Konrad-Zuse-Zentrum für Informationstechnik Berlin
Takustraße 7, D-14195 Berlin-Dahlem, Germany

Norbert Ascheuer Matteo Fischetti

Martin Grötschel

A polyhedral study of the asymmetric travelling salesman problem with time windows

Preprint SC 97–11 (February 1997)

A polyhedral study of the asymmetric travelling salesman problem with time windows

Norbert Ascheuer [*] Matteo Fischetti [†] Martin Grötschel [*]

Berlin, February 1997

Abstract

The asymmetric travelling salesman problem with time windows (ATSP-TW) is a basic model for scheduling and routing applications. In this paper we present a formulation of the problem involving only 0/1-variables associated with the arcs of the underlying digraph. This has the advantage of avoiding additional variables as well as the associated (typically very ineffective) linking constraints. In the formulation, time window restrictions are modelled by means of "infeasible path elimination" constraints. We present the basic form of these constraints along with some possible strengthenings. Several other classes of valid inequalities derived from related asymmetric travelling salesman problems are also described, along with a lifting theorem. We also study the ATSP-TW polytope, P_{TW}, defined as the convex hull of the integer solutions of our model. We show that determining the dimension of P_{TW} is strongly $\mathcal{NP}$–complete problem, even if only one time window is present. In this latter case, we provide a minimal equation system for P_{TW}. Computational experiments on the new formulation are reported in a companion paper [5] where we show that it outperforms alternative formulations on some classes of problem instances.

Keywords : Asymmetric Travelling Salesman Problem,Time Windows, Polyhedral Theory, Valid Inequalities

1 Introduction

The *Asymmetric Travelling Salesman Problem with Time Windows (ATSP-TW)*, in the "open path" version considered in this paper, can be described as follows. Consider a complete digraph $G = (V, A)$ on $n := |V|$ nodes, having nonnegative arc *costs* c_{ij} and *set-up times* t_{ij} associated with each arc $(i, j) \in A$. Nodes correspond to jobs to be processed in sequence and without preemption on a single machine. Arcs correspond to job transitions, the associated set-up times t_{ij} giving the changeover time needed to process node j right after node i. A *processing time* $p_i \geq 0$, a *release date* $r_i \geq 0$ and a *deadline* $d_i \geq r_i$ are given for every node $i \in V$. The release date r_i denotes the earliest possible, the deadline d_i the latest possible starting time for processing node $i \in V$. The interval $[r_i, d_i]$ is called the *time window* for node i. The time window is called *active* if $r_i > 0$ or $d_i < +\infty$; a time window of the type $[0, +\infty)$ is called *relaxed*. We deal with the case where waiting times are allowed, i.e.,

[*]Konrad–Zuse–Zentrum für Informationstechnik Berlin (ZIB), Takustr. 7, 14195 Berlin–Dahlem, Germany
[†]Dipartimento di Matematica ed Informatica, University of Udine, Italy (work supported by M.U.R.S.T. and C.N.R.,Italy)

processing may reach a node $i \in V$ earlier than r_i and wait until the node is released. We assume that costs and set-up times, as well as processing times, release dates and deadlines are integer values, and allow $d_i = +\infty$ for some nodes $i \in V$.

The *minimal time delay* for processing node j immediately after node i is given by

$$\vartheta_{ij} := p_i + t_{ij}.$$

In several applications the triangle inequality on ϑ is satisfied, i.e.,

$$\vartheta_{ij} \leq \vartheta_{ik} + \vartheta_{kj}, \text{ for all } i, j, k \in V, |\{i, j, k\}| = 3. \tag{1.1}$$

The problem is to find a min-cost Hamiltonian path satisfying the time window restrictions, i.e., a node sequence with minimal total cost such that, for every node $i \in V$ the start time for processing (visiting) node $i \in V$ lies within the given time window $[r_i, d_i]$.

By introducing an additional node (depot) the problem can easily be transformed into its "closed tour" version in which one is interested in finding a min-cost Hamiltonian tour through all nodes, starting at the depot node at time 0 and satisfying the time window restrictions for all other nodes.

Note that ATSP-TW reduces to the standard asymmetric TSP when the time windows are relaxed, hence ATSP-TW with general time windows is an $\mathcal{NP}$–hard problem. In fact, Savelsbergh [33] showed that it is already strongly $\mathcal{NP}$–complete to find a feasible solution for the problem; this also follows from Garey and Johnson [26], who showed that it is strongly $\mathcal{NP}$–complete to find a feasible schedule for non–preemptive single machine scheduling with release times and deadlines. Furthermore, Tsitsiklis [35] showed that the problem with general time windows remains strongly $\mathcal{NP}$–complete, even if the nodes correspond to points on a line and all processing times equal 0.

The research in this paper was motivated by a joint project with industry that had the aim to minimize the time needed for the unloaded moves of a stacker crane in an automated storage system, see Ascheuer [3] and Abdel–Hamid et al. [1] for details. As already mentioned, the ATSP-TW (in its "open path" version) can be interpreted as a *one machine scheduling problem* with time-windows and sequence dependent setup-times. We refer to Queyranne and Schulz [31] for a comprehensive survey on polyhedral approaches to scheduling problems.

Some different versions of the ATSP-TW are discussed in the literature on vehicle routing problems. For example, ATSP-TW is a subproblem in the *"cluster–first, route–second approach"*, where the nodes that have to be visited are first clustered according to some heuristic criterion, and then routed by solving an instance of ATSP-TW for each cluster. In other applications soft time windows are considered, i.e., a violation of the time windows is allowed but results in an additional penalty cost. For a survey on time–constrained routing and scheduling problems see Desrochers et al. [19, 20], among others.

To our knowledge only a few papers describe exact methods for solving to optimality the (symmetric) TSP with time windows. Christofides et al. [17] and Baker [7] describe branch&bound algorithms, whereas Dumas et al. [21] solve the problem through dynamic programming. Polyhedral methods have turned out to be an appropriate tool to solve the unconstrained ATSP (Fischetti and Toth [25]) and the ATSP with precedence constraints (Ascheuer et al. [4, 6]). Hence our attempt of extending these methods to take time windows into account.

In this paper we address a polyhedral study of the ATSP-TW. The paper is organized as follows. Section 2 gives the notation used throughout and introduces the basic model we use, in which the time window restrictions are modelled by means of "infeasible path elimination" constraints. The polytope associated with the feasible solutions to this model is addressed in Section 3, where we investigate the polytope dimension in the very special case in which only one of the time windows is active. We show that, even in this oversimplified case, determining the polytope dimension is a strongly $\mathcal{NP}$-complete problem. Several classes of additional valid inequalities are introduced in Section 4. In particular, we describe different lifted forms of the basic infeasible path elimination constraints, new inequalities associated with concatenation of feasible paths, a lifting procedure, and a strengthened version of the (π, σ)–inequalities proposed by Balas et al. [12] for the ATSP with precedence constraints. Some conclusions are finally drawn in Section 5.

2 Notation and Modelling

The ATSP-TW can be modelled in various ways. A "standard model" (cf. Desrochers et al. [18], among others) involves binary arc variables x_{ij} as well as node variables τ_i, indicating the time when node i is visited. The time window restrictions are modelled with the help of the bound constraints $r_i \leq \tau_i \leq d_i$; linking between the $x-$ and $\tau-$variables is provided by a generalization of the Miller-Tucker-Zemlin inequalities, namely:

$$\tau_i + \vartheta_{ij} - (1 - x_{ij}) \cdot M \leq \tau_j \quad \forall i, j \in V, i \neq j$$

where M is a sufficiently large positive value. Note that these constraints involve a "big M"–term that is known to cause computational problems.

