STRONG BLOCKS AND THE OPTIMUM BRANCHING PROBLEM

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A complete characterization of the branching polytope on an arbitrary digraph in terms of linear inequalities was given by J. Edmonds, c.f. [2]. This description was refined by R. Giles (c.f. [3]) who showed which of these inequalities are essential, i.e. define facets of the branching polytope.
R. Giles characterized the facets of the polytope which is obtained by intersecting two matroid polytopes. As the branching polytope can be viewed as the intersection of two matroid polytopes (c.f. [3]) he obtaines this characterization by an appropriate graphical interpretation of the matroidal conditions.
This note aims at a purely graph theoretical and linear algebraical proof of the facial characterization of the branching polytope which is at the same time elementary and in its basic idea applicable to many other situations in polyhedral combinatorics.

1. NOTATION

With some minor exceptions (defined below) we use the graph theoretical terminology of Berge [1]. We consider directed, loop-free graphs only and call them *digraphs*, denoted by $G = (V,E)$, where V is the set of *nodes* and E the family of arcs. Multiple arcs are allowed. A node v with $d_G^+(v) = d_G^-(v) = 1$ is called a *carrier*. A *branching* is a forest with the property that every node is the terminal node of at most one arc, an *arborescence* is a connected branching. For our purpose blocks of a digraph must contain at least two nodes. Digraphs which have at least two nodes and are strongly connected and 2-connected are called *strong blocks*. The digraph obtained from G by removing one arc e is denoted by $G - e$. A strong block G with the property that the removal of any arc e destroys at least one of the two defining properties is called *minimal*, i.e., $G - e$ is not a strong block for all $e \in E$. For $W \subset V$ the family of arcs in E having both end nodes in W is denoted by $E(W)$. For $F \subset E$ the set of nodes that are contained in at least one arc of F is denoted by $V(F)$.

A *polytope* $P \subset \mathbb{R}^m$ is the convex hull of finitely many vectors of $\mathbb{R}^m$. The dimension of $P(\dim P)$ is the maximal number of affinely independent points in P minus one. A linear inequality $ax \leq a_0$ is called *valid* with respect to P if $P \subset \{x \in \mathbb{R}^m : ax \leq a_0\}$. A valid inequality $ax \leq a_0$ is called a *facet* of P if $\dim(P \cap \{x : ax = a_0\}) = \dim P - 1$.

A useful criterion for checking linear independence is the following

LEMMA 1.1. *Let $x_1, \ldots, x_k, y \in \mathbb{R}^m$ and let $ax \leq a_0$, $bx \leq b_0$ be two inequalities. If $ax_i = a_0$, $bx_i = b_0$, for $i = 1, \ldots, k$, and $ay = a_0$, $by \neq b_0$ holds, then y is linearly independent from $x_1, \ldots, x_k$.*

PROOF: Suppose there are $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that $y = \sum_{i=1}^{k} \lambda_i x_i$, then

$$a_0 = ay = a\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i (ax_i) = \left(\sum_{i=1}^{k} \lambda_i\right) a_0 \quad \text{contradicts}$$

$$b_0 = by = b\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i (bx_i) = \left(\sum_{i=1}^{k} \lambda_i\right) b_0. \ \square$$

Lemma 1.1. is often applicable in polyhedral combinatorics when a set of lineary independent incidence vectors x_i, y is constructed which all satisfy $ax = a$ and where all x_i have a common component (e.g. of ones) while y has a different entry (zero) in this component.

For variables indexed by a set E the summation $\sum_{e \in E} x_e$ is abbreviated by $x(E)$.

2. THE BRANCHING POLYTOPE AND ITS TRIVIAL FACETS

Let $G = (V,E)$ be a digraph, and let $\mathfrak{B}$ be the set of all branchings (which are sets of arcs) in G . With each $B \in \mathfrak{B}$ we associate an *incidence vector* $x^B \in \mathbb{R}^{|E|}$ by setting $x_e^B = 1$ if $e \in B$, $x_e^B = 0$ if $e \notin B$. The convex hull of the incidence vectors of branchings

(2.1.) $$\tilde{P}_B(G) := \mathrm{conv}\{x^B \in \mathbb{R}^{|E|} : B \in \mathfrak{B}\}$$

is called the *branching polytope* of G. Clearly, if a weight function $c : E \to \mathbb{R}$ is given, the problem of finding an optimum branching can be solved via the linear program

(2.2.) $$\max cx, \quad x \in \tilde{P}_B(G).$$

Edmonds [2] proved that a certain system of inequalities is a complete linear characterization of $\tilde{P}_B(G)$ by displaying an algorithm that produces a branching of total weight c^*, and showing that there exists a solution of the corresponding dual linear program with the same objective value c^*. His system of inequalities, however, is quite redundant. By applying his theory of the intersection of two matroid polytopes Giles [3] could show which of these inequalities are essential, i.e. define facets of $\tilde{P}_B(G)$.

