Lecture Notes

Fare Planning

Marc Pfetsch

Zuse Institute Berlin

pfetsch@zib.de

Technische Universität Berlin
Fakultät II, Institut für Mathematik
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Ganzzahlige Optimierung im Öffentlichen Verkehr

Ralf Borndörfer, Christian Liebchen, Marc Pfetsch

Contents

Chapter 1

Fare Planning

1.1 What is Fare Planning?

In this chapter we deal with the problem to optimize fares for a public transport system. We assume that we are given a price system and we want to optimize with respect to different objectives, such as maximization of the revenue, profit, or the number of passengers. The price system includes structural decisions, for instance, whether we have zone tariffs or distance dependent fares and it includes the types of different tickets, such as single tickets, monthly tickets etc.

Currently, fares in public transport system are planned through a political process, i.e., they are subject to negotiations. The main question often is: To what extent can one raise fares under political and social constraints? The goal of fare planning, as we will present it in the following, is to introduce mathematical optimization into this process. With the models of this chapter one can reach quantitative results and new fare systems can be tested in silico. New and more complicated fare systems are likely to appear in the future, when they are made practical through technical innovations like electronic ticketing.

The outline of this chapter is as follows. We first fix notation and then present several different nonlinear models for fare planning. These models are based on so-called *demand functions*, i.e., functions that determine the number of passengers that want to travel with a given ticket type for a given fare. We will discuss how we obtain such a demand function using a so-called *logit model*, which is a special kind of *discrete choice model*. Then we present computational results.

Details can be found in Borndörfer, Neumann, and Pfetsch [2].

1.2 Basic Models

The *fare planning problem* involves a traffic network $G = (V, E)$, where the nodes V represent locations and the edges E connections that can be used for travel. Given is a set $D \subseteq V \times V$ of *origin-destination pairs* (OD-pairs or traffic relations). We assume fixed passenger routes, i.e., for every OD-pair $(s, t) \in D$ there is a unique path Q_{st} through the traffic network the passengers will use. In our case, the passengers use the time-minimal path.

Furthermore, we are given a finite set $\mathcal{C}$ of *travel choices*. Examples of

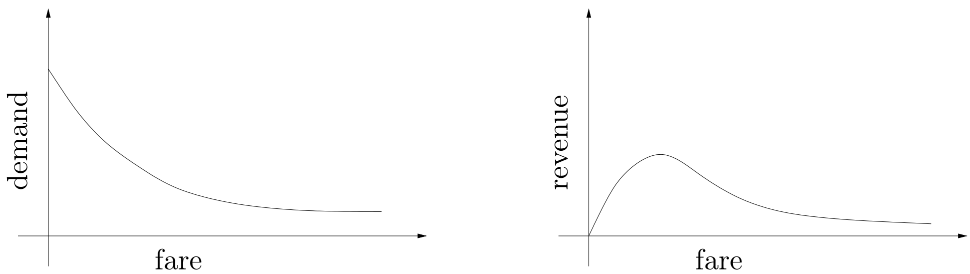

Figure 1.1: Examples for a demand and revenue function.

travel choices that we have in mind are: single or monthly tickets, distance dependent fares, etc. Travel choices may also include the number of trips during a time horizon, e.g., 30 trips during a month with a monthly ticket.

We consider $n \in \mathbb{N}$ nonnegative fare variables $x_1, \dots, x_n$, which we call *fares* in the following. A *fare vector* is a vector $\boldsymbol{x} \in \mathbb{R}^n_+$ of such fares.

The model involves *price functions* $p^i_{st} : \mathbb{R}^n \to \mathbb{R}_+$ and *demand functions* $d^i_{st} : \mathbb{R}^n \to \mathbb{R}_+$ for each OD-pair $(s, t) \in D$ and each travel choice $i \in \mathcal{C}$. The price functions $p^i_{st}(\boldsymbol{x})$ determine the price for traveling with travel choice i from s to t depending on the fare vector $\boldsymbol{x}$. All p^i_{st} appearing in this paper are affine functions and hence differentiable. The demand functions $d^i_{st}(\boldsymbol{x})$ measure the amount of passengers that travel from s to t with travel choice i, depending on the fare vector $\boldsymbol{x}$. To simplify notation, we use

$$d_{st}(\boldsymbol{x}) := \sum_{i \in \mathcal{C}} d^i_{st}(\boldsymbol{x}).$$

We assume that d_{st} is *nonincreasing*, i.e.,

$$\boldsymbol{x}^1 \leq \boldsymbol{x}^2 \quad \Rightarrow \quad d_{st}(\boldsymbol{x}^1) \geq d_{st}(\boldsymbol{x}^2).$$

