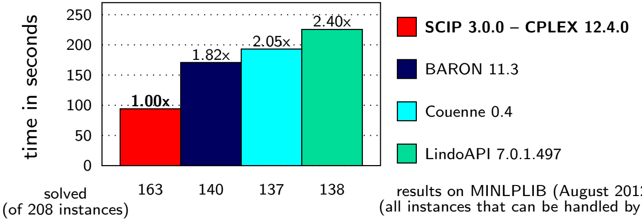
- For general MINLPs, global covers become larger and larger.

- one of the fastest non-commercial MIP solvers



results by H. Mittelmann (10/Jan/2013)

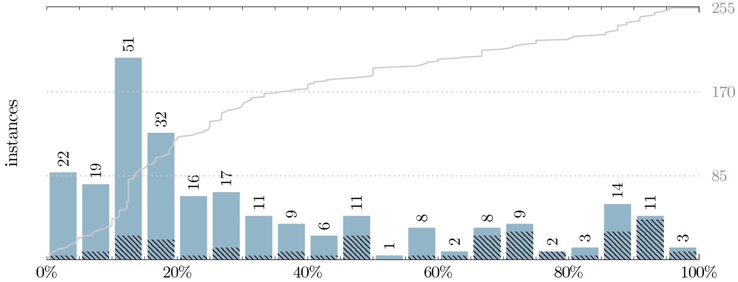
- MINLP solver benchmark



results on MINLPLIB (August 2012)
(all instances that can be handled by all solvers)

How big are minimum covers?

- ▷ SCIP 3.0 on 255 instances from MINLPLib



- ▷ $\leqslant$ 14% on 85, $\leqslant$ 36% on 170 instances
- ▷ similar on MIQCPs and general MINLPs

Test set

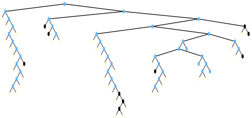
- 149 MIQCPs from GloMIQO test set

Comparison to other heuristics

- Undercover: solution for 76 instances (typically less than 0.1 sec)
- root heuristics: Baron 65, Couenne 55, SCIP 98
- lower success rate on general MINLPs

Extensions

- cover structure can be further exploited
- Undercover branching [B., Gleixner. Proc. of SEA 2013]



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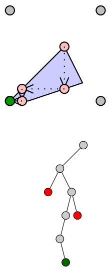
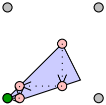
The motivation(s)

First motivation:

- many MIP heuristics based on rounding
- How likely to succeed?
- How good can they get?





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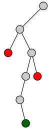
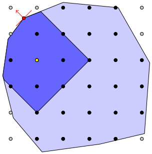
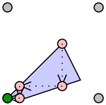


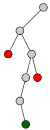


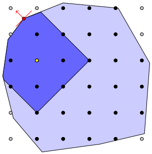
Second motivation:

- Large Neighborhood Search
  - e.g., RINS, Local Branching
- typically requires incumbent solution


BOINK!
BOINK!







Definition (rounding)

Let $\bar{x} \in [L, U]$. The set

$$\mathcal{R}(\bar{x}) := \{x \in [L, U] \mid x_j \in \{\lfloor \bar{x}_j \rfloor, \lceil \bar{x}_j \rceil\} \; \forall j \in \mathcal{I}\}$$

is called the set of *roundings* of $\bar{x}$.

Definition (optimal rounding)

Let $\bar{x} \in [L, U]$ and $\tilde{x} \in \mathcal{R}(\bar{x})$.

1. We call $\tilde{x}$ a *feasible rounding* of $\bar{x}$, if $g_i(\tilde{x}) \leqslant 0 \; \forall \, i \in \mathcal{M}$.
2. We call $\tilde{x}$ an *optimal rounding* of $\bar{x}$, if $\tilde{x} \in \operatorname{argmin}\{d^T x \mid x \in \mathcal{R}(x), g_i(x) \leqslant 0 \; \forall \, i \in \mathcal{M}\}$.
3. We call $\bar{x}$ *roundable* if it has a feasible rounding.

