NOTES ON TAYLOR EXPANSION

Consider a function $f(x) : \mathbb{R} \to \mathbb{R}$ that is infinitely differentiable at point $a \in \mathbb{R}$, i.e. there exists the nth derivative evaluated at $x = a$:

$$f^{(n)}(a) = \frac{d^n}{dx^n} f(x)\bigg|_{x=a} \qquad \forall n = 0, 1, 2, \cdots \infty \,. \tag{1}$$

Then, the function $f(x)$ is approximated at specific point x, in the neighbourhood of a, as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \,. \end{aligned} \tag{2}$$

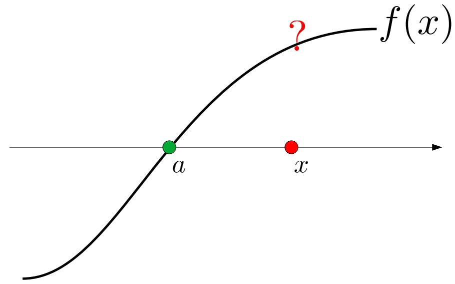


FIG. 1: We know the value of the function and all its derivatives at a (green dot), the Taylor expansion approximates the function at x (red dot).

Alternative definition

The Taylor expansion is often defined in a different way. Consider the point $x \in \mathbb{R}$, and assume that $f(x)$ is infinitely differentiable at this point. The Taylor expansion at $x + \Delta x$, where Δx is a small interval, is

$$\begin{aligned} f(x+\Delta x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}\Delta x^n \\ &= f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \frac{f'''(x)}{3!}\Delta x^3 + \cdots \,. \end{aligned} \tag{3}$$

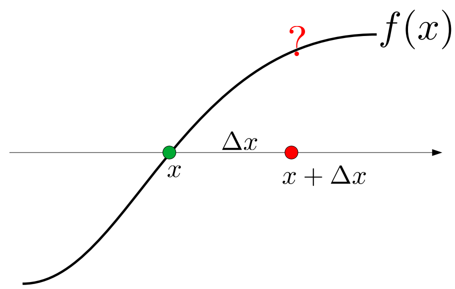


FIG. 2: We know the value of the function and all its derivatives at x (green dot), the Taylor expansion approximates the function at $x + \Delta x$ (red dot).

Example

Consider the exponential function

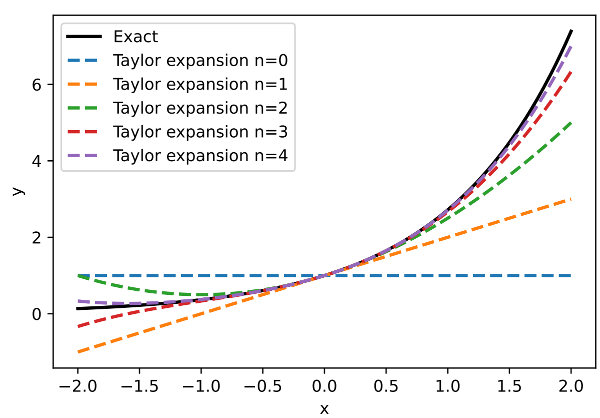
$$f(x) = e^x \,, \tag{4}$$

and let approximate e^x around the point $a = 0$. The nth derivative of e^x is always the same:

$$f^{(n)}(x) = \frac{d^n}{dx^n} e^x = e^x \quad \forall n = 0, 1, 2, \cdots \infty \,. \tag{5}$$

and its value in $a = 0$ is always

$$f^{(n)}(a) = \frac{d^n}{dx^n} e^x \bigg|_{x=0} = e^0 = 1 \,. \tag{6}$$



Then the Taylor expansion is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \,. \tag{7}$$