It follows that the demand is maximized for $\boldsymbol{x} = \mathbf{0}$. Note that the components d^i_{st} will not be nonincreasing in general. In our examples, demand functions are also differentiable and, in particular, continuous. See Figure 1.1 for an illustration.

The *revenue* $r(\boldsymbol{x})$ is calculated as:

$$r(\boldsymbol{x}) := \sum_{(s,t) \in D} \sum_{i \in \mathcal{C}} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}).$$

The first model for the fare planning problem maximizes revenue:

$$\begin{array}{lll} (\text{MAX-R}) \qquad & \max & \sum_{(s,t) \in D} \sum_{i \in \mathcal{C}} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) \\ & s.t. & \boldsymbol{x} \geq \mathbf{0}\,. \end{array}$$

The model assumes a fixed level of service, that is constant costs. Additional constraints on $\boldsymbol{x}$ can also be included into the model, e.g., upper bounds on the fares, because of political reasons. Since the demand function is nonincreasing, lower bounds on the fare could ensure a certain demand.

The next model is more realistic, since it allows a variable level of service. This means to include the costs for the service into the model. We do this by also planning the frequencies of the lines. In principle, we would like to include a complete line planning model. Due to the complexity of line planning, in this context, however, we can only include a simplified version. That is, we consider a *line pool* $\mathcal{L}$ and compute a *continuous frequency* $f_\ell \geq 0$ for each line $\ell \in \mathcal{L}$. We are given parameters $c_\ell \geq 0$ for the operating costs of line $\ell \in \mathcal{L}$. We assume that the lines are symmetric, i.e, f_ℓ is the frequency for the back and forth direction. Finally, we are given *vehicle capacities* $\kappa_\ell \geq 0$ for each line.

With these additional assumptions we can maximize the *profit* (revenue minus costs) under the restriction of sufficient transportation capacity on each edge and including a fixed subsidy $\mathcal{S} \geq 0$:

$$
\begin{array}{lll}
(\text{Max-P}) \qquad \max & \displaystyle\sum_{(s,t)\in D} \sum_{i\in\mathcal{C}} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) - z & \\
s.t. & \displaystyle\sum_{\ell\in\mathcal{L}} c_\ell \, f_\ell - \mathcal{S} \leq z & \\
& \displaystyle\sum_{\substack{(s,t)\in D\\ e\in Q_{st}}} d_{st}(\boldsymbol{x}) \leq \sum_{\ell: e\in\ell} f_\ell \, \kappa_\ell & \forall e \in E \\
& \boldsymbol{x} \geq \boldsymbol{0} & \\
& \boldsymbol{f} \geq \boldsymbol{0} & \\
& z \geq 0. &
\end{array}
$$

Because z is nonnegative and is minimized ($-z$ is maximized), we have

$$
z = \max\Big\{\sum_{\ell\in\mathcal{L}} c_\ell \, f_\ell - \mathcal{S}, 0\Big\}.
$$

Therefore it is guaranteed that the subsidy can only be used for compensating the costs.

Note. If the costs for transporting passengers are smaller than the revenue, then optimal fares for Max-P with $\mathcal{S} = 0$ are optimal for Max-P with $\mathcal{S} > 0$ and conversely, as long as the subsidy is smaller than the "optimal" costs of model Max-P. This holds, because of the following reasoning. Under the given conditions, we have

$$
\sum_{\ell\in\mathcal{L}} c_\ell \, f_\ell - \mathcal{S} = z.
$$

Substituting this into the objective of model Max-P leaves a constant $\mathcal{S}$, which does not influence optimal solutions. This shows the claim.