Recently, Maffioli and Sciomachen [29] and van Eijl [36] proposed a different model involving $|A|$ additional variables y_{ij} which give the time when node i is left in direction to node j (with $y_{ij} = 0$ whenever $x_{ij} = 0$). The time window restrictions are then modelled via the constraints $r_i x_{ij} \leq y_{ij} \leq d_i x_{ij}$, whereas the linking between $x-$ and $y-$variables is established through the inequalities

$$\sum_{i=1}^{n} (y_{ij} + \vartheta_{ij} x_{ij}) \leq \sum_{k=1}^{n} y_{jk} \quad \forall j = 1, ..., n.$$

The computational results reported in Bogatsch [14] indicate that this model outperforms the standard one, mainly on instances where time windows are not too tight.

In this section we provide a different way of modelling the ATSP-TW as an integer linear program, which avoids additional variables as well as "big M"–coefficients. The time window constraints are modelled implicitly by a class of *infeasible path constraints*. The infeasibility is not restricted to the case of time windows: capacity constraints, e.g., might be modelled as well in a similar vein.

We next introduce the main notation used in the sequel. Given a node set $W \subseteq V$, let

$$A(W) := \{(i,j) \in A \mid i, j \in W\}$$

denote the set of all arcs with tail and head in W. For any two node sets $U, W \subseteq V$ let,

$$(U : W) := \{(i,j) \in A \mid i \in U, j \in W\}$$

denote the set of arcs with tail in U and head in W. To simplify notation, we write $(W : j)$ and $(j : W)$ instead of $(W : \{j\})$ and $(\{j\} : W)$, respectively. If $U = \emptyset$ or $W = \emptyset$, then $(U : W) = \emptyset$. Given a node set $W \subset V, W \neq \emptyset$, we also define

$$\begin{array}{lcl} \delta^-(W) & := & \{(i,j) \in A \mid i \in V \setminus W, j \in W\}, \\ \delta^+(W) & := & \{(i,j) \in A \mid i \in W, j \in V \setminus W\}, \\ \delta(W) & := & \delta^-(W) \cup \delta^+(W). \end{array}$$

The arc set $\delta(W)$ is called a *cut*. To simplify notation we write $\delta^-(v), \delta^+(v)$, and $\delta(v)$, instead of $\delta^-(\{v\}), \delta^+(\{v\})$, and $\delta(\{v\})$, respectively. The numbers $|\delta^-(v)|, |\delta^+(v)|$, and $|\delta(v)|$ are called the *indegree*, *outdegree*, and *degree* of node $v \in V$. A node with degree zero is called *isolated*.

For notational convenience, a path P consisting of the arc set $\{(v_i, v_{i+1}) \mid i = 1, ..., k-1\}$ is sometimes denoted by $P = (v_1, v_2, ..., v_k)$. If not stated differently, the path P is always intended to be open and *simple*, i.e., $|P| = k - 1$ and $v_i \neq v_j$ for $i \neq j$. Moreover we let

$$[P] := \{(v_i, v_j) \in A \mid 1 \leq i < j \leq k\}$$

denote the *transitive closure* of $P = (v_1, ..., v_k)$.

Given a path $P = (v_1, ..., v_k)$, the *earliest starting time* t_{v_i} at node v_i $(i = 1, ..., k)$ along P is computed as

$$\begin{array}{lcl} t_{v_1} & := & r_{v_1} \\ t_{v_i} & := & \max\{t_{v_{i-1}} + \vartheta_{v_{i-1}v_i}, r_{v_i}\} \quad \text{for } i = 2, ..., k \end{array}$$

Notice that the formula introduces appropriate "waiting times" $w_{v_i} = \max\{0, r_{v_{i-1}} - (t_{v_{i-1}} + \vartheta_{v_{i-1}v_i})\}$ whenever a node $v_i, i = 2, ...k$, is reached before its release date. If no such waiting times are needed, i.e., $w_i = 0$ for all $i = 2, ...k$, the path is called *minimal*. We denote by $\vartheta(P) = t_{v_k}$ the earliest starting time at the last node of P. Notice that for minimal paths one has $\vartheta(P) = r_{v_1} + \sum_{i=1}^{k-1} \vartheta_{v_i v_{i+1}}$. To simplify notation, we sometimes write $\vartheta(v_1, v_2, ...v_k)$ instead of $\vartheta(P)$ for $P = (v_1, v_2, ...v_k)$.

A Hamiltonian path $P = (v_1, ..., v_n)$ is called *feasible* if each node is visited within its time window, i.e., $r_{v_i} \leq t_{v_i} \leq d_{v_i}$ for $i = 1, ..., n$. A path $P = (v_1, ..., v_k)$ where $2 \leq k \leq n$, is said to be *infeasible* if it does not occur as a subpath in any feasible Hamiltonian path. Deciding whether a given path P is feasible is clearly an $\mathcal{NP}$–complete problem, even when P contains only one node, as in this case it amounts to deciding whether a feasible Hamiltonian path exists. Easily checkable and obvious sufficient conditions for infeasibility are given in the following lemma.

Lemma (2.1).
A given path $P = (v_1, ..., v_k)$ is infeasible, if at least one of the following conditions holds:

- *(i) P violates the deadline for its last node v_k, i.e., $\vartheta(P) > d_{v_k}$.*
- *(ii) The triangle inequality (1.1) on ϑ is satisfied and there exists a node w not covered by P such that both paths $P_1 = (w, v_1, ..., v_k)$ and $P_2 = (v_1, ..., v_k, w)$ violate the given deadline on their last node, i.e., $\vartheta(P_1) > d_{v_k}$ and $\vartheta(P_2) > d_w$.*

In case condition (ii) above is satisfied, we say that node w cannot be *covered by (an extension of) path P*. If the triangle inequality on ϑ is not satisfied the condition can easily be generalized by considering the ϑ–shortest paths from w to v_1 and from v_k to w.

Time windows give rise to precedences among the nodes. For example, whenever the ϑ–shortest path from j to i is longer than $d_i - r_j$ we can conclude that i has to precede j in any feasible solution. Let $i \prec j$ denote the fact that i has to precede j in any feasible solution and let $G_P = (V, R)$ denote the *precedence graph* where each arc $(i, j) \in R$ represents a precedence relationship $i \prec j$. Without loss of generality we assume G_P to be acyclic and transitively closed. Moreover, let

$$\begin{aligned} \pi(v) &:= \{i \in V | (i, v) \in R\}, \\ \sigma(v) &:= \{j \in V | (v, j) \in R\}. \end{aligned}$$

represent the set of the *predecessors* and *successors* of a node $v \in V$, respectively.

We next describe the basic model we propose, written for the "open-path" version of ATSP-TW. A similar model can easily be defined for the "close tour" version of the problem. For each arc $(i, j) \in A$ we introduce a binary variable $x_{ij} \in \{0, 1\}$ with the interpretation:

$$x_{ij} = \begin{cases} 1, & \text{if } (i,j) \in A \text{ is in a feasible Hamiltonian path,} \\ 0, & \text{otherwise.} \end{cases}$$

For any $Q \subseteq A$ we write $x(Q)$ for $\sum_{(i,j) \in Q} x_{ij}$. The model reads

$$\begin{array}{llll} & \min \sum\limits_{(i,j)\in A} c_{ij} x_{ij} & & \\ \text{s.t.} & x(A) & = n - 1 & & (2.2) \\ & x(\delta^-(i)) & \le 1 & \forall\ i \in V & (2.3) \\ & x(\delta^+(i)) & \le 1 & \forall\ i \in V & (2.4) \\ & x(A(W)) & \le |W| - 1 & \forall\ W \subset V, |W| \ge 2 & (2.5) \\ & x(P) & \le |P| - 1 & \forall \text{ infeasible path } P & (2.6) \\ & x_{ij} & \in \{0, 1\} & \forall\ (i, j) \in A & (2.7) \end{array}$$

Constraints (2.5) forbid the occurrence of subtours, whereas the *infeasible path constraints* (2.6) avoid infeasible paths to be part of the solution. Observe that the model retains its validity even if infeasibility in (2.6) is checked just as in Lemma 2.1 (i). The formulation of the infeasible path constraints, as stated in (2.6) is very weak; in Section 4.1 we present possible strengthenings of these inequalities.