Input: MINLP P

begin

$\bar{x} \leftarrow$ relaxation optimum;

Fix all integral variables
$x_i := \bar{x}_i \quad$ for all $i : \bar{x}_i \in \mathbb{Z}$;

Reduce domain of fractional variables
$x_i \in \{\lfloor \bar{x}_i \rfloor; \lceil \bar{x}_i \rceil\}$;

Solve the resulting sub-MINLP;

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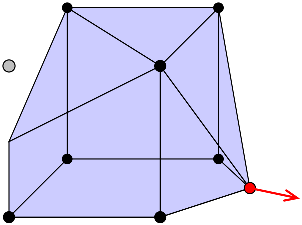

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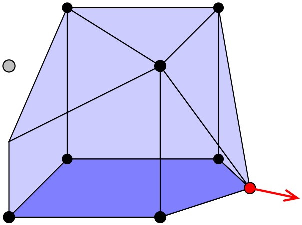

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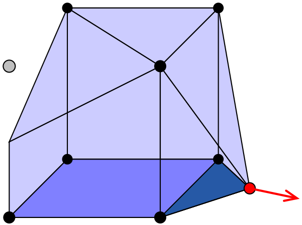

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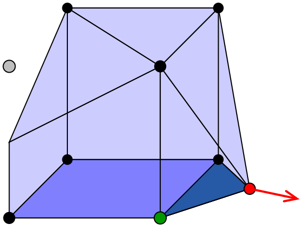


Lemma

Let the starting point $\bar{x}$ be feasible for a relaxation.

1. If the sub-MINLP is infeasible, then no feasible rounding of $\bar{x}$ exists.
2. If not, the optimum of the sub-MINLP is the optimal rounding of $\bar{x}$.

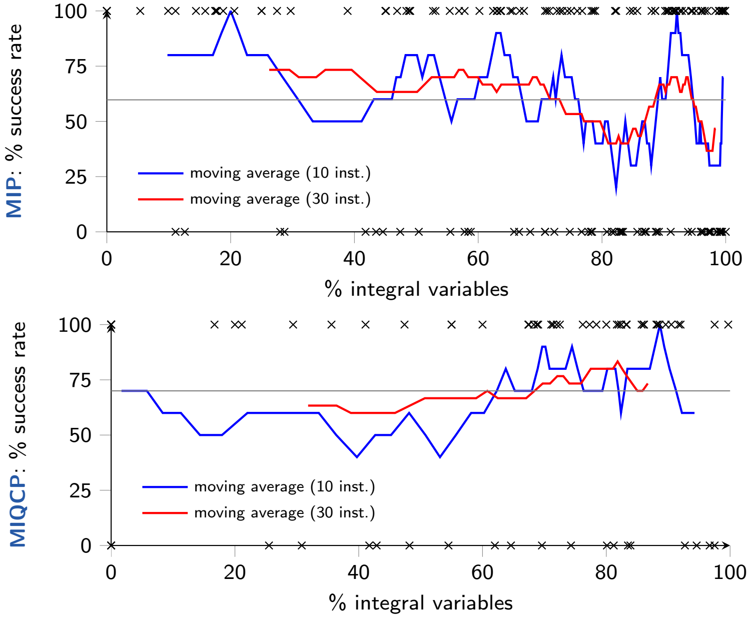
Corollary

$\bar{x}$ either has an optimal rounding or is not roundable.