These two models are meant to improve the profit of the public transport system. We now consider a model that also covers social issues of public transport. The objective is to maximize the number of passengers such that the public transport system is not a "losing deal". More precisely: In the case of zero subsidy, the objective is to maximize the number of passengers such that the costs have to be smaller than the revenue; in the case of positive subsidy the costs have to be smaller than the revenue plus subsidy.

$$
\begin{aligned}
(\text{MAX-D}) \qquad & \max \sum_{(s,t)\in D} d_{st}(\boldsymbol{x}) \\
& s.t. \sum_{(s,t)\in D} \sum_{i\in\mathcal{C}} p^i_{st}(\boldsymbol{x}) \cdot d^i_{st}(\boldsymbol{x}) \geq \sum_{\ell} c_\ell \, f_\ell - \mathcal{S} \\
& \qquad \sum_{\substack{(s,t)\in D \\ e\in Q_{st}}} d_{st}(\boldsymbol{x}) \leq \sum_{\ell: e\in\ell} f_\ell \, \kappa_\ell \qquad \forall e \in E \\
& \qquad \boldsymbol{x} \geq \mathbf{0} \\
& \qquad \boldsymbol{f} \geq \mathbf{0}.
\end{aligned}
$$

Note. Without the first side constraint, the optimal solution would be $\boldsymbol{x} = 0$ since the d_{st} are nonincreasing.

All introduced models are nonlinear programs that may be quite hard to solve in general. Nevertheless, in our examples all functions are differentiable and we managed to compute the optimum.

1.3 Demand Functions

Our approach to fare planning is based on the assumption that passenger behavior in response to fares can be given by the demand functions d^i_{st}. This is a necessary, but in practice quite strong assumption.

There are several issues that are discussed in the literature.

- For d^i_{st} to exist, passengers need *full knowledge* of the situation and act *rationally* with respect to the change of fares. It follows that demand functions are nonincreasing. The assumption on full knowledge and rationality is clearly unrealistic.
- Passenger behavior in reality is asymmetric, i.e., passengers behave differently to increasing and decreasing fares. In particular, if a fare is raised and lowered back to the original value, passengers do not behave as before, at least not immediately.
- In general, passengers need time to adjust to fare changes.
- A principle drawback is that demand functions cannot be measured, since (ceteris paribus) experiments cannot be carried out, and surprising effects significantly influence the situation. For instance, in many experiments with zero fares the main passenger increase is caused by induced traffic and by passengers that used a bike or went by foot before.

These are valid arguments that demand functions cannot model the "truth". Nevertheless, we will follow large parts of the economic and public transport literature that take the point of view that demand functions can be used to predict reality with reasonable accuracy.

1.3.1 Elasticity Demand Functions

The perhaps best known class of demand functions arise from constant elasticity models and are also called Cobb-Douglas functions, see Cerwenka [3]. They play a prominent role in the economic literature on public transport fares, see Oum, Waters, and Yong [7], and Goodwin [5].

The *elasticity* is the relative change in demand divided by the relative change in fares. For a (continuously) differentiable function $d : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ (with $\mathbb{R}_{>0} := \{x \in \mathbb{R} \,:\, x > 0\}$), we get for $x_0 > 0$:

$$\epsilon(x_0) = \lim_{x \to x_0} \frac{\frac{d(x)-d(x_0)}{d(x_0)}}{\frac{x-x_0}{x_0}} = \frac{x_0}{d(x_0)} \frac{d(x)-d(x_0)}{x-x_0} = x_0 \frac{d'(x_0)}{d(x_0)}.$$

In the public transport literature the elasticity is often assumed to be constant, e.g., $\varepsilon = -0.3$, a value which is usually attributed to Curtin and Simpson [4]. Constant elasticities are designed for use in a small neighborhood around some point. In fact, for $\varepsilon < 0$ Cobb-Douglas functions are undefined at zero and hence not applicable for situations where a fare can be reduced to zero. We will not use Cobb-Douglas functions in the following.

1.4 Discrete Choice Models

A popular type of demand functions arises from a discrete choice analysis, see, e.g., Ben-Akiva and Lerman [1] or Maier and Weiss [6]. Here, passengers choose among a number of travel alternatives the one with highest utility. The so-called *logit models* include randomness in passenger preferences, which captures fuzziness in decision changes of passengers and renders the resulting demand functions continuous. We will introduce the basics of this approach in the following.