We want to remark here that objective functions such as minimizing the makespan or minimizing the waiting times cannot be incorporated easily in this model. On the other hand, note that no "big M"–term is required in the model. Moreover, only arc variables are present, which implies that no linking constraints between arc and node variables are necessary. This suggests that (a strengthening of) this model can be superior to alternative ones, at least for some problem classes. The computational results presented in a companion paper [5] confirm this expectation.

3 Polyhedral analysis

In this section we aim at analyzing the dimension of the *ATSP-TW polytope*

$$P_{TW} := conv\{x \in \mathbb{R}^A \mid x \text{ satisfies } (2.2)\text{–}(2.7)\},$$

defined as the convex hull of the characteristic vectors of all feasible Hamiltonian paths on the complete digraph $G = (V, A)$. Recall that it is a strongly $\mathcal{NP}$–complete problem to find a

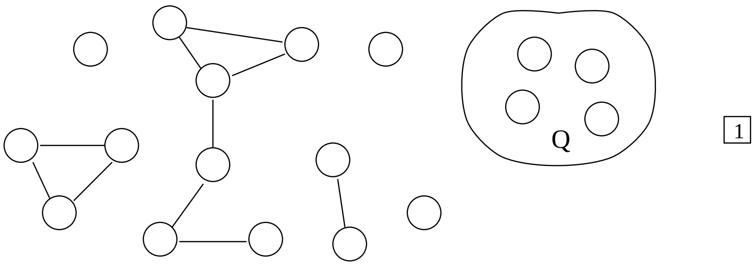

Figure 3.1 The feasibility graph G_F (with node 1 added)

feasible ATSP-TW solution, i.e., to decide (for the general case) whether P_{TW} is nonempty. We will therefore restrict ourselves to study "simpler" special cases, in particular the one where only one time window is active. Rather unexpectedly, even in this oversimplified case the polyhedral analysis is quite difficult, in that P_{TW} lies on a number of hyperplanes whose defining equations have no counterpart in the pure ATSP. Moreover, the solution of a min–cost Hamiltonian path problem is required to compute the dimension of P_{TW}, i.e., determining this dimension is a strongly $\mathcal{NP}$–complete problem even when all time windows but one are relaxed.

In the remaining part of this section we consider the case in which only the time window $[r_1, d_1]$ associated with node 1 (say) is active. All other time windows are relaxed to $[0, +\infty)$. Furthermore, we assume that the triangle inequality (1.1) on ϑ is satisfied.

Partition the node set as follows :

$$V = \{1\} \cup Q \cup W$$

where

$$Q := \{j \in V \setminus \{1\} \mid \vartheta(j,1) = r_j + \vartheta_{j1} > d_1\}$$

contains the nodes that cannot be sequenced before node 1 without violating the time window for node 1, and $W := V \setminus (\{1\} \cup Q)$. We define an undirected *feasibility graph* $G_F = (V \setminus \{1\}, E)$, where

$$E := \{ij \mid i,j \in V \setminus \{1\}, i \neq j, \min\{\vartheta(i,j,1), \vartheta(j,i,1)\} \leq d_1\}$$

contains all node pairs i, j such that either $(i,j,1)$ or $(j,i,1)$, or both, are feasible paths. Note that the nodes of Q (possibly among others) are isolated in G_F, i.e., they are "incompatible" with all other nodes in $j \in V \setminus \{1\}$, see Figure 3.1 for an illustration.

Consider the following equations:

$$\begin{aligned}
x_{j1} &= 0, \quad \forall\, j \in Q && (3.1)\\
x(\delta^+(1)) &= 1, \quad \text{if node 1 cannot be the final node} \\
&\qquad\quad \text{of a feasible Hamiltonian path} && (3.2)\\
\sum_{j \in S_h} x(\delta^-(j)) &= |S_h| - x(S_h : 1), \quad h = 1, ..., m && (3.3)
\end{aligned}$$

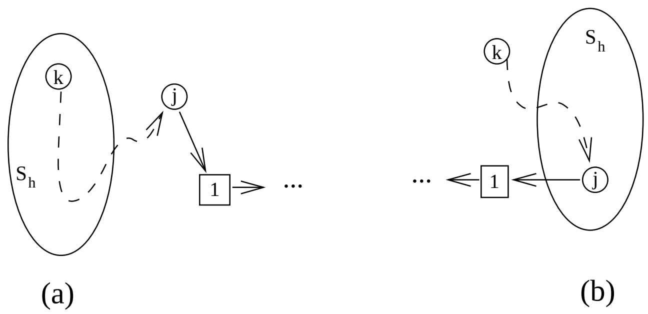

Figure 3.2 Illustration for the proof of Lemma 3.4

where $S_1, ..., S_m$ are the m (say) connected components of G_F. Note that adding up the m equations (3.3) leads to

$$\sum_{h=1}^{m}(\sum_{j\in S_h} x(\delta^-(j))) = \sum_{h=1}^{m}|S_h| - \sum_{h=1}^{m} x(S_h:1),$$

which can be rewritten as $x(A) = n-1$, i.e., equation (2.2) is a linear combination of (3.3). Furthermore, note that checking the condition in (3.2) requires the solution of a min–cost Hamiltonian path problem.

Lemma (3.4).
Equations (3.1)–(3.3) are valid for P_{TW}.

Proof. Validity is obvious for equations (3.1) and (3.2).
As to equations (3.3), take any connected component S_h of G_F and note that no edge in G_F crosses the cut induced by S_h.
Consider the characteristic vector x of any feasible Hamiltonian path, and let k be its starting node, i.e., $x(\delta^-(k)) = 0$. The left–hand side of (3.3) reads

$$\sum_{j\in S_h} x(\delta^-(j)) = \begin{cases} |S_h|-1, & \text{if } k \in S_h, \\ |S_h|, & \text{otherwise.} \end{cases}$$

Hence (3.3) is violated if and only if one of the following two cases occur (recall that $1 \notin S_h$ as node 1 is not part of G_F).

(i) $k \in S_h$ and $x(S_h : 1) = 0$, see Figure 3.2 (a): Let j be such that $x_{j1} = 1$ (note that $k \in S_h$ implies $k \neq 1$, i.e., node 1 has indeed a predecessor j in the path); since $x(S_h : 1) = 0$ we must have $j \notin S_h$, which implies that k and j are "compatible", i.e., $kj \in E$, a contradiction.

(ii) $k \notin S_h$ and $x(S_h : 1) = 1$, see Figure 3.2 (b): Let $j \in S_h$ be such that $x_{j1} = 1$ and observe that $kj \in E$, again a contradiction.

□

It is important to observe here that equations (3.1)–(3.3) are valid for the case of general time windows as well, node 1 being any node with active time window.

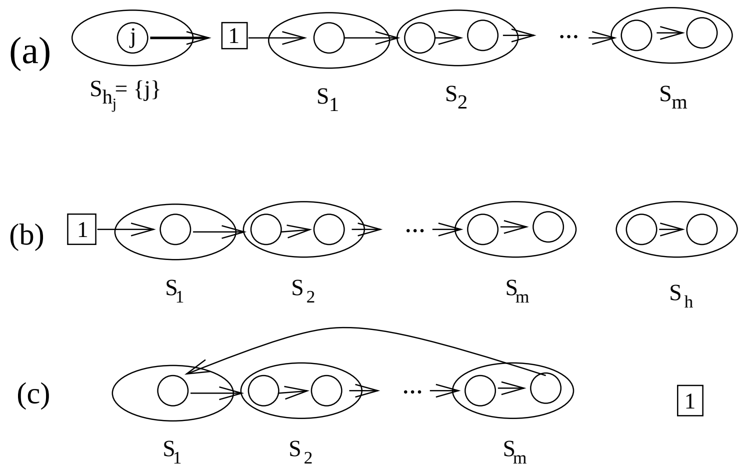

Figure 3.3 Constructions for the proof of Lemma 3.5

Lemma (3.5).
Equations (3.1)–(3.3) are linearly independent.

