Network Design and Operation (WS 2015)

Excercise Sheet 8

Submission: Mo, 14. December 2015, tutorial session

Exercise 1

Proposition 1. The vector x^0 is a vertex of the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0_n\}$ if and only if the vector $(x^0, y^0) \in \mathbb{R}^{n+m}$ such that:

$$\begin{pmatrix} x^0 \\ y^0 \end{pmatrix} = \begin{pmatrix} x^0 \\ b - Ax^0 \end{pmatrix}$$

is a vertex of the polyhedron $P^{STD} = \{(x^0, y^0) \in \mathbb{R}^{n+m} : Ax + Iy = b,\ x \geq 0_n,\ y \geq 0_m\}$.

Prove that if (x^0, y^0) is a vertex of P^{STD} then x^0 is a vertex of P.

Exercise 2

Let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a polyhedron and assume that A has full column rank. Additionally, denote by $I = \{1, \ldots, m\}$ the set of rows of A. Prove that $x^* \in P$ is a vertex of P if and only if there exists a set $B \subseteq I$ such that $|B| = n$, A_B is invertible and $A_B x^* = b_B$ (here, we denote by A_B and b_B the rows of A and the components of b indexed by B, respectively).

Exercise 3

Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix. Prove that A is totally unimodular if and only if the polyhedron

$$P = \{x \in \mathbb{R}^n : \beta \leq Ax \leq b,\ \ell \leq x \leq u\}$$

is integral for all vectors $\beta, b \in \mathbb{R}^m$, $\ell, u \in \mathbb{R}^n$

Exercise 4

Let A be an $m \times n$ binary matrix (i.e, $A \in \{0, 1\}^{m \times n}$) and denote by J the set of the column indices. Prove that if A possesses the property:

$$\forall j, k, \ell \in J :\ j \leq k \leq \ell, \ \text{ if } a_{ij} = 1 \text{ and } a_{i\ell} = 1 \text{ then } a_{ik} = 1$$

then A is totally unimodular.

Exercise 5

Proposition 2. Let $A \in \{-1, 0, 1\}^{m \times n}$ be a matrix such that each of its column has at most two non-zero entries.

The matrix A is totally unimodular *if and only if* there exists a partition (I_1, I_2) of the set of rows I of A such that:

$$a_{i_1,j} \cdot a_{i_2,j} \leq 0 \iff \{i_1, i_2\} \subset I_1 \text{ or } \{i_1, i_2\} \subset I_2$$

for all $i_1, i_2 \in I$ with $i_1 \neq i_2$ and $a_{i_1},\ a_{j_2} \neq 0$

Derive a polynomial time algorithm that can test whether a generic matrix A satisfies the above condition

Hint: checking that a graph is bipartite can be checked in polynomial time

Exercise 6

Let A be an $m \times n$ binary matrix (i.e, $A \in \{0,1\}^{m \times n}$). The matrix A possesses the *(column-wise) consecutive-ones property* when, after a possible reordering of the rows of A, the non-zero entries appear consecutively in each column.

Prove that A possessing the (column-wise) consecutive-ones property is totally unimodular by exploiting the following proposition.

Proposition 3. Let $A \in \{-1,0,1\}^{m \times n}$ be a matrix. A is totally unimodular if for any subset I' of the rows, there exists a partition (I_1, I_2) such that each column $j \in J$ satisfies:

$$\left| \sum_{i \in I_1} a_{ij} - \sum_{i \in I_2} a_{ij} \right| \leq 1$$