In a discrete choice model for public transport, each passenger chooses among a finite set A of *alternatives* for travel, e.g., single ticket, monthly ticket, bike, car travel, etc. Associated with each alternative $a \in A$ and each OD-pair $(s,t) \in D$ is a *utility* U^a_{st} which may depend on the passenger. Each utility is the sum of an observable part, the deterministic utility V^a_{st}, and a random utility, the disturbance term ν^a_{st}, i.e.:

$$U^a_{st} = V^a_{st} + \nu^a_{st}.$$

Assuming that each passenger chooses the alternative with the highest utility,

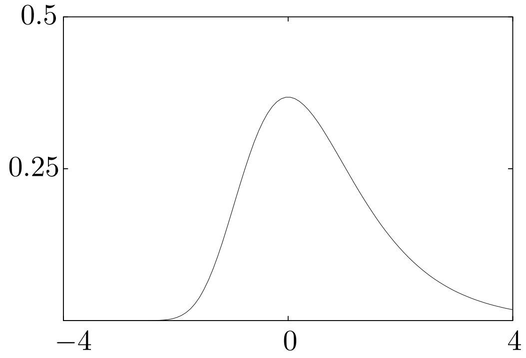

Figure 1.2: Gumbel density function $g(x)$.

the probability of choosing alternative $a \in A$ is

$$P^a_{st} := \mathbb{P}\big[V^a_{st} + \nu^a_{st} = \max_{b \in A}(V^b_{st} + \nu^b_{st})\big]. \tag{1.1}$$

1.4.1 Logit Models

In a *logit model* the disturbance terms ν^a_{st} are assumed to be independently and identically distributed according to the *Gumbel distribution* $G(\eta, \mu)$, which is defined by the density function

$$g(x) = \mu e^{-\mu(x-\eta)} \exp(-e^{-\mu(x-\eta)}),$$

where η is a location parameter and $\mu > 0$ is a scale parameter. The distribution function is then

$$G(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} \mu e^{-\mu(t-\eta)} \exp(-e^{-\mu(t-\eta)})\,dt = \exp(-e^{-\mu(x-\eta)}).$$

The Gumbel distribution resembles the Gauß distribution (see Figure 1.2) and we have:

Lemma 1.1. *The Gumbel distribution has the following properties.*

(a) The mean is $\eta + \gamma/\mu$*, where* γ *is the Euler constant,* $\gamma \approx 0.577$*. The variance is* $\pi^2/(6\mu^2)$*.*

(b) If v *is Gumbel distributed with parameters* (η, μ)*, then* $\alpha \cdot v + c$ *is Gumbel distributed with parameters* $(\alpha\eta + c, \mu/\alpha)$*, for* $\alpha > 0$*,* $c \in \mathbb{R}$*.*

(c) If v_1 *and* v_2 *are independent Gumbel distributed variables with parameters* (η_1, μ) *and* (η_2, μ)*, respectively, then* $v_1 - v_2$ *is logistically distributed, i.e., according to the following distribution function:*

$$F(x) := \frac{1}{1 + e^{\mu(\eta_2 - \eta_1 - x)}} \tag{1.2}$$

(d) *If $v_1, \dots, v_n$ are independent Gumbel distributed variables with parameters $(\eta_1, \mu), \dots, (\eta_n, \mu)$, respectively, then* $\max\{v_1, \dots, v_n\}$ *is Gumbel distributed with parameters:*

$$\Big(\frac{1}{\mu} \ln \sum_{j=1}^{n} e^{\mu\, \eta_j}, \mu\Big).$$

We skip the proof here, but will now derive the logit model, where we assume $\eta = 0$ in the following.

Proposition 1.2. *Let ν_{st}^a be independent Gumbel distributed variables with parameters $(0, \mu)$. Then the probability P_{st}^a that alternative a for OD-pair $(s,t) \in D$ is chosen is*

$$P_{st}^a = \frac{e^{\mu V_{st}^a}}{\sum_{b \in A} e^{\mu V_{st}^b}} \,. \tag{1.3}$$

Proof. We fix $(s,t) \in D$. By Lemma 1.1 (b) it follows that $U_{st}^a = V_{st}^a + \nu_{st}^a$ is Gumbel distributed with parameters (V_{st}^a, μ). From (1.1) we know that

$$P_{st}^a := \mathbb{P}\big[V_{st}^a + \nu_{st}^a \geq \max_{b \in A \setminus \{a\}} (V_{st}^b + \nu_{st}^b)\big].$$