Proof. It is sufficient to exhibit for each equation $\alpha x = \alpha_0$ of the family (3.1)–(3.3) a point $x \in \mathbb{R}^A$ satisfying all the equations in the family except $\alpha x = \alpha_0$.

For each equation (3.1) let x to be the characteristic vector of any path of the form $(j, 1, \phi[S_1], ..., \phi[S_{h_j-1}], \phi[S_{h_j+1}], ..., \phi[S_m])$, where $S_{h_j} = \{j\}$ is the singleton component of G_F containing node j and $\phi[S_i]$ denotes any permutation of the nodes in S_i; see Figure 3.3(a).

For each equation (3.3) let x to be the characteristic vector of the two paths $(1, \phi[S_1], ..., \phi[S_{h-1}], \phi[S_{h+1}], ..., \phi[S_m])$ and $(\phi[S_h])$; see Figure 3.3(b).

As to equation (3.2) let x be the characteristic vector of any cycle on $n-1$ nodes of the form $(\phi[S_1], ..., \phi[S_m])$; see Figure 3.3(c). □

We now show that (3.1)–(3.3) define a minimal equation system for the polytope P_{TW}, i.e., that no other independent valid equation exists. Let

$$\mu = \begin{cases} 1, & \text{if the condition in (3.2) is satisfied, i.e., node 1 cannot} \\ & \text{be the final node of a feasible Hamiltonian path,} \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (3.6).
Consider any ATSP-TW instance defined on a complete digraph $G = (V, A)$ with $n \geq 4$ nodes. If only the time window for one node is active, then

$$dim(P_{TW}) = |A| - (|Q| + m + \mu).$$

Proof. As before, we assume w.l.o.g. that only the time window of node 1 is active. Lemmas (3.4) and (3.5) imply that $dim(P_{TW}) \leq |A| - (|Q| + m + \mu)$. We give a direct proof that this bound is tight, consisting of exhibiting $|A| - (|Q| + m + \mu) + 1$ affinely independent vertices of P_{TW}.

Consider first the face F of P_{TW} induced by $x(\delta^-(1)) \geq 0$, containing all feasible Hamiltonian paths starting with node 1. As we are assuming $d_j = +\infty$ for all $j \neq 1$, every Hamiltonian path starting with node 1 is indeed feasible. It is then easy to see that there is a 1-1 correspondence between the vertices of F and the Hamiltonian tours of G; in particular, $dim(F)$ equals the dimension of the "closed-tour" ATSP polytope on G. Hence $dim(F) = |A| - 2n + 1$, and a minimal equation system for F is given by

$$
\begin{array}{llll}
(i) & x_{j1} & = 0 & \forall\, j \in V \setminus \{1\}, \\
(ii) & x(\delta^-(j)) & = 1 - x_{j1} & \forall\, j \in V \setminus \{1\}, \\
(iii) & x(\delta^+(1)) & = 1. &
\end{array}
$$

(Note that, because of (i), the right–hand side in (ii) is in fact 1.) Therefore, there exist $dim(F) + 1 = |A| - 2n + 2$ affinely independent vertices of this face. To prove our claim we need $2n - 1 - |Q| - m - \mu$ additional affinely independent points, which we define in the following way:

A. For each $j \in (V \setminus \{1\}) \setminus Q$, we define the point $x^j \in P_{TW}$ associated with any feasible Hamiltonian path of the form $(j, 1, ...)$. The affine independence of each such point follows from equations *(i)*, in that $x^j_{j1} = 1$ whereas $x_{j1} = 0$ for all other points defined previously. This construction produces $n - 1 - |Q|$ new points.

B. For each $h = 1, 2, ..., m$, in turn, consider the component S_h and let $T_h \subseteq E$ be any tree spanning S_h. Choose any root node $\rho \in S_h$ and give an orientation to the edges in T_h so as to obtain a directed tree (arborescence) rooted at ρ. Then scan the nodes $v \in S_h \setminus \{\rho\}$ in any sequence visiting each node after its father node in the arborescence, and let f_v be the father node of v in the arborescence. Since $f_v v \in E$, there exists a feasible Hamiltonian path of the form $(v, f_v, 1, ...)$ or $(f_v, v, 1, ...)$, whose characteristic vector y^v satisfies all the equations *(ii)* except those with $j = v$ and $j = f_v$. Because of the particular sequence in visiting the nodes, all the points constructed before the current y^v satisfy the equation *(ii)* written for $j = v$, hence each new point is affinely independent from the previous ones. The above construction produces $|S_h| - 1$ new points y^v for each component S_h, i.e., $n - 1 - m$ new points in total.

C. If $\mu = 1$, we are done; otherwise, the last point to be constructed is associated with any feasible Hamiltonian path of the form $(..., 1)$. The affine independence of this point follows from the fact that equation *(iii)* is satisfied by all the previous points in which node 1 always appears in position 1, 2, or 3 (recall $n \geq 4$).

□

As already mentioned, an implication of Theorem 3.6 is that determining the dimension of P_{TW} is a difficult problem even when all time windows except one are relaxed. To be more specific, consider the following decision problem:

PROBLEM "DIMENSION":

INSTANCE:	Any ATSP-TW instance defined on a complete digraph $G = (V, A)$ with integer triangular arc length $\vartheta_{ij} \geq 0$.
QUESTION:	$dim(P_{TW}) = \|A\| - 1$?

Theorem (3.7).
Problem DIMENSION is strongly $\mathcal{NP}$–complete even if all time windows but one are relaxed.

Proof. Because of the equation $x(A) = n - 1$ we know that $dim(P_{TW}) \leq |A| - 1$, hence the answer "yes" can be certified concisely by exhibiting $|A|$ affinely independent vertices of P_{TW}. This shows that DIMENSION belongs to the class $\mathcal{NP}$. We next show that the problem is indeed strongly $\mathcal{NP}$–complete through a reduction from the following well-known strongly $\mathcal{NP}$–complete problem, HP:

> Given a complete digraph $\tilde{G} = (\tilde{V}, \tilde{A})$ with integer triangular arc lengths $\tilde{\vartheta}_{ij} \geq 0$ and an integer bound L, does $\tilde{G}$ contain a Hamiltonian path $\tilde{P}$ with $\sum_{(i,j)\in\tilde{P}} \tilde{\vartheta}_{ij} \leq L$?

Note that it is not restrictive to assume that $min\{\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ji}\} \leq L$ holds for all $i, j \in \tilde{V}, i \neq j$, since otherwise the answer to problem HP is trivially "no".

Given any instance of HP one can define, in polynomial time, a new complete digraph $G = (V, A)$ with node set $V := \tilde{V} \cup \{1\}$, and set up the following ATSP-TW instance on G:

- $p_j := 0, r_j := 0$, for all $j \in V$
- $d_j := \begin{cases} +\infty, & \text{for } j \in \tilde{V}, \\ L, & \text{for } j = 1 \end{cases}$
- $t_{ij} := \begin{cases} \tilde{\vartheta}_{ij}, & \text{for all } (i,j) \in \tilde{A}, \\ 0, & \text{for all } (i,j) \in \delta^-(1) \\ max\{\tilde{\vartheta}_{ij} : (i,j) \in \tilde{A}\}, & \text{for all } (i,j) \in \delta^+(1). \end{cases}$

This instance has just one active time window, namely the one associated with node 1, and the values $\vartheta_{ij} := p_i + t_{ij} (= t_{ij})$ satisfy the triangle inequality, as required. In addition, the feasibility graph G_F is complete since $min\{\vartheta(i,j,1), \vartheta(j,i,1)\} = min\{\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ji}\} \leq L = d_1$ holds for all $i, j \in \tilde{V}, i \neq j$. It then follows from Theorem 3.6 that $dim(P_{TW}) = |A| - 1 - \mu$, hence problem DIMENSION has an affirmative answer if and only if $\mu = 0$, a condition equivalent to the existence of a Hamiltonian path $\tilde{P}$ in $\tilde{G}$ with $\sum_{(i,j)\in\tilde{P}} \tilde{\vartheta}_{ij} \leq L$. This proves the claim. □

4 Classes of valid inequalities

In this section we summarize known classes of inequalities valid for P_{TW}, present strengthenings of these inequalities, and state new classes of valid inequalities.

