HOMOGENIZATION OF POLYHEDRA

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A subset P of $\mathbf{R}^n$ is called a <u>polyhedron</u> if there exists an (m,n)-matrix A and a vector $b \in \mathbf{R}^m$ such that $P = \{x \in \mathbf{R}^n \mid Ax \leq b\}$. If A is an (m,n)-matrix and $b \in \mathbf{R}^m$ we denote by <u>$P(A,b)$</u> the polyhedron $\{x \in \mathbf{R}^n \mid Ax \leq b\}$. We shall always assume that $\underline{M} := \{1,2,\ldots,m\}$ is the set of row indices of A and that $\underline{N} := \{1,2,\ldots,n\}$ is the set of column indices of A. The i-th row of the inequality system $Ax \leq b$ is denoted by $A_{i.}\, x \leq b_i$.

By the well-known theorem of Weyl there is an equivalent definition of polyhedra $P \subseteq \mathbf{R}^n$, namely <u>$P = \mathrm{conv}(V) + \mathrm{cone}(E)$</u>, where V and E are finite subsets of $\mathbf{R}^n$, <u>$\mathrm{conv}(V)$</u> denotes the convex hull of V, and <u>$\mathrm{cone}(E)$</u> the conical hull of E, i.e. the set of nonnegative linear combinations of elements of E.

A bounded polyhedron is called a <u>polytope</u>, and a polyhedron P with the property that $\lambda x \in P$ for all $x \in P$ and all $\lambda \geq 0$ is called a <u>polyhedral cone.</u> For any set $S \subseteq \mathbf{R}^n$, $\mathrm{rec}(S) := \{y \in \mathbf{R}^n \mid x + \lambda y \in S$ for all $x \in S$ and all $\lambda \geq 0\}$ denotes the <u>recession cone</u> of S. It is well-known that if $P = P(A,b) = \mathrm{conv}(V) + \mathrm{cone}(E)$ is a polyhedron then $\mathrm{rec}(P) = P(A,0) = \mathrm{cone}(E)$, and therefore $\mathrm{rec}(P)$ is a polyhedral cone.

Let $P \subseteq \mathbf{R}^n$ be a polyhedron. We call $cx \leq c_0$ a <u>valid inequality</u> for P if $cx \leq c_0$ holds for all $x \in P$. A subset $F \subseteq P$ is a face of P if $F = \{x \in P \mid cx = c_0\}$ holds for some valid inequality $cx \leq c_0$ of P, and this face F is <u>proper</u> if $F \neq P$.

Denote by $F'(P)$ the set of nonempty faces of a polyhedron P. Set

$$F_0 := \bigcap_{F \in F'(P)} F \qquad \text{and} \qquad F(P) := F'(P) \cup \{F_0\}$$

Then it is easy to see that $F(P)$ is a finite lattice under set inclusion.

If $P = P(A,b) \subseteq \mathbf{R}^n$ is a polyhedron then we introduce the mapping (2^S denotes the power set of S)

$$\text{eq}: \ 2^P \longrightarrow 2^M$$

$$P \supseteq F \longmapsto \{i \in M \mid A_{i.}\, x = b_i \ \text{ for all } \ x \in F\} \subseteq M,$$

thus <u>$eq(F)$</u> is the set of all row indices of A such that the corresponding inequalities are binding for F, $eq(F)$ is called the <u>equality set of F.</u>

If the polyhedron P is given as $P = \text{conv}(V) + \text{cone}(E) \subseteq \mathbf{R}^n$ we can define similar mappings as follows: Given a vector $x \in \text{conv}(V) + \text{cone}(E)$, then we say that $u \in V$ convexly supports x with respect to (V,E) is x has a representation $x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$ such that $\lambda_u > 0$, and we say that $f \in E$ <u>conically</u> supports x if x has a representation $x = \sum_{v \in V} \lambda_v v + \sum_{e \in E} \mu_e e$ such that $\lambda_f > 0$. We define for $F \subseteq P$

$$ex_V(F) := \{v \in V \mid v \ \text{ supports some vector } \ x \in F \ \text{ convexly with respect to } (V,E)\}$$

$$ex_E(F) := \{e \in E \mid e \ \text{ supports some vector } \ x \in F \ \text{ conically with respect to } (V,E)\},$$