For both the sufficiency part and the necessity part we shall give new proofs using elementary graph theory and linear algebra only.

The branching polytope $\tilde{P}_B(G)$ has full dimension $|E|$ because the empty set and all sets consisting of a single arc are branchings and their incidence vectors are affinely independant. Therefore, no hyperplane contains $\tilde{P}_B(G)$, and for any valid inequality $ax \leq a_o$ there is a vector $y \in \tilde{P}_G(B)$, such that $ay < a_o$.

In order to prove that a valid inequality $ax \leq a_o$ is a facet of $\tilde{P}_B(G)$ it is sufficient to show that

$$\underline{H_a} := \tilde{P}_B(G) \cap \{x : ax = a_o\}$$

contains $|E|$ affinely independent incidence vectors of branchings. Let $\underline{B_a} := \{B \in \mathfrak{B} : x^B \in H_a\}$. For a valid inequality $ax \leq a_o$ we define the induced digraph $G_a = (V_a, E_a)$ by setting $E_a := \{e \in E : a_e \neq o\}$ and $V_a := V(E_a)$.

For convenience we will say that a *set of branchings is affinely or linearly independant* if the corresponding incidence vectors are affinely or linearly independent.

It is obvious that the inequalities

(2.3)	$x_e \geq o$	for all $e \in E$,
(2.4)	$x(\omega^-(v)) \leq 1$	for all $v \in V$

are valid with respect to $\tilde{P}_B(G)$ and it is easily seen which of the inequalities (2.3.) and (2.4.) are facets of $\tilde{P}_B(G)$.

DEFINITION 2.1. *Let $G = (V,E)$ be a digraph, then*

$$V^- := \{v \in V : \exists w \in V \text{ such that } \Gamma_G^-(v) = \{w\} \text{ and } v \in \Gamma_G^-(w)\} \cup \{v \in V : \Gamma_G^-(v) = \emptyset\}. \ \square$$

PROPOSITION 2.2. *Let $G = (V,E)$ be a digraph, then*

a) $x_e \geq 0$ is a facet of $\tilde{P}_B(G)$ for all $e \in E$,

b) $x(\omega^-(v)) \leq 1$ is a facet of $\tilde{P}_B(G)$ if and only if $v \in V - V^-$.

PROOF: a) The zero vector and all but one unit vectors satisfy $x_e \geq 0$ with equality, they are contained in $\tilde{P}_B(G)$, and are affinely independent.

b) If $\Gamma_G^-(v) = \emptyset$ then $x(\omega^-(v)) = 0 \cdot x \leq 1$ is clearly not a facet.

If $\Gamma_G^-(v) = \{w\}$ and $v \in \Gamma_G^-(w)$ then $x(\omega^-(v)) \leq 1$ is dominated by $x(E(\{v,w\}) \leq 1$ and therefore not a facet.

Let $v \in V - V^-$, then for every $f \notin \omega^-(v)$ there is an arc $e_f \in \omega^-(v)$ such that $\{f,e_f\}$ is a branching. The $|E|$ branchings $\{e\}$ for all $e \in \omega^-(v)$ and $\{f,e_f\}$ for all $f \in E - \omega^-(v)$ are obviously linearly independent and satisfy $x(\omega^-(v)) \leq 1$ with equality. □

Because of their simple structure the inequalities of type (2.3), (2.4) which define facets of $\tilde{P}_B(G)$ are called *trivial*.

3. THE OTHER FACETS

In general, $\tilde{P}_B(G)$ does not only have trivial facets. To get a complete linear description of $\tilde{P}_B(G)$ we will use the following technique: We assume that $ax \leq a_o$ is a nontrivial facet of $\tilde{P}_B(G)$ and conclude from this assumption that (a,a_o) and the induced digraph G_a have to have certain properties that render a universal characterization.

The following facts about branchings are well known. Let G be a digraph and B be a branching of G. If v is a root of B and $e \in \omega^-(v)$ then $A := B \cup \{e\}$ is a branching if and only if A does not contain a circuit containing e.

In the following let $ax \leq a_o$ be a nontrivial facet of $\tilde{P}_B(G)$, and let $G_a = (V_a, E_a)$, H_a, B_a be defined as above. If $x \in \tilde{P}_B(G)$ and $0 \leq y \leq x$ then clearly $y \in \tilde{P}_B(G)$. This property implies that all coefficients of a are nonnegative, i.e. $a_e > 0$ for all $e \in E_a$.