In the last section we have seen that it is a difficult problem to establish the dimension of P_{TW} even for the simple case where only one time window is active. Another problem for the polyhedral study is that, even for fixed n and fixed time windows, little changes in the set-up times t_{ij} may result in dramatical changes of the polytope structure, e.g., increasing just one such coefficient may make many feasible paths or even the whole instance infeasible. Therefore, proving that certain classes of inequalities are facet defining for P_{TW} appears a very difficult task; hence we only prove the validity of the constraints we propose. With the help of a computer program [16] we have verified for small instances with $n \leq 5$ that these inequalities are indeed facet defining for most instances.

Not many classes of valid inequalities for the time–constrained ATSP can be found in the literature. If we ignore the setup times t_{ij}, ATSP-TW is related to *single machine scheduling problems*, which require sequencing a set of jobs on a single machine with release times and deadlines. Balas [8] introduced inequalities for this class of problems, that can be incorporated in the "standard" ATSP-TW models in which node variables are used. Applegate and Cook [2] performed computational experiments with several of these classes for the job shop scheduling problem in which however a different objective function (minimization of the makespan) is considered. We refer to Queyranne and Schulz [31] for a comprehensive survey on polyhedral approaches to machine scheduling problems.

It can be seen easily that all classes of inequalities that are valid for the unconstrained ATSP–polytope (the convex hull of Hamiltonian tours in G) have a version valid for P_{TW}, obtained by possibly modifying the inequality to take care of the "open path" formulation we consider. The ATSP–polytope has been studied extensively, among others, by Grötschel [27], Grötschel and Padberg [28], Balas [9], Fischetti [22, 23, 24], Balas and Fischetti [10, 11], Chopra and Rinaldi [15], and Queyranne and Wang [32]. Similarly, all classes of valid inequalities derived for the precedence–constrained ATSP (cf. Balas et al. [12]) are valid for P_{TW}, by considering the precedence relationships implied by the time windows. For some of these inequalities we will present strengthened versions taking into account the time windows explicitely.

4.1 Infeasible path constraints

These constraints express the fact that certain paths are infeasible, i.e., they violate a time window constraint. For a given infeasible path $P = (v_1, ..., v_k)$ the basic version of these inequalities is given by $x(P) \le |P| - 1$. There exist however several possibilities to strengthen these inequalities. We next discuss in detail some of these lifted inequalities.

4.1.1 Tournament constraints

Lemma (4.1).
For all infeasible simple paths $P = (v_1, ..., v_k)$ the tournament constraint

$$x([P]) := \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x_{v_i v_j} \le k - 2 \quad (= |P| - 1) \tag{4.2}$$

is valid for P_{TW}; see Figure 4.4 for an illustration.

Proof. Because of the degree inequalities $x(\delta^-(j)) \le 1$, condition $\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x_{v_i v_j} > k - 2$ would imply $\sum_{j=i+1}^{k} x_{v_i v_j} = 1$ for all $i = 1, ..., k-1$. This in turn implies $x_{v_{k-1} v_k} = 1$, $x_{v_{k-2} v_{k-1}} + x_{v_{k-2} v_k} = 1$ (i.e., $x_{v_{k-2} v_{k-1}} = 1$), etc. But then one would have $x_{ij} = 1$ for all $(i, j) \in P$, impossible because of the infeasibility of P. □

Notice that the above proof also shows that, for any simple path P, $x([P]) = |P|$ iff $x_{ij} = 1$ for all $(i, j) \in P$.

The validity of a tournament constraint only depends on the infeasibility of a single path P. In the case when other infeasible paths through the nodes $v_1, ..., v_k$ exist, the inequality can be further lifted in several ways. For example, in the tournament constraint represented in Figure 4.4, the coefficient of variable $x_{v_3 v_1}$ can be lifted to 1 if both paths (v_2, v_3, v_1, v_4) and (v_3, v_1, v_2, v_4) happen to be infeasible.

Given a node set $W \subseteq V$, let $\Phi[W]$ denote a generic permutation of the nodes in W.

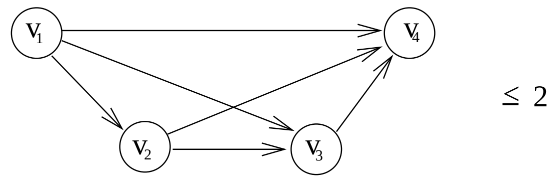

Figure 4.4 The support graph of a tournament constraint on $k = 4$ nodes

Theorem (4.3).
For each node set $Q = \{v_1, ..., v_{k-1}\} \subset V$ and each node $v_k \in V \setminus Q$ such that all paths of the form $(\Phi[Q], v_k)$ are infeasible, the inequality

$$x(A(Q)) + x(Q : v_k) \leq k - 2 \quad (= |Q| - 1) \tag{4.4}$$

is valid for P_{TW}.

Proof. Let x be any vertex of P_{TW}, and let $G_x = (V, \{(i, j) \in A : x_{ij} = 1\})$ be its support graph. If vertex x violates (4.4), then $x(A(Q)) = |Q| - 1$ and $x(Q : v_k) = 1$, hence G_x contains a path of the form $(\Phi[Q], v_k)$, impossible since any such path is infeasible by assumption. □

Theorem (4.5).
For each node set $S = \{v_2, ..., v_{k-1}\} \subset V$ and any two nodes $v_1, v_k \in V \setminus S, v_1 \neq v_k$, such that all paths of the form $(v_1, \Phi[S], v_k)$ are infeasible, the inequality

$$x(v_1 : S) + x(A(S)) + x(S : v_k) + x_{v_1 v_k} \leq k - 2 \quad (= |S|) \tag{4.6}$$

is valid for P_{TW}.

Proof. Analogous to that of the previous theorem: if x violates (4.6), then $x(v_1 : S) = x(S : v_k) = 1$ and $x(A(S)) = |S| - 1$, hence G_x would contain an infeasible path of the form $(v_1, \Phi[S], v_k)$. □

Note that inequality (4.4) is a strengthening of the subtour elimination constraint $x(A(Q)) \leq |Q| - 1$. Moreover both inequalities (4.4) and (4.6) are a strengthening of the tournament constraints (4.2) associated with all the infeasible paths of the form $(\Phi[Q], v_k)$ and $(v_1, \Phi[S], v_k)$, respectively.

It is not easy to decide whether all the paths of the form $(\Phi[Q], v_k)$ and $(v_1, \Phi[S], v_k)$ are infeasible, as required for the validity of inequalities (4.4) and (4.6). Easily checkable sufficient conditions are given by the next lemma; tighter conditions can be derived in a similar way.

Lemma (4.7).

(a) Take any $Q \subset V$ and $v_k \in V \setminus Q$. If

$$\min_{v_i \in Q}\{r_{v_i}\} + \sum_{v_i \in Q} \min\{\vartheta_{v_i v_j} \mid v_j \in Q \cup \{v_k\}\} > d_k$$

then all the paths of the form $(\Phi[Q], v_k)$ are infeasible.

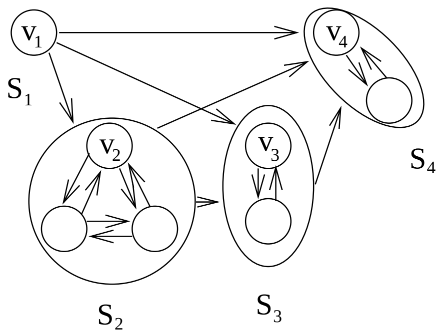

Figure 4.5 A generalized tournament constraint

(b) Take any $S \subset V$ and $v_1, v_k \in V \setminus S, v_1 \neq v_k$. If

$$r_{v_1} + \min\{\vartheta_{v_1 v_j} \mid v_j \in S\} + \sum_{v_i \in S} \min\{\vartheta_{v_i v_j} \mid v_j \in S \cup \{v_k\}\} \quad > \quad d_k$$

then all the paths of the form $(v_1, \Phi[S], v_k)$ are infeasible.