Choose $a \in A$ and define

$$U_{st}^\star := \max_{b \in A \setminus \{a\}} U_{st}^b.$$

Then Lemma 1.1 (d) shows that $U_{st}^\star$ is Gumbel distributed with parameters $(V_{st}^\star, \mu)$, where

$$V_{st}^\star := \frac{1}{\mu} \ln \sum_{b \in A \setminus \{a\}} e^{\mu\, V_{st}^b}.$$

Then, we can write $U_{st}^\star = V_{st}^\star + \nu_{st}^\star$, where $\nu_{st}^\star$ is Gumbel distributed with parameters $(0, \mu)$. It follows that

$$P_{st}^a = \mathbb{P}\big[V_{st}^a + \nu_{st}^a \geq V_{st}^\star + \nu_{st}^\star\big] = \mathbb{P}\big[\nu_{st}^\star - \nu_{st}^a \leq V_{st}^a - V_{st}^\star\big].$$

By Lemma 1.1 (c), $\nu_{st}^\star - \nu_{st}^a$ is logistically distributed. Applying (1.2) and using that $\eta = 0$ for both variables, we get

$$\begin{aligned} P_{st}^a = F(V_{st}^a - V_{st}^\star) &= \frac{1}{1 + e^{\mu(V_{st}^\star - V_{st}^a)}} = \frac{e^{\mu V_{st}^a}}{e^{\mu V_{st}^a} + e^{\mu V_{st}^\star}} \\ &= \frac{e^{\mu V_{st}^a}}{e^{\mu V_{st}^a} + e^{\ln \sum_{b \in A \setminus \{a\}} e^{\mu\, V_{st}^b}}} = \frac{e^{\mu V_{st}^a}}{\sum_{b \in A} e^{\mu\, V_{st}^b}}, \end{aligned}$$

which proves the claim. ∎

1.5 Application to Fare Planning

We apply the above logit model to fare optimization as follows. We consider a time horizon T and assume that a passenger who travels from s to t performs a random number of trips $X_{st} \in \mathbb{Z}_+$ during T, i.e., X_{st} is a discrete random variable. We assume that passengers do not mix alternatives, i.e., the same travel alternative is chosen for all trips. Furthermore, we assume an upper bound N on X_{st}. Let the alternatives have utilities

$$U_{st}^{a,k}(\boldsymbol{x}) = V_{st}^{a,k}(\boldsymbol{x}) + \nu_{st}^a$$

that depend on the fare vector $\boldsymbol{x}$ and the number of trips k.

Let A' be the set of public transport alternatives. Then the travel choices are $\mathcal{C} = A' \times \{1, \dots, N\}$. We write $d_{st}^{a,k}(\boldsymbol{x})$ for the amount of passengers traveling k times during T with alternative a from s to t and similarly $p_{st}^{a,k}(\boldsymbol{x})$ for the price of this travel. It follows that

$$d_{st}^{a,k}(\boldsymbol{x}) \;=\; \rho_{st} \cdot P_{st}^a(\boldsymbol{x}, k) \cdot \mathbb{P}[X_{st} = k] \;=\; \rho_{st} \cdot \frac{e^{\mu V_{st}^{a,k}(\boldsymbol{x})}}{\sum_{b \in A} e^{\mu V_{st}^{b,k}(\boldsymbol{x})}} \cdot \mathbb{P}[X_{st} = k], \tag{1.4}$$

where ρ_{st} is the entry of the OD-matrix corresponding to $(s,t) \in D$. The revenue can then be written as:

$$r(\boldsymbol{x}) = \sum_{(s,t) \in D} \sum_{a \in A'} \sum_{k=1}^{N} p_{st}^{a,k}(\boldsymbol{x}) \cdot d_{st}^{a,k}(\boldsymbol{x}) = \sum_{(s,t) \in D} \sum_{i \in \mathcal{C}} p_{st}^{i}(\boldsymbol{x}) \cdot d_{st}^{i}(\boldsymbol{x}).$$

This formula expresses the *expected* revenue over the probability spaces for X_{st} and disturbance terms ν_{st}^a.