Computational results

Results when using an LP relaxation:

	MIPLIB	MIQCP	MINLPLib
integrality (% vars)	71.7	59.9	63.5
roundability (% inst)	59.7	70.0	61.9


MIP: % success rate
100
75
50
25
0
moving average (10 inst.)
moving average (30 inst.)
0
20
40
60
80
100
% integral variables
MIQCP: % success rate
100
75
50
25
0
moving average (10 inst.)
moving average (30 inst.)
0
20
40
60
80
100
% integral variables

Results when using an LP relaxation:

	MIPLIB	MIQCP	MINLPLib
integrality (% vars)	71.7	59.9	63.5
roundability (% inst)	59.7	70.0	61.9

Further findings

- cuts: no effect on integrality, bad for roundability
- NLP solutions much less integral, similar roundability

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RENS as primal heuristic

- defensive version succeeds in $\approx 50\%$ (MIP and MIQCP)
- inferior to portfolio, superior to single best
- mixed runtimes: mostly ≤ 5 sec, for MIP sometimes ≈ 1 min

From structures to heuristics to global solvers

1. Undercover: the largest sub-MIP [B., Gleixner. Math. Prog., 2013]
2. RENS: the optimal rounding [B. Math. Prog. C, under review]
3. Measuring the impact of primal heuristics [B. OR Letters, accepted]

And what about the global solvers?

How important are primal heuristics in global solvers?

A MIP software vendor says:

Our advanced MIP heuristics for quickly finding feasible solutions often produce good quality solutions where other solvers fall flat, leading to some of our biggest wins vs. the competition.

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- one vendor: 6% improvement
- other vendor: 9% improvement
- non-commercial solver: 15 % improvement

$\rightsquigarrow$ **not important at all**

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So, what is wrong here?

Comparing performance

How to measure the added value of a primal heuristic?

- time to optimality t_{solved}, number of branch-and-bound nodes
  - very much depends on dual bound
- time to best solution t_{opt}
  - nearly optimal solution might be found long before
- time to first solution t_1
  - disregards solution quality
- performance profiles
  - depend on t_{solved}, hence on dual bound
  - not an absolute number
- primal integral

Let $\tilde{x}$ be a solution, $\tilde{x}_{\text{opt}}$ be an optimum, $\mathrm{t}_{\max} \in \mathbb{R}_{\geq 0}$ be a timelimit.

Primal gap $\gamma \in [0, 1]$ of $\tilde{x}$:

$$\gamma(\tilde{x}) := \begin{cases} 0, & \text{if } \left|c^T \tilde{x}_{\text{opt}}\right| = \left|c^T \tilde{x}\right| = 0, \\ 1, & \text{if } c^T \tilde{x}_{\text{opt}} \cdot c^T \tilde{x} < 0, \\ \frac{\left|c^T \tilde{x}_{\text{opt}} - c^T \tilde{x}\right|}{\max\{\left|c^T \tilde{x}_{\text{opt}}\right|, \left|c^T \tilde{x}\right|\}}, & \text{else.} \end{cases}$$

Primal gap function $p\colon [0, \mathrm{t}_{\max}] \mapsto [0, 1]$:

$$p(t) := \begin{cases} 1, & \text{if no incumbent until point } t, \\ \gamma(\tilde{x}(t)), & \text{with } \tilde{x}(t) \text{ incumbent at point } t. \end{cases}$$

- step function, changes at points t_i when new incumbent found
- $p(0) = 1$, $p(t) = 0$ for all $t \geq t_{\text{opt}}$
- monotonically decreasing

Primal integral $P(T)$ of $T \in [0, \mathrm{t}_{\max}]$:

$$P(T) := \int_{t=0}^{T} p(t)\, dt = \sum_{i=1}^{I} p(t_{i-1}) \cdot (t_i - t_{i-1}),$$