Proof. Obvious from the definitions. □

4.1.2 Generalized tournament constraints

The tournament constraint (4.2) can in some cases be generalized by using a clique lifting technique akin to that described in Balas and Fischetti [11, 12], in which each node v_i is replaced by a clique S_i of "clones". We obtain the following result.

Theorem (4.8).
Let $S_1, ..., S_k$ be $k \geq 2$ disjoint node sets, and assume that any path of the form $(\Phi[S_1], ..., \Phi[S_k])$ is infeasible. Then the inequality

$$\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x(S_i : S_j) + \sum_{i=1}^{k} x(A(S_i)) \leq k - 2 + \sum_{i=1}^{k} (|S_i| - 1) = \sum_{i=1}^{k} |S_i| - 2$$

is valid for P_{TW}.

Proof. The inequality can only be violated by a point $x \in P_{TW}$ if $x(A(S_i)) = |S_i| - 1$ for $i = 1, ..., k$, and $\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} x(S_i : S_j) = k - 1$, where this latter condition implies $x(S_1 : S_2) = ... = x(S_{k-1} : S_k) = 1$. But this would imply that the support graph of x contains an infeasible path of the form $(\phi[S_1], ..., \Phi[S_k])$, a contradiction. □

Figure 4.5 gives an illustration of a generalized tournament constraint based on infeasible paths on four cliques.

A simple case in which the assumption of Theorem 4.8 is satisfied arises when the triangle inequality (1.1) on ϑ holds, and there exists $v_i \in S_i$ $(i = 1, \ldots, k)$ such that the path $P =$

$(v_1, ..., v_k)$ is infeasible because of condition *(i)* in Lemma 2.1, or because of condition *(ii)* in the same lemma (provided that $w \notin \cup_{i=1}^{k} S_i$).

A different generalization of tournament constraints can be obtained along the following lines. Suppose we are given a family $\mathcal{P} = \{P_1, P_2, ..., P_k\}$ of node disjoint simple paths, and let ω be any permutation of the indices of $\mathcal{P}$. The path $P = (P_{\omega(1)}, P_{\omega(2)}, ..., P_{\omega(k)})$ is called a *concatenation* of the paths in $\mathcal{P}$. Now it may happen that the paths $P_1, P_2, ..., P_k$ are feasible in themselves but that there is no way to connect them in a feasible way. This observation leads to the following result.

Theorem (4.9).
Let $\mathcal{P} = \{P_1, P_2, ..., P_k\}$ be a family of node disjoint simple paths and assume that the triangle inequality (1.1) on ϑ is satisfied. If $\vartheta(P) > d_w$ for any concatenation $P = (...., w)$ of the paths in $\mathcal{P}$, then the inequality

$$\sum_{i=1}^{k} x([P_i]) \leq \sum_{i=1}^{k} |P_i| - 1$$

is valid for P_{TW}.

Proof. In order to violate the inequality a feasible solution $x \in P_{TW}$ must satisfy $x([P_i]) = |P_i|$ for all $i = 1, ..., k$. As observed in the proof of Theorem 4.1, this implies $x_{uv} = 1$ for all $(u, v) \in P_i$ and $i = 1, ..., k$, a contradiction. □

4.1.3 Other lifted path inequalities

Given an infeasible path $P = (v_1, ..., v_k)$, one possible way of strengthening the basic infeasible path constraint $x(P) \leq |P| - 1$ into a tournament constraint $x([P]) \leq |P| - 1$ has been already presented. There are however several other ways to lift the basic infeasible path constraint, whose validity is only based on the assumption of the infeasibility of path $P = (v_1, ..., v_k)$.

Theorem (4.10).
If $P = (v_1, v_2, ..., v_k)$ is an infeasible path, then the following inequalities are valid for P_{TW}:

$$\begin{array}{llll}
(a) & x(P) + \sum_{j=1}^{k-2} x_{v_j v_k} + \sum_{j=2}^{k-2} \sum_{l=1}^{j-1} x_{v_j v_l} & \leq k-2 \\
(b) & x(P) + \sum_{j=3}^{k} x_{v_1 v_j} + \sum_{j=3}^{k-1} \sum_{l=j+1}^{k} x_{v_l v_j} & \leq k-2 \\
(c) & x(P) + \sum_{j=2}^{k-1} \sum_{l=1}^{j-1} x_{v_j v_l} & \leq k-2 \\
(d) & x(P) + \sum_{j=2}^{k-1} \sum_{l=j+1}^{k} x_{v_l v_j} & \leq k-2.
\end{array}$$

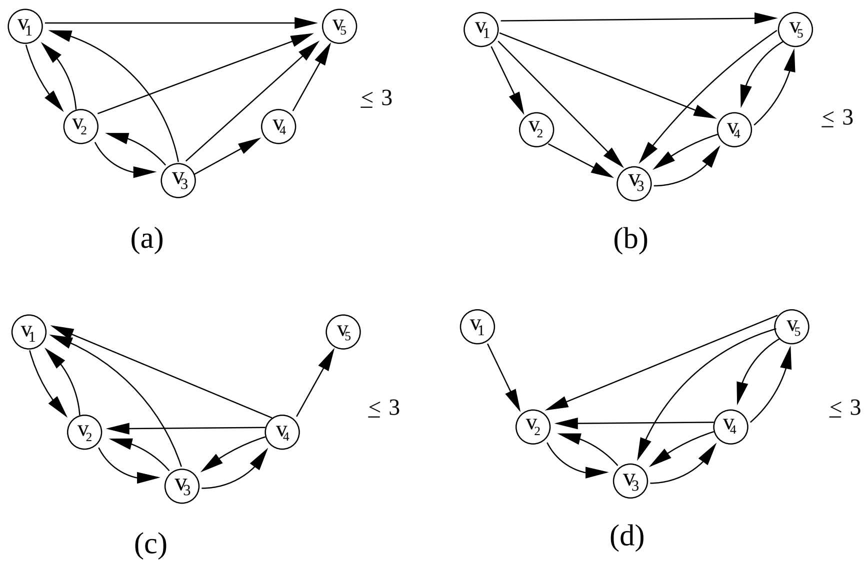

Figure 4.6 Lifted infeasible path constraints on $k = 5$ nodes

Proof.
Let $ax \leq k-2$ be any of the inequalities *(a)–(d)* above. Clearly, for any $x \in P_{TW}$ we have $ax \leq x(A(\{v_1, ..., v_k\}) \leq k-1$. We claim that $ax = k-1$ implies $x_{ij} = 1$ for all $(i,j) \in P$, from which the thesis will follow immediately. Indeed, let $x \in P_{TW}$ be any point such that $x_{v_h v_{h+1}} = 0$ for an arbitrary $h \in \{1, ..., k-1\}$. We have to show that $ax \leq k-2$ holds for all cases *(a)* to *(d)* above.

(a) If $h = k-1$, then $ax \leq \sum_{i=1}^{k-2} x(\delta^+(v_i)) \leq k-2$. Otherwise $ax \leq x(A(\{v_1, ..., v_h\}) + \sum_{i=h+1}^{k-2} x(\delta^+(v_i)) + x(\delta^-(v_k)) \leq (h-1) + (k-2-h) + 1 = k-2$;

(b) If $h = 1$, then $ax \leq \sum_{i=3}^{k} x(\delta^-(v_i)) \leq k-2$. Otherwise $ax \leq x(A(\{v_{h+1}, ..., v_k\}) + \sum_{i=3}^{h} x(\delta^-(v_i)) + x(\delta^+(v_1)) \leq (k-h-1) + (h-2) + 1 = k-2$;

(c) We have $ax \leq x(A(\{v_1, ..., v_h\})) + \sum_{i=h+1}^{k-1} x(\delta^+(v_i)) \leq (h-1) + (k-1-h) = k-2$.