and combining these notions we define the mapping

$$ex : 2^P \longrightarrow 2^V \times 2^E$$

$$P \supseteq F \longmapsto ex(F) := (ex_V(F),\ ex_E(F)) \subseteq (V,E).$$

The set $ex(F)$ is called the <u>extreme set</u> of P. Clearly, if F is a face of P then $ex_V(F)$ contains all vertices of P contained in F, and $ex_E(F)$ contains all extreme vectors of F, and thus $F = \text{conv}(ex_V(F)) + \text{cone}(ex_E(F))$. Note that for the empty face of P we have $ex_V(\emptyset) = \emptyset$, $ex_E(\emptyset) = \emptyset$, and that $F \subseteq P$ is empty if an only if $ex_V(F) = \emptyset$.

As usual cone polarity is denoted by "o", i.e. for a set $S \subseteq \mathbf{R}^n$, $S^o := \{y \in \mathbf{R}^n \mid yx \leq 0 \ \text{ for all } \ x \in S\}$. With this notion we can define τ-homogenization as follows:

<u>Definition</u> *Let* $S \subseteq \mathbb{R}^n$, $T \subseteq \mathbb{R}^{n\ 1}$ *and* $\tau \in \{-1,0,1\}$,

then we call the set

$$\tau\text{-}hog(S) := \{\binom{x}{\tau} \in \mathbb{R}^{n+1} \mid x \in S\}^{oo}$$

the <u>*τ-homogenization*</u> *of* S. *The set*

$$\tau\text{-}dhog(T) := \{x \in \mathbb{R}^n \mid \binom{x}{\tau} \in T\}$$

is called the <u>*τ-dehomogenization*</u> *of* T.

The idea of homogenization is very natural and has been implicitly considered in many papers. It was known to Minkowski and for instance employed by Goldman (1956), but we could not find out the first explicit use of it. To our knowledge Stoer-Witzgall (1970) were the first who developed homogenization techniques in a broader sense. We found some extensions of their notions very useful.

For sets $S, S_1, S_2 \subseteq \mathbb{R}^n$, $T, T_1 T_2 \subseteq \mathbb{R}^{n+1}$ and $\tau \in \{-1,0,1\}$, $\delta \in \{-1,1\}$ the following calculation rules are abvious

a) *If* $S_1 \subseteq S_2$, *then* $\tau\text{-}hog(S_1) \subseteq \tau\text{-}hog(S_2)$.

b) $\tau\text{-}dhog(T_1 \cap T_2) = \tau\text{-}dhog(T_1) \cap \tau\text{-}dhog(T_2)$

c) $\delta(\tau\text{-}hog(S)) = (\delta\tau)\text{-}hog(\delta S)$.

d) $\delta(\tau\text{-}dhog(T)) = (\delta\tau)\text{-}dhog(\delta T)$.

Note that τ-dhog and τ-dhog could have been defined for any $\tau \in \mathbb{R}$, but a moment's reflection shows that the cases $\tau \in \{-1,0,1\}$ are the essential ones and all other τ-homogenizations (τ-dehomogenizations) can be obtained from the above given ones by simple scaling. The τ-homogenization of a polyhedron can be characterized as follows:

<u>Theorem</u> *Let* $P = P(A,b) = conv(V) + cone(E)$ *be a nonempty polyhedron and let* $\tau \in \{-1,1\}$. *Then*

$$\begin{aligned}
\tau\text{-}hog(P) &= \{\binom{x}{z} \in \mathbb{R}^{n+1} \mid Ax - \tau bz \leq 0,\ \tau z \geq 0\} \\
&= P(B_\tau, 0),\ where\ B_\tau := \begin{pmatrix} A, & \tau b \\ 0, & -\tau \end{pmatrix} \\
&= cone\ (\{\binom{v}{\tau} \mid v \in V\} + \{\binom{e}{0} \mid e \in E\}) \\
&= cone\ (D_\tau),\ where\ D_\tau = \begin{pmatrix} V, E \\ \tau 1^T, 0^T \end{pmatrix}
\end{aligned}$$

The next result shows how information about P can be derived from information about τ-hog(P).