LEMMA 3.1 *G_a is strongly connected.*

PROOF: It is clear that G_a has to be connected. Suppose $\omega(W) = \omega^-(W)$ is a cocircuit in G_a. Let $S \subset W$ be the set of nodes which are terminal nodes of at least one arc of $\omega^-(W)$. Let $B \in B_a$ be any branching and suppose there is a node $v \in S$ such that v is a root of B. By assumption v is the terminal node of an arc $e \in \omega^-(W)$. The arc set $A:=B\cup\{e\}$ cannot contain a circuit containing e since there is no arc going from W to $V-W$, thus A is a branching. But then $ax^A = ax^B + a_e > a_o$ contradicts the validity of $ax \leq a_o$. Therefore, if $B \in B_a$ then x^B satisfies $x(\omega^-(v)) \leq 1$ with equality for all $v \in S$. This contradicts the assumption that $ax \leq a_o$ is nontrivial. ◻

LEMMA 3.2. *G_a is 2-connected.*

PROOF: Suppose G_a contains an articulation point v. Then there are induced subdigraphs $G_1 = (V_1, E(V_1))$, $G_2 = (V_2, E(V_2))$ satisfying $V_1 \cup V_2 = V_a$, $V_1 \cap V_2 = \{v\}$, $E(V_1) \neq \phi \neq E(V_2)$, $E(V_1) \cup E(V_2) = E_a$. Since $ax \leq a_o$ is nontrivial there is a branching $B \in B_a$ rooted at v. Define $B_1 := B \cap E(V_1)$, $B_2 := B \cap E(V_2)$, and $b_1 := ax^{B1}$, $b_2 := ax^{B2}$.

Let $A \in B_a$ be any branching and $A_1 := A \cap E(V_1)$, $A_2 := A \cap E(V_2)$. By construction, $D_1 = B_1 \cup A_2$ and $D_2 = B_2 \cup A_1$ are branchings of G_a, i.e. $ax^{D1} \leq a_o$, $ax^{D2} \leq a_o$. Since $A, B \in B_a$, equality has to hold in both of these inequalities. This implies $ax^{A1} = b_1$ and $ax^{A2} = b_2$, hence $ax \leq a_o$ cannot be a facet. Contradiction. ◻

LEMMA 3.3. *All branchings $B \in B_a$ are arborescences of G_a.*

PROOF: Suppose $B \in B_a$ is not an arborescence of G_a. We may assume that B has two components, say B_1 and B_2, such that $V_1 \cup V_2 = V_a$ holds for $V_1 := V(B_1), V_2 := V(B_2)$.
Let $b_1 := ax^{B1}$, $b_2 = ax^{B2}$. The sets $E_1 = E(V_1) \cup \omega^-(V_1)$, $E_2 = E(V_2) \cup \omega^-(V_2)$ define a partition of E_a.

Let $A \in B_a$ be any branching and $A_1 = A \cap E_1$, $A_2 = A \cap E_2$. Then $D_1 = B_1 \cup A_2$ and $D_2 = B_2 \cup A_1$ are branchings of G_a. It follows from $A, B \in B_a$ and $ax^{D_1} \leq a_o$, $ax^{D_2} \leq a_o$ that $ax^{A_1} = b_1$ and $ax^{A_2} = b_2$ have to hold. This is a contradiction to $ax \leq a_o$ being a facet. □

Since every arborescence of G_a contains exactly $|V_a| - 1$ arcs we get

PROPOSITION 3.4. *All nontrivial facets of $\tilde{P}_B(G)$ are of the form*

$$(3.1) \qquad x(E(W)) \leq |W| - 1$$

where $(W, E(W))$ is a an induced subdigraph of G which is a strong block.

PROOF: Given a facet $ax \leq a_o$ then by Lemma 3.3 all vectors $x \in H_a$ satisfy $x(E_a) = |V_a| - 1$. Since facets of fully dimensional polytopes are unique up to a constant factor this implies that $a = \alpha\, x(E_a)$ and $a_o = \alpha(|V_a| - 1)$ for some $\alpha > 0$. Since all branchings $(V_a, E(V_a))$ contain at most $|V_a| - 1$ arcs, the inequality $x(E(V_a)) \leq |V_a| - 1$ is a valid inequality for $\tilde{P}_B(G)$, and as $x(E_a) \leq x(E(V_a))$ holds we necessarily have $E_a = E(V_a)$. Thus, any nontrivial facet is of the above form. By Lemma 3.1 and 3.2 (V_a, E_a) is a strong block. □

To prove the converse of Proposition 3.4 we utilize a result on minimal strong blocks which states that every minimal strong block contains at least two carriers (c.f. [5]).