(d) We have $ax \leq x(A(\{v_{h+1}, ..., v_k\})) + \sum_{i=2}^{h} x(\delta^-(v_i)) \leq (k-h-1) + (h-1) = k-2$.

□

Figure 4.6 gives examples of the inequalities *(a)–(d)* for the infeasible path $P = (v_1, ..., v_5)$. As in the case of tournament constraints these inequalities can further be lifted when other infeasible paths through the nodes of P exist; see Ascheuer [3] for details.

4.2 A lifting procedure

In this section we describe a lifting procedure, called *V–lifting*, that can be used to construct new families of valid infeasible path inequalities for the ATSP-TW.

Suppose we are given two valid inequalities with integer coefficients, say $\alpha x \leq \alpha_0$ and $\beta x \leq \beta_0$, such that $\beta_0 = \alpha_0 + 1$ and $\beta_{ij} \geq \alpha_{ij}$ for all $(i,j) \in A$. Furthermore, assume that

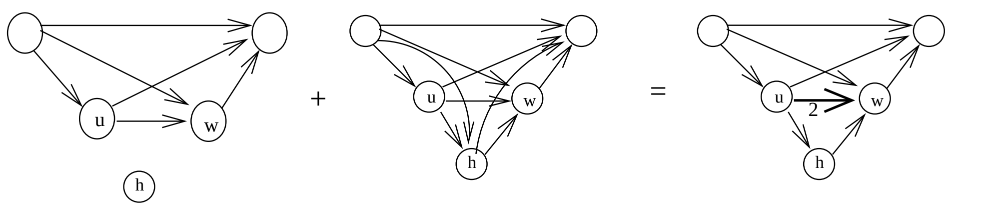

Figure 4.7 V-lifted tournament constraint

there exist 3 distinct nodes, say u, w and h, such that $\beta_{uh} \geq \alpha_{uh}+1$ and $\beta_{hw} \geq \alpha_{hw}+1$. By adding up and then rounding the following valid inequalities weighted by $\frac{1}{2}$

$$\begin{array}{lll} \alpha x & \leq & \alpha_0 \\ \beta x & \leq & \beta_0 = \alpha_0 + 1 \\ x_{uh} + x_{hw} + 2x_{uw} & \leq & 2 \end{array}$$

one obtains the new valid inequality

$$\alpha x + x_{uh} + x_{hw} + x_{uw} \leq \alpha_0 + \lfloor \frac{3}{2} \rfloor = \alpha_0 + 1$$

in which the α–coefficient of the 3 arcs $(u,h),(h,w)$ and (u,w) is increased by 1 at the "expense" of an increase of 1 of the right–hand side α_0.

Notice that the correctness of the construction only depends on the validity of the inequality $x_{uh} + x_{hw} + 2x_{uw} \leq 2$ (which follows from the degree inequalities), hence it holds for the unconstrained ATSP polytope as well.

V-lifting can easily be applied to tournament constraints, as illustrated in Figure 4.7. To this end, let $\alpha x := x([P]) \leq \alpha_0 := |P| - 1$ be the tournament constraint associated with any infeasible path $P = (v_1, ..., v_k)$. Take a node $h \notin \{v_1, ..., v_k\}$ and assume that the path $P' = (v_1, ..., v_i, h, v_{i+1}, ..., v_k)$ is infeasible for a certain index i (when ϑ satisfies the triangle inequality, this is likely to be the case for any choice of h and i). Then let $\beta x \leq \beta_0$ be the tournament inequality $x([P']) \leq |P'| - 1$ associated with P', define $u := v_i$ and $w := v_{i+1}$, and obtain through V-lifting the new inequality

$$x([P]) + x_{v_i h} + x_{h v_{i+1}} + x_{v_i v_{i+1}} \leq |P|$$

in which the coefficient of arc (v_i, v_{i+1}) is raised to 2. By iterating the procedure one can obtain V-lifted tournament inequalities with several coefficients 2.

4.3 Strengthened predecessor/successor–inequalities

In Balas et al. [12] the so-called (π, σ)–inequalities (for predecessor–successor inequalities) have been presented for the precedence–constrained ATSP. These inequalities can be strengthened for ATSP-TW by taking time-windows into account explicitly.

The (π, σ)–inequalities can be described as follows. We are given two disjoint node sets $X, Y \subset V$ such that $i \prec j$ for all pairs $i \in X, j \in Y$. Furthermore, we are given a node set

$S \subset V$ such that X is contained in S and Y in its complement $\overline{S} := V \setminus S$. In order not to violate the precedence relationships among the nodes in X and Y, a feasible path cannot cross the cut $(S : \overline{S})$ only through arcs incident with $W := \pi(X) \cup \sigma(Y)$; see Section 2 for the definition of $\pi(\cdot)$ and $\sigma(\cdot)$. This observation leads to the (π, σ)*–inequality*

$$x(S \setminus W : \overline{S} \setminus W) \geq 1. \tag{4.11}$$

Due to the time window restrictions, some paths from $S \setminus W$ to $\overline{S} \setminus W$ might be infeasible, which may result into a reduction of some left–hand side coefficients.

Theorem (4.12).
Let X and Y be two disjoint node sets such that $i \prec j$ for all $i \in X$ and $j \in Y$, and define $W := \pi(X) \cup \sigma(Y)$. Assume that the triangle inequality (1.1) on ϑ is satisfied and define

$$\tilde{W} := W \cup \{k \in V \setminus (X \cup Y) : \exists i \in X \text{ and } j \in Y \text{ s.t. } \vartheta(i,k,j) > d_j\}$$

and

$$Q := \{(u,v) \in \delta^{+}(S) \mid \exists i \in X \text{ and } j \in Y \text{ s.t. } \vartheta(i,u,v,j) > d_j\}.$$

Then for all $S \subset V$ such that $X \subseteq S$ and $Y \subseteq \overline{S}$ the inequality

$$x((S \setminus \tilde{W} : \overline{S} \setminus \tilde{W}) \setminus Q) \geq 1$$

is valid for P_{TW}.

Proof. For any feasible Hamiltonian path, the subpath P from the node of X visited last, say node i^*, to the node of Y visited first, say node j^*, cannot traverse any node of W without violating a precedence relationship. Furthermore, P cannot traverse any node $k \in \tilde{W} \setminus W$ since otherwise it would contain an infeasible path of the form $(i, ..., i^*, ..., k, ..., j^*, ..., j)$, where $i \in X$ and $j \in Y$ are the two nodes whose existence is required in the definition of $\tilde{W}$. Finally, P cannot use any arc $(u,v) \in Q$, since all paths of the form $(i, ..., i^*, ..., u, v, ..., j^*, ..., j)$, are infeasible. Thus, an arc in $(S \setminus \tilde{W} : \overline{S} \setminus \tilde{W}) \setminus Q$ has to be used to leave S, from which the validity of the inequality follows. $\square$

5 Conclusions

The asymmetric travelling salesman problem with time windows (ATSP-TW) is a very important basic model for scheduling and routing applications. We have studied the polyhedral structure of a possible formulation of the problem, in which only 0–1 arc variables are considered. This has the advantage of avoiding additional variables and the associated (typically very ineffective) linking constraints. A drawback is that the model cannot accommodate objective functions depending on makespan or waiting times easily.

Time windows are modelled by means of infeasible path elimination constraints which forbid the occurrence of paths leading to a violation of some deadlines. Similar constraints can also be used to model any other kind of path infeasibility. We have described the basic form of these constraints, and we have introduced some possible strengthenings. Several other classes of cuts have been described as well, derived from related ATSP problems.

We have also studied the ATSP-TW polytope, P_{TW}, whose vertices are the characteristic vectors of the Hamiltonian paths satisfying the time-window restrictions. To our knowledge, this polytope has never been studied before. Even on a complete graph with triangular set-up times, finding a feasible ATSP-TW solution is a difficult problem. Hence deciding whether P_{TW} is nonempty is an $\mathcal{NP}$–complete problem in the general case. We have then studied a very simple special case of the problem, in which only one of the time windows is active. Rather unexpectedly, even under these assumptions the determination of the polytope dimension is far from trivial, in that the polytope lies in a family of hyperplanes having no counterpart in the pure ATSP-case. Moreover, we have shown that the exact determination of the polytope dimension is a difficult problem as it requires the solution of a min-cost Hamiltonian path problem. As a consequence, a deep analysis of the facial structure of P_{TW} appears a very difficult task.