<u>Theorem</u> *Let* P *be a nonempty polyhedron and* $\tau \in \{-1,1\}$ *then*

$$P = \tau\text{-}dhog(\tau\text{-}hog(P)) \quad \text{and}$$

$$rec(P) = o\text{-}dhog(\tau\text{-}hog(P)).$$

This theorem shows that τ-hog(P) in a sense contains both P and its recession cone rec(P). It is also possible to describe the faces of τ-hog(P) which correspond to faces of P, namely these are exactly those faces which have a nonempty intersection with $\{\binom{x}{z} \in \mathbf{R}^{n+1} \mid z = \tau\}$. For the sake of clarity we shall shorten our notation and denote by

$$\underline{H_\tau(P)} := \{F \in F(\tau\text{-hog}(P)) \mid \exists x \in \mathbf{R}^n \ \binom{x}{\tau} \in F\} \cup \{\{0\}\}$$

the set of all faces of the τ-homogenization which have a nonempty intersection with the τ-hyperplane $\{\binom{x}{z} \in \mathbf{R}^{n+1} \mid z = \tau\}$ or which equal the face $\{0\}$ of τ-hog(P). By

$$\underline{H_0(P)} := \{F \in F(\tau\text{-hog}(P)) \mid F \subseteq \mathbf{R}^n \times \{0\}\}$$

we denote all those faces of the τ-homogenization which lie completely in $\{\binom{x}{z} \in \mathbf{R}^{n+1} \mid z = 0\}$. Clearly,

$$F(\tau\text{-hog}(P)) = H_0(P) \cup H_\tau(P)$$

and $H_0(P)$ as well as $H_\tau(P)$ are lattices with respect to set inclusion.

<u>Theorem</u> *Let* P *be a nonempty polyhedron and let* $\tau \in \{-1,1\}$. *Then the mappings*

(a) $\tau\text{-}hog : F(P) \to H_\tau(P)$

(b) $\tau\text{-}dhog : H_\tau(P) \to F(P)$

(c) $0\text{-}hog : F(rec(P)) \to H_0(P)$

(d) $0\text{-}dhog : H_0(P) \to F(rec(P))$

are bijections with

$(\tau\text{-}hog)^{-1} = \tau\text{-}dhog$ *and*

$(0\text{-}hog)^{-1} = 0\text{-}dhog$. *In particular*

(e) *for all nonempty faces* F *of* $P = P(A,b)$ *we have* $\tau\text{-}hog(F) = \{\binom{x}{z} \in P(B_\tau,0) \mid (B_\tau)_{eq(F)}.\binom{x}{z} = 0\}$.

(f) *for all nonempty faces* F *of* $P = conv(V) + cone(E)$ *we have* $\tau\text{-}hog(F) = cone(\{\binom{v}{\tau} \mid v \in ex_V(F)\} + \binom{e}{o} \mid e \in ex_E(F)\})$

(g) *for all nonempty faces* F *of* $rec(P) = P(A,0)$ *and* $I := eq(F) \cup \{m+1\}$ *(here* eq *is taken with respect to* $P(A,0)$*) we have* $0\text{-}hog(F) = \{\binom{x}{z} \in P(B_\tau,0) \mid (B_\tau)_{I.} \binom{x}{z} = 0\}$

(h) *for all nonempty faces* F *of* $rec(P) = cone(E)$ $0\text{-}hog(F) = cone(\{\binom{e}{0} \mid e \in ex_E(F)\})$ *(here* ex *is taken with respect to* $cone(E)$*).*

Corollary *Let* P *be a nonempty polyhedron and let* $\tau \in \{-1,1\}$*. The face lattice* $F(P)$ *is isomorphic to the face lattice* $H_\tau(P)$ *and the face lattice* $F(rec(P))$ *is isomorphic to the face lattice* $H_0(P)$*.*

Corollary *If* P *is a polytope, then the face lattices* $F(P)$ *and* $F(\tau\text{-}hog(P))$ *are isomorphic.*

Proposition *Let* P *be a nonempty polyhedron and let* $\tau \in \{-1,1\}$*. Then*

(a) *for all* $F \in F(P)$

$$dim(\tau\text{-}hog(F)) = dim(F) + 1.$$

(b) *for all* $F \in F(rec(P))$

$$dim(0\text{-}hog(F)) = dim(F).$$

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