Exercise 12: Implementing the Lin-Kernighan heuristic for the TSP

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Outline

The Traveling Salesman Problem

Given

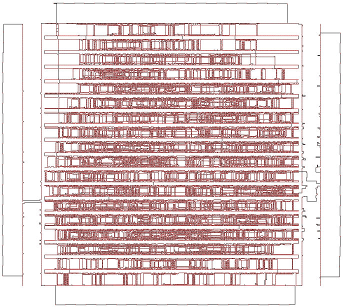
- Complete undirected graph $G = (V, E)$
- Metric edge costs $c_e \geq 0$ for all $e \in E$.

Problem

- Find a hamiltionian cycle with minimal cost.

Lin-Kernighan (LK)

- **Most famous and best** local search approach for the sym. TSP.
- Developed by Shen Lin and Brian Kernighan in 1973.
- Best **exact** solver for the TSP is `Concorde` (Applegate, Bixby, Chvátal, Cook).
- Best LK-Code today: Keld Helsgaun.
- `Concorde` + Code of Helsgaun, 2006: `pla85900` solved **(world record)**. (Applegate, Bixby, Chvátal, Cook, Espinoza, Goycoolea, Helsgaun)



k-opts

- k-opt neighborhood for tour x: $\mathcal{N}_k(x)$ consists of all tours, which can be constructed from x by deleting and adding k edges.

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Observations

- Two hamiltionian cycles only differ in k edges $(2 \leq k \leq n)$, i.e.: $x_{opt} \in \mathcal{N}_k(x)$ for every x.
- **Problem 1:** k-optimality in $\mathcal{N}_k$ can only be tested in $\mathcal{O}(n^k)$.
- **Problem 2:** k is unknown.
- **Approach:** Choose an **efficient searchable** neighborhood such that k can be choosen dynamically.

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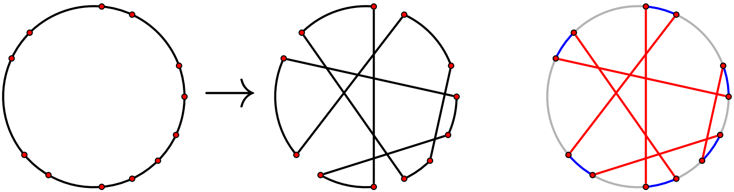
$\Rightarrow$ **Sequential k-opt moves.**

Definition: Sequential k-opt move

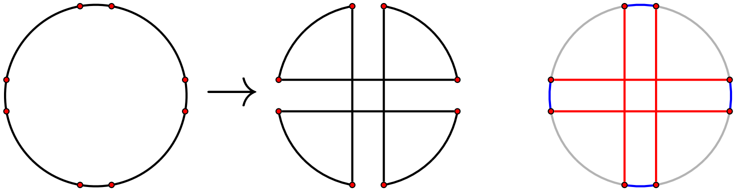
- A k-opt move is called sequential if it can be described by a path alternating between deleted and added edges.

Example

Sequential 6-opt

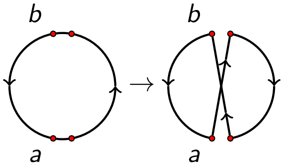


Double-Bridge-Move (4-opt)



Lin-Kernighan

Flip operations



- **Operation:**
  $\textbf{flip}(\textbf{next}(a), \textbf{prev}(b))$
- **Gain g_t of flip t:**

$$g_t = c(a, \text{next}(a)) + c(\text{prev}(b), b) - c(\text{next}(a), b) - c(a, \text{prev}(b))$$

Lin-Kernighan

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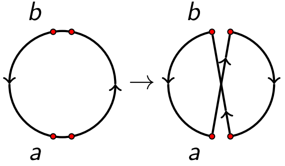

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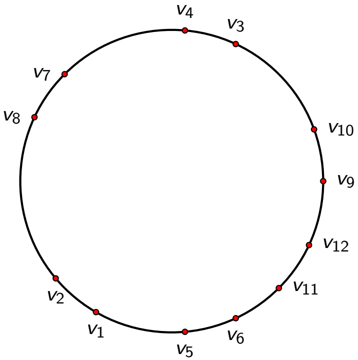
Lin-Kernighan

- Choose a fix starting node a and construct the alternating path by a sequence of flip operations of the form flip(next(a), prev(b)).
- The goal within this construction is to obtain

$$\sum_{i=1}^{k} g_{t_i} > 0.$$

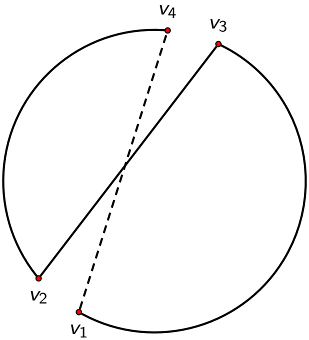
Example



$v_1 \underline{v_2 - v_8 v_7 - v_4} v_3 - v_{10} v_9 - v_{12} v_{11} - v_6 v_5 -$