Note that $r(\boldsymbol{x})$ is continuous and even differentiable if the deterministic utilities $V_{st}^{a,k}$ (and the price functions $p_{st}^{a,k}(\boldsymbol{x})$) have this property. This is, for instance, the case for affine functions as customary in discrete choice models, see also the example below.

1.6 Computational Results

In this section, we will present computational results for two different fare systems.

1.6.1 Fare System 1

For the first fare system we use the following:

- We distinguish two tariff zones.

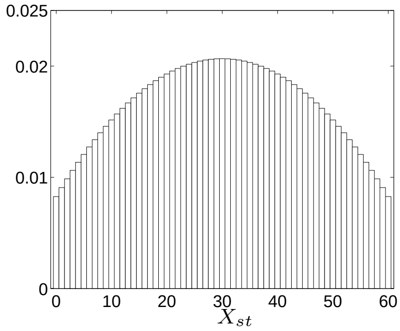

Figure 1.3: Probabilities for the number of travels.

- Travel choices: monthly ticket (M), single ticket (S), car (C); we use: $A = \{M, S, C\}$.
- Fare variables $x = (x_s, x_m)$ (where x_s and x_m are the fares for a single and monthly ticket, respectively).
- Gumbel parameter $\mu = \frac{1}{30}$, $\nu = 0$
- The probabilities for each number of trips can be seen in Figure 1.3, where the maximum number of trips is $N = 60$.

The price functions for one tariff zone are:

- $p_{st}^{S,k}(x_s, x_m) = x_s \cdot k$
- $p_{st}^{M,k}(x_s, x_m) = x_m$
- $p_{st}^{C,k}(x_s, x_m) = q_F + \ell_{st}^C \cdot q_V \cdot k, \quad \text{with } q_F = 100,\ q_V = 0.1.$

Hence, for alternative "single ticket" one has to pay the fare for a single ticket times the number of trips k. For a monthly ticket one pays the monthly ticket price only. For using a car one pays a combination of a fixed cost q_F and a distance dependent price $\ell_{st}^C \cdot q_V$, where ℓ_{st}^C is the length of the trip for a car and q_V are operating cost.

We use the following (deterministic) utilities for one tariff zone:

- $V_{st}^{S,k}(x_s, x_m) = -(x_s \cdot k) - 0.1 \cdot t_{st} \cdot k$
- $V_{st}^{M,k}(x_s, x_m) = -(x_m) - 0.1 \cdot t_{st} \cdot k$
- $V_{st}^{C,k}(x_s, x_m) = -(q_F + \ell_{st}^C \cdot q_V \cdot k) - 0.1 \cdot t_{st}^C \cdot k + y_{st}$

Here, t_{st} and t_{st}^C are the travel times for using public transport and car, respectively. The parameters y_{st} measure the "comfort" of car travel and are calibrated such that the model for the current fares yields the original demand data. The minus signs in the utilities are used, because we want the utility to decrease when the fares or travel times increase. The two parts are weighted by 0.1, i.e., one minute of traveling time is worth 0.1 monetary units.