LEMMA 3.5. *Every strong block $G = (W, F)$ contains $|F|$ linearly independent arborescences.*

PROOF: By induction on $|F|$. A strong block with two arcs necessarily has two nodes, and the lemma clearly holds for $|V| = 2$.

Now let $G = (W,F)$ be a strong block with $m + 1$ arcs and $n \geq 3$ nodes. By our induction hypothesis the lemma is true for strong blocks with m arcs. There are two possibilities: either G is a minimal strong block or not.

1) G is not minimal, i.e. there exists an arc $e \in F$ such that $G - e$ is a strong block. By the induction hypothesis $G - e$ contains m linearly independent arborescences. As G is strongly connected there is an arborescence B in G containing e. Since none of the m arborescences in $G - e$ contains e, B is linearly independent of these by Lemma 1.1, hence $|F| = m+1$ linearly independent arborescences of G are found.

2) If G is a minimal strong block then by a result of [5] G contains a carrier $v \in V$. Let (u,v), $(v,w) \in F$. Define $G' := (V-\{v\}, (F-\{(u,v),(v,w)\}) \cup \{(u,w)\})$. G' is clearly a strong block with m arcs and - by induction hypothesis - G' contains m linearly independent arborescences $A_1', \ldots, A_m'$.

Let $A_i := (A_i' - \{(u,w)\}) \cup \{(u,v),(v,w)\}$ if $(u,w) \in A_i'$, and

$A_i := A_i' \cup \{(u,v)\}$ if $(u,w) \notin A_i'$

The arc sets A_i are linearly independent arborescences of G by construction, and they all contain the arc (u,v). As G is strongly connected, G contains an arborescence A_{m+1} with root v, hence, A_{m+1} does not contain (u,v). By Lemma 1.1. A_{m+1} is linearly independent of the arborescences $A_1 \ldots A_m$ and the Lemma is proved. ◻

The above Lemma immediately implies

PROPOSITION 3.6. *Let $G = (V,E)$ be a digraph and $W \subset V$, $|W| \geq 2$, such that $(W,E(W))$ is a strong block, then*

$$x(E(W)) \leq |W| - 1 \qquad (3.1)$$

is a facet of $\tilde{P}_B(G)$.

Proof: If $(W,E(W))$ is a strong block then by Lemma 3.5 it contains a set I of $|E(W)|$ linearly independent arborescences, i.e., branchings of G, the incidence vectors of which satisfy (3.1) with equality.

Let $f \in E - E(W)$, and let v be the terminal node of f. If $v \in W$, then because of strong connectivity there is an arborescence A_f in $(W,E(W))$ with root v. If $v \notin W$ take any arborescence A_f in $(W,E(W))$. Define $B_f := A_f \cup \{f\}$ for all $f \in E - E(W)$. By construction the incidence vectors of the branchings B_f satisfy (3.1) with equality. Clearly the set $I \cup \{B_f : f \in E - |E(W)|\}$ is a set of $|E|$ linearly independent branchings, which proves the Proposition. □

Summing up the Propositions 2.2, 3.4, 3.6 we obtain

THEOREM 3.7. ([2], [3]) *Let $G = (V,E)$ be a digraph. Then a complete and non-redundant characterization of the branching polytope $\tilde{P}_B(G)$ is given by*

(3.3) $x_e \geq 0$ *for all* $e \in E$

(3.4) $x(\omega^-(V)) \leq 1$ *for all* $v \in V - V^-$

(3.5) $x(E(W)) \leq |W| - 1$ *for all* $W \in V$ *such that* $(W,E(W))$ *is a strong block.* □

The induction technique used in Lemma 3.5 can be used in several other problems of polyhedral combinatorics. In general the induction in the case of a non-minimal situation is straightforeward while for the case of a minimal object (e.g. a minimal strong block) a problem-specific result (like the existence of a carrier) has to be exploited.
We would like to mention an application of the above result. The n-city asymmetric travelling salesman problem can be represented as the intersection of the branching problem and a partition - matroid problem on a modified complete digraph K'_{n+1} on $n + 1$ nodes (c.f. [4]). It was shown in [4] that all inequalities of types (2.3), (2.4), (3.1) that are facets of $\tilde{P}_B(K'_{n+1})$ are also facets of the monotone

asymmetric travelling salesman polytope P_T^n, hence, the asymmetric travelling salesman problem inherits the polyhedral properties of the optimum branching problem.

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