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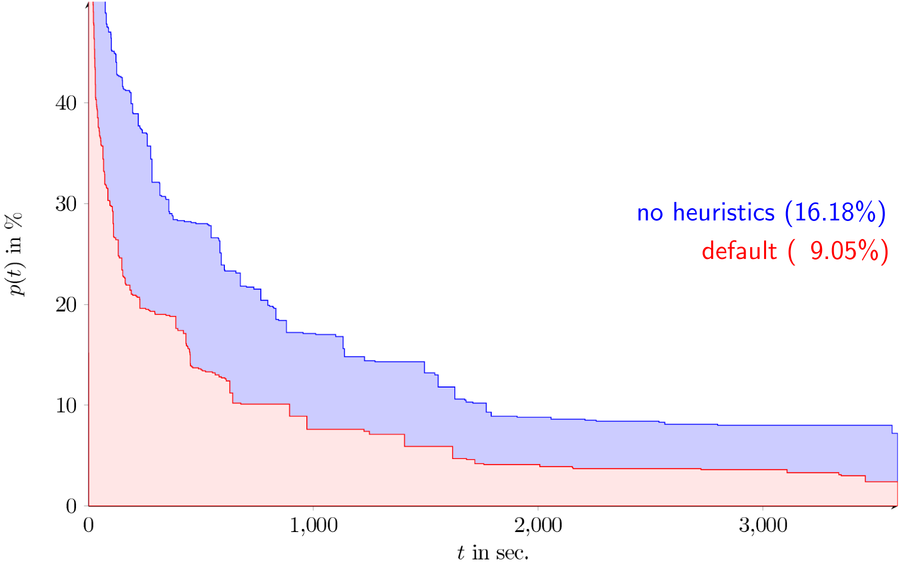
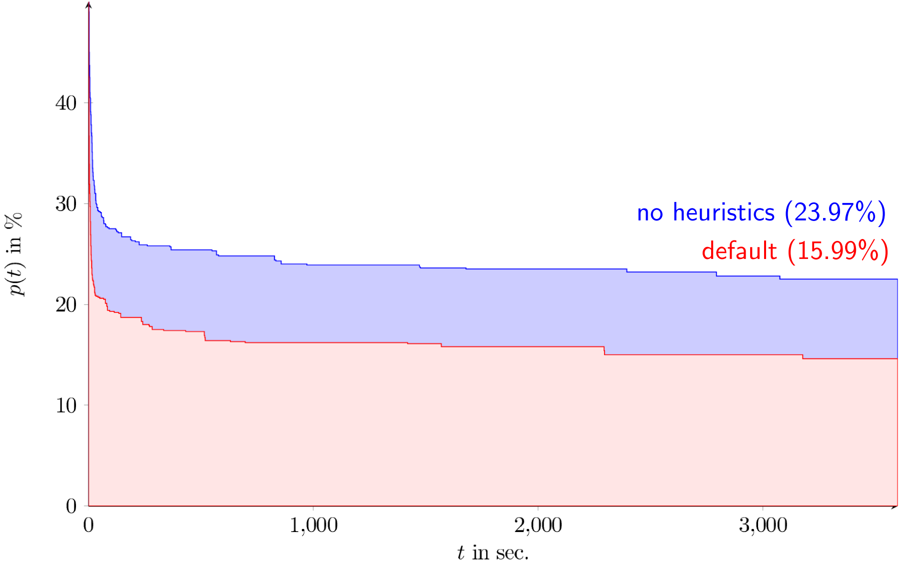
- performance profiles
  - depend on t_{solved}, hence on dual bound
  - not an absolute number
- primal integral $P(\mathrm{t}_{\max})$
  - favors finding good solutions early
  - considers each update of incumbent
  - $P(\mathrm{t}_{\max})/\mathrm{t}_{\max}$ “average solution quality”
  - expected quality of the incumbent, if stopped arbitrarily

Average primal integral: SCIP / MIP



Average primal integral: SCIP / MINLP



Undercover & RENS:

- LNS start heuristics for MINLP
- exploit generic problem structures
- successful application as root node heuristics

Primal integral:

- new performance measure, captures overall solution process
- more robust, though not immune, against randomness
- overall impact of primal heuristics (w.r.t. $P(\mathrm{t}_{\mathrm{max}})$) significant

slides and technical reports of presented papers: http://www.zib.de/berthold

SCIP Optimization Suite: http://scip.zib.de

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