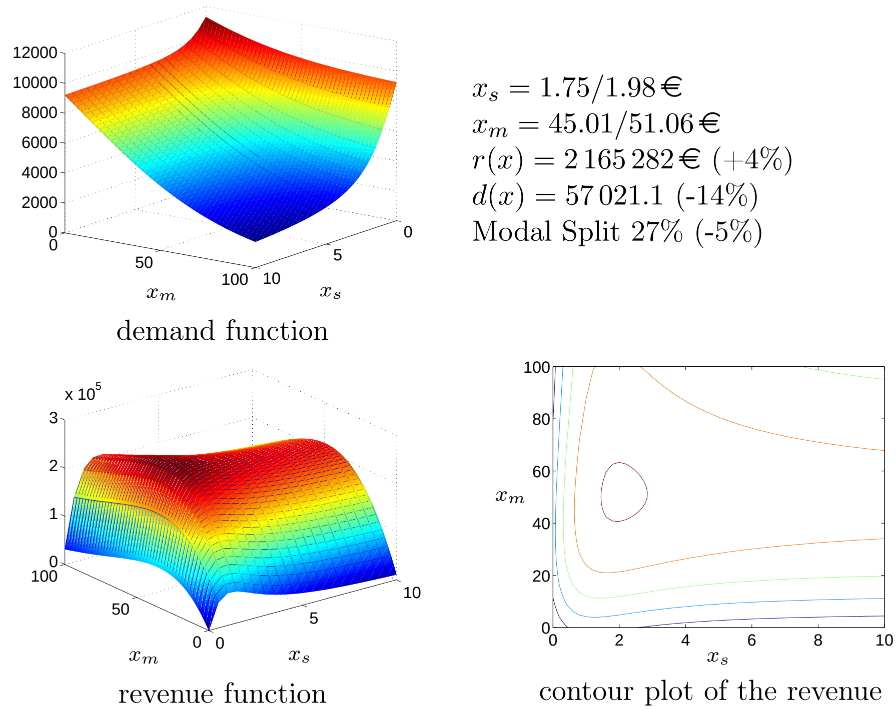

Figure 1.4: Results for maximizing demand MAX-D. The fares are given for the two tariff zones. The pictures show results for tariff zone 2. The comparison is with respect to the status quo.

Using (1.4), the revenue can be computed as:

$$r(x) = \sum_{(s,t)} \sum_{a \in \{S,M\}} \sum_{k=1}^{N} \rho_{st} \cdot \frac{p_{st}^{a,k}(x_s, x_m) \cdot e^{\mu V_{st}^{a,k}(x_s,x_m)}}{\sum\limits_{b \in \{\mathrm{M,S,C}\}} e^{\mu V_{st}^{b,k}(x_s,x_m)}} \cdot \mathbb{P}(X_{st} = k).$$

Note that we sum only over the public transport alternatives (S and M) and that this expresses the expected revenue subject to both probability distributions (for the number of trips and the disturbance terms). This revenue has to be combined for the two tariff zones.

The demand and revenue functions and optimization results for fare system 1 are shown in Figure 1.4. Table 1.1 shows the optimization results of all models presented previously.

Figure 1.5 shows the distribution of passengers for cars and public transport. One can see that much more passengers use the car than public transport. One can also see the space distribution of the passengers.

Figure 1.6 shows a plot of the objective function when we vary subsidy in model MAX-D. Here $x = 0$ is the optimal solution when we have that $\mathcal{S}$ is larger than $\approx 3\,200\,000$. If $x = 0$ we have $\approx 112\,000$ (of totally $209\,315$)

Table 1.1: Results for all models with fare system 1. We use a subsidy of 1 000 000 for the models marked with *, otherwise $\mathcal{S} = 0$.

	x_s	x_m	revenue	demand	cost
Status quo	1.45	32.50	1 831 499	60 627.0	1 914 519
	2.20	49.50	254 818	5876.0	
Max-R	1.75	45.01	1 909 843	51 038.8	1 662 187
	1.98	51.06	255 439	5 982.3	
Max-P	3.96	64.66	1 613 537	29 819.2	912 876
	7.93	87.59	170 892	2 310.8	
Max-P*	3.46	62.23	1 683 464	32 560.5	1 000 000
	6.93	83.25	183 202	2 597.7	
Max-D	1.09	32.42	1 771 871	64 988.3	2 027 026
	2.09	53.03	255 154	5 783.7	
Max-D*	0.57	18.98	1 293 622	80 034.0	2 527 431
	1.13	37.95	233 809	7 651.9	

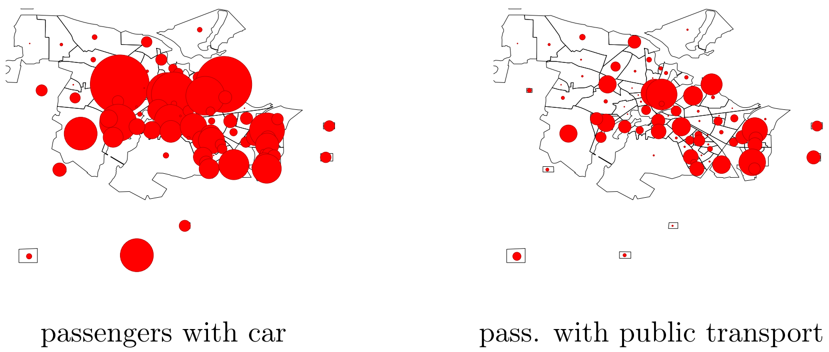

Figure 1.5: Results for Max-D with $\mathcal{S} = 1\,000\,000$. The size of the circles are proportional to the number of passengers that start their travel at the corresponding districts (which can be seen in the background).

