Exercise 2

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Random walk and binomial distribution

Consider a one-dimensional lattice of equally spaced points. Assume that a random walker is at position $x_0 = 0$ at time 0, and that he either jumps to the right with probability $p = \frac{1}{2}$ or to the left with probability $p = \frac{1}{2}$. After N timesteps he arrives at position m, where $m \in \mathbb{Z}$ denotes the node of the grid that he reached: $m > 0$ if the walker reached a site on the right of the starting position, $m < 0$ if the walker reached a site on the left of the starting position. The probability to reach the position m after N timesteps is given by the binomial distribution:

$$P_d(m, N) = \begin{cases} 0 & \text{if } N \text{ even and } m \text{ odd} \\ 0 & \text{if } N \text{ odd and } m \text{ even} \\ \frac{N!}{\frac{N+m}{2}! \cdot \frac{N-m}{2}!} \left(\frac{1}{2}\right)^N & \text{else} \end{cases} . \tag{1}$$

Pen-and-paper exercise

- Show that the expected position is

  $$\langle m \rangle = 0 ,$$

  and that the variance of the position is

  $$\langle m^2 \rangle = N .$$

- For very large N, the distribution function becomes continuous:

  $$P_c(m, N) = \sqrt{\frac{2}{N\pi}} \exp\left(-\frac{m^2}{2N}\right) . \tag{2}$$

  Derive eq. 2 using the Stirling's approximation:

  $$N! \approx \left(\frac{N}{e}\right)^N \sqrt{2\pi N} , \tag{3}$$

  where $e = 2.17828....$

Computational exercise

- Write a program to simulate the one-dimensional random walk X_n.
- Generate 4 sets of trajectories, each of 10000 trajectories, for $N = 3, 6, 30, 6000$.
- For each set of trajectories build the histogram using the final positions X_N.
- Compare the histograms with the analytical distribution defined in eqs. 1 and 2.