$\texttt{flip}(v_2, v_4)$

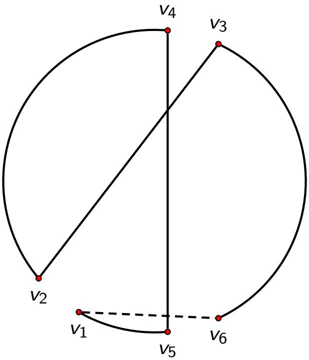
Example



$$v_1 \underline{v_4 - v_7 v_8 - v_2 v_3 - v_{10} v_9 - v_{12} v_{11} - v_6} v_5 -$$

$\texttt{flip}(v_4, v_6)$

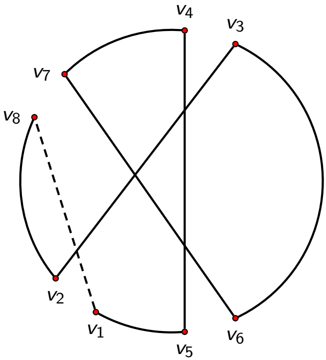
Example



$$v_1\,\underline{v_6 - v_{11}\,v_{12} - v_9\,v_{10} - v_3\,v_2 - v_8}\,v_7 - v_4\,v_5 -$$

$\texttt{flip}(v_6, v_8)$

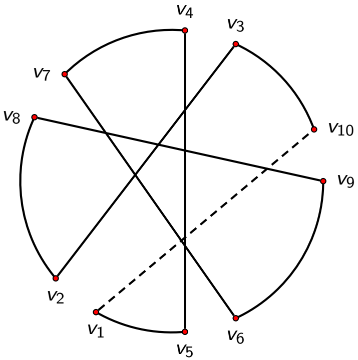
Example



$$v_1 \underline{v_8 - v_2 v_3 - v_{10}} v_9 - v_{12} v_{11} - v_6 v_7 - v_4 v_5 -$$

$$\texttt{flip}(v_8, v_{10})$$

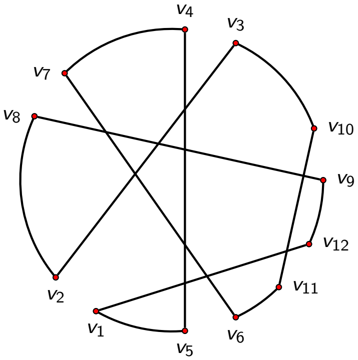
Example



$v_1\, \underline{v_{10} - v_3\, v_2 - v_8\, v_9 - v_{12}}\, v_{11} - v_6\, v_7 - v_4\, v_5 -$

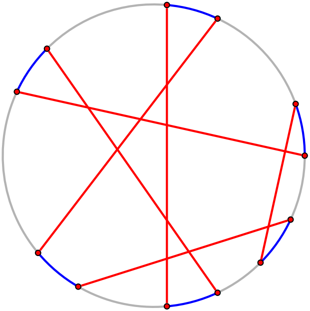
$\texttt{flip}(v_{10}, v_{12})$

Example



$v_1 v_{12} - v_9 v_8 - v_2 v_3 - v_{10} v_{11} - v_6 v_7 - v_4 v_5 -$

Example



Implementation details

- Backtracking
- Neighborhoodgraph
  - k-Nearest graph
  - α-Nearest graph
  - Delaunay triangulation
- Mak-Morton-Moves
- `alternateStep()`
- Swaps
- Bentley-Marking-Scheme
- Kicking-Strategic
  - Double-Bridge-Moves
  - ...

Neighborhood graphs for the choice of the edge $\{a, \text{next}(a)\}$

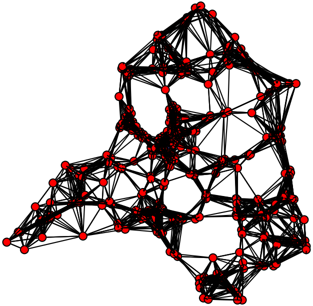


Figure: Neighborhood graph: 15 nearest neighbors

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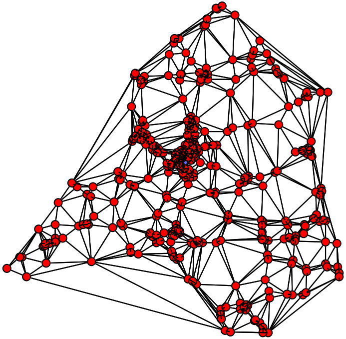

Figure: Neighborhood graph: Delaunay triangulation