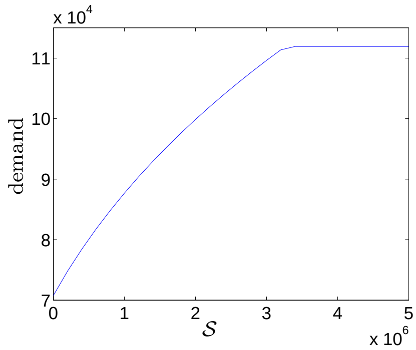

Figure 1.6: Changing subsidy $\mathcal{S}$ for the model Max-D.

Table 1.2: Computational results for all models with fare system 2. We use a subsidy of 1 000 000 for the parts marked with * and $\mathcal{S} = 0$ otherwise.

	x_b	x_d	revenue	demand	cost
MAX-R	27.34	0.26	1 901 102	59 673.9	1 456 154
MAX-P	33.30	0.65	1 568 256	33 989.0	728 189
MAX-P*	30.88	0.45	1 778 231	44 216.2	1 000 000
MAX-D	20.68	0.18	1 822 608	71 253.3	1 822 608
MAX-D*	12.94	0.09	1 402 984	88 144.1	2 402 984

passengers that use public transport.

1.6.2 Fare System 2

In the second fare system we do not distinguish the two tariff zones, but introduce a distance dependent fare. We have the following travel choices: standard ticket (S), reduced ticket (R), car (C) and define $A = \{S, R, C\}$. For the reduced ticket one pays a basic fare x_b once in the beginning and then only has to pay half the distance dependent fare x_d (which is also used for the standard ticket). More precisely, the price functions are as follows:

- $p_{st}^{S,k}(x_b, x_d) = x_d \cdot \ell_{st} \cdot k$
- $p_{st}^{R,k}(x_b, x_d) = x_b + \frac{1}{2} x_d \cdot \ell_{st} \cdot k$
- $p_{st}^{C,k}(x_b, x_d) = q_F + \ell_{st}^C \cdot q_V \cdot k, \quad$ with $q_F = 100$, $q_V = 0.1$.

Here, ℓ_{st} is the distance for traveling from s to t with public transport.

The (deterministic) utilities are defined as follows:

- $V_{st}^{S,k}(x_b, x_d) = -(x_d \cdot \ell_{st} \cdot k) - 0.1 \cdot t_{st} \cdot k$
- $V_{st}^{R,k}(x_b, x_d) = -(x_b + \frac{1}{2} x_d \cdot \ell_{st} \cdot k) - 0.1 \cdot t_{st} \cdot k$
- $V_{st}^{C,k}(x_b, x_d) = -(q_F + \ell_{st}^C \cdot q_V \cdot k) - 0.1 \cdot t_{st}^C \cdot k + y_{st}$

These utilities are set up similar to fare system 1.

The (expected) revenue function can be computed as follows:

$$r(x) = \sum_{(s,t)} \sum_{a \in \{S,R\}} \sum_{k=1}^{N} \rho_{st} \cdot \frac{p_{st}^{a,k}(x_b, x_d) \cdot e^{\mu V_{st}^{a,k}(x_b, x_d)}}{\sum\limits_{b \in \{\mathrm{S,R,C}\}} e^{\mu V_{st}^{b,k}(x_b, x_d)}} \cdot \mathbb{P}(X_{st} = k)$$

Table 1.2 shows optimization results for all models and Table 1.3 presents a comparison between the two fare systems. One can see that with model MAX-D one can attract more passengers with fare system 2 than with fare system 1, but with less revenue. The same holds for model MAX-R.

Table 1.3: Comparison between fare system 1 and 2. We use $\mathcal{S} = 0$.

		revenue	demand	costs
Status quo		2 072 106	66 503.0	3 597 604
Max-R	fare system 1	2 165 282	57 021.1	1 662 187
	fare system 2	1 901 102	59 673.9	1 456 154
Max-D	fare system 1	2 027 026	70 772.0	2 027 026
	fare system 2	1 822 608	71 253.3	1 822 608

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