In the companion paper [5] the polyhedral analysis herein presented is used to design a branch–and–cut algorithm for the exact solution of ATSP-TW, whose performance is evaluated computationally on several real-world test problems. An outcome is that the new approach outperforms some previous LP-based methods on loosely–constrained classes of problem instances.

Acknowledgment

We are grateful for the suggestions Andreas Schulz made on this topic.

References

[1] A. Abdel-Hamid, N. Ascheuer, and M. Grötschel. Order picking in an automatic warehouse: Solving on–line asymmetric TSP's. Technical report, Konrad–Zuse–Zentrum für Informationstechnik, Berlin, 1997. (In preparation).

[2] D. Applegate and W. Cook. A computational study of the job–shop scheduling problem. *ORSA Journal on Computing*, 3:149–156, 1991.

[3] N. Ascheuer. *Hamiltonian Path Problems in the On-line Optimization of Flexible Manufacturing Systems*. PhD thesis, Technische Universität Berlin, Germany, 1995. Available as Technical Report TR 96-03, Konrad-Zuse-Zentrum für Informationstechnik Berlin, at URL: http://www.zib.de/ZIBbib/Publications/.

[4] N. Ascheuer, L. Escudero, M. Grötschel, and M. Stoer. A cutting plane approach to the sequential ordering problem (with applications to job scheduling in manufacturing). *SIAM Journal on Optimization*, 3:25–42, 1993.

[5] N. Ascheuer, M. Fischetti, and M. Grötschel. Solving ATSP with time windows by branch-and-cut. Technical report, ZIB Berlin, 1997. (In preparation).

[6] N. Ascheuer, M. Jünger, and G. Reinelt. A branch & cut algorithm for the asymmetric Hamiltonian path problem with precedence constraints. Technical report, ZIB Berlin, 1997. (In preparation).

[7] E.K. Baker. An exact algorithm for the time–constrained traveling salesman problem. *Operations Research*, 31(5):938–945, 1983.

[8] E. Balas. On the facial structure of scheduling polyhedra. *Mathematical Programming Study*, 24:179–218, 1985.

[9] E. Balas. The asymmetric assignment problem and some new facets of the travelling salesman polytope on a directed graph. *SIAM Journal on Discrete Mathematics*, 3:425–451, 1989.

[10] E. Balas and M. Fischetti. The fixed–outdegree 1–arborescence polytope. *Mathematics of Operations Research*, 17:1001–1018, 1992.

[11] E. Balas and M. Fischetti. A lifting procedure for the asymmetric traveling salesman polytope and a large new class of facets. *Mathematical Programming*, 58:325–352, 1993.

[12] E. Balas, M. Fischetti, and W. Pulleyblank. The precedence–constrained asymmetric traveling salesman polytope. *Mathematical Programming*, 68:241–265, 1995.

[13] M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser, editors. *Network Routing*, volume 8 of *Handbooks in Operations Research and Management Science*. Elsevier Science B.V., Amsterdam, 1995.

[14] S. Bogatsch. Ein Branch&Cut-Algorithmus für das asymmetrische Travelling Salesman Problem mit Zeitfenstern – ein Vergleich verschiedener Modellierungen. Master's thesis, Technical University of Berlin, 1996.

[15] S. Chopra and G. Rinaldi. The graphical asymmetric travelling salesman polyhedron: Symmetric inequalities. *SIAM Journal on Discrete Mathematics*, 9(4):602–624, 1996.

[16] T. Christof. PORTA – A polyhedron representation transformation algorithm, version 1.2.1. Code written by T. Christof (Univ. Heidelberg), revised by A. Löbel and M. Stoer, available from the ZIB electronic library eLib by anonymous ftp from `elib.zib.de`, in the directory /pub/mathprog/polyth, 1994.

[17] N. Christofides, A. Mingozzi, and P. Toth. State-space relaxation procedures for the computation of bounds to routing problems. *Networks*, 11:145–164, 1981.

[18] M. Desrochers and G. Laporte. Improvements and extensions to the Miller–Tucker–Zemlin subtour elimination constraints. *Operations Research Letters*, 10(1):27–36, 1991.

[19] M. Desrochers, J.K. Lenstra, M.W.P. Savelsbergh, and F. Soumis. Vehicle routing with time windows: Optimization and approximation. In B.L. Golden and A.A. Assad, editors, *Vehicle routing: Methods and studies*, pages 65–84. North–Holland, 1988.

[20] Jacques Desrosiers, Yvan Dumas, Marius M. Solomon, and Francois Soumis. *Time Constrained Routing and Scheduling*, chapter 2, pages 35–139. Volume 8 of Ball et al. [13], 1995.

[21] Y. Dumas, J. Desrosiers, E. Gelinas, and M.M. Solomon. An optimal algorithm for the traveling salesman problem with time windows. *Operations Research*, 43(2):367–371, 1995.

[22] M. Fischetti. Facets of the asymmetric traveling salesman polytope. *Mathematics of Operations Research*, 16:42–56, 1991.

[23] M. Fischetti. Three facet–lifting theorems for the asymmetric traveling salesman polytope. In E. Balas, G. Cornuéjols, and R. Kannan, editors, *Integer Programming and Combinatorial Optimization*, pages 260–273. Carnegie Mellon University, Pittsburgh, USA, 1992.

[24] M. Fischetti. Clique tree inequalities define facets of the asymmetric traveling salesman polytope. *Discrete Applied Mathematics*, 56:9–18, 1995.

[25] M. Fischetti and P. Toth. A polyhedral approach to the asymmetric traveling salesman problem. Technical report, University of Bologna, April 1994. (To appear in *Management Science*).

[26] M.R. Garey and D.S. Johnson. Two–processor scheduling with start–times and deadlines. *SIAM Journal on Computing*, 6:416–426, 1977.

[27] M. Grötschel. *Polyedrische Charakterisierungen kombinatorischer Optimierungsprobleme*. Hain, Meisenheim am Glan, 1977.

[28] M. Grötschel and M. Padberg. Polyhedral theory. In E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, and D.B. Shmoys, editors, *The Traveling Salesman Problem*. John Wiley & Sons, 1985.

[29] F. Maffioli and A. Sciomachen. A mixed-integer model for solving ordering problems with side constraints. Technical Report Research report, University of Genoa, Italy, 1993.

[30] G.L. Nemhauser and M.W.P. Savelsbergh. A cutting plane algorithm for the single machine scheduling problem with release dates. In M. Akguk, H. Hamacher, and S. Tufecki, editors, *Combinatorial Optimization: New Frontiers in the Theory and Practice*, NATO ASI Series F: Computer and Systems Sciences 82, pages 63–84. Springer–Verlag, 1992.

[31] M. Queyranne and A. Schulz. Polyhedral approaches to machine scheduling. Technical Report 408/1994, Technische Universität Berlin, 1994.

[32] M. Queyranne and Y. Wang. Symmetric inequalities and their composition for asymmetric travelling salesman polytopes. *Mathematics of Operations Research*, 20:838–863, 1995.

[33] M.W.P. Savelsbergh. Local search for routing problems with time windows. *Annals of Operations Research*, 4:285–305, 1985.

[34] M.W.P. Savelsbergh. The vehicle routing problem with time windows: minimizing route duration. *ORSA Journal on Computing*, 4:146–154, 1992.

[35] J. Tsitsiklis. Special cases of traveling salesman and repairman problems with time windows. *Networks*, 22:263–282, 1992.

[36] C.A. van Eijl. A polyhedral approach to the delivery man problem. Technical Report 95–19, Department of Mathematics and Computer Science, Eindhoven University of Technology, 1995.