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ON THE SYMMETRIC TRAVELLING SALESMAN PROBLEM: SOLUTION OF A 120-CITY PROBLEM

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The polytope associated with the symmetric travelling salesman problem has been intensively studied recently (cf. [7, 10, 11]). In this note we demonstrate how the knowledge of the facets of this polytope can be utilized to solve large-scale travelling salesman problems. In particular, we report how the shortest roundtrip through 120 German cities was found using a commercial linear programming code and adding facetial cutting planes in an interactive way.

Key words: Travelling Salesman Problem, Cutting Planes, Facets, Computation.

1. Introduction

The travelling salesman problem (TSP) is one of the oldest and most intensively studied combinatorial optimization problems, but the first approach towards a solution of the TSP due to Dantzig et al. [3] does not seem to have attracted much attention in the sequel. Their idea was to generate a "good" solution heuristically, to formulate the TSP as a linear programming problem in zero–one variables, and to try to prove optimality of the heuristically obtained tour using cutting planes. Their procedure contained "artistic" and interactive parts and did not result in a straightforward algorithm. Although they were able to solve a 49-city problem, it was not clear how good their cutting planes really were with respect to proving optimality or obtaining a good lower bound. These are some of the reasons why the branch-and-bound techniques that came up in the sixties superceded the cutting planes approach for the TSP. The various branch-and-bound algorithms (for a survey see [1]) were and still are highly successful, but it became evident that due to the fact that the computational work grows exponentially with the number of cities, there are bounds on the size of the problems solvable with these methods.

S. Hong [12] seems to be one of the first to have rediscovered the appeal of the cutting planes approach. He automatized some of the interactive parts of the Dantzig et al. procedure and incorporated further cutting planes. He reported good results on moderately sized problems. So did Miliotis [14] who combined the Dantzig et al. linear programming idea with branch-and-bound techniques.

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Stimulated by the work of Edmonds (e.g. [5, 6]) and others polyhedral combinatorics, i.e. the study of polyhedra associated with combinatorial optimization problems, became a field of intensive research during the last decade and soon grew into a powerful tool for the solution of combinatorial optimization problems.

With regard to the travelling salesman problem intensive studies of the facet-structure of polytopes related to the TSP were carried out, cf. [2, 7–13, 15], and tremendously large classes of inequalities essential for the characterization of these polytopes were discovered. However, it also turned out that it is very unlikely that a complete description of these polytopes can ever be obtained (cf. [7]).

Based on the results in [7, 10–12] concerning the symmetric travelling salesman problem Padberg and Hong [15] developed a quite sophisticated cutting plane algorithm which clearly proved the usefulness of this approach, in particular, they solved a symmetric 318-city problem within 0.26% of optimality, a result which seems to be far outside the range of all presently known branch-and-bound methods.

This paper also aims at validating the usefulness of facetial cutting planes and presents the solution of a real-world symmetric 120-city travelling salesman problem gained in the same interactive fashion which was used in 1954 by Dantzig et al. [3]. The only difference is that our procedure due to the theoretical work in [7, 10, 11] could be based on a much better knowledge of the underlying polytope and that better LP-routines and computers were available.

2. Notation

All graphs $G=[V,E]$ considered are undirected, have no loops and no multiple edges. The node set V is assumed to be $\{1,2,\dots,n\}$; edges $e\in E$ are denoted by $\{i,j\}$ where $i\neq j$. A graph on n nodes is called *complete* if $E=\{\{i,j\}: i, j\in V,\ i\neq j\}$ and will be denoted by $K_n=[V,E]$. A set C of $k\geq 3$ edges, $C=\{\{v_1,v_2\},\{v_2,v_3\},\dots,\{v_{k-1},v_k\},\{v_k,v_1\}\}$ where $v_i\neq v_j$ if $i\neq j$, is called a *cycle* of length k. Cycles of length n are called *tours*, cycles of length $k<n$ are sometimes also called *subtours*. If $C_1, C_2,\dots, C_r$ are cycles such that every node is contained in exactly one cycle C_i, then the union of these cycles is called perfect 2-matching or simply *2-matching*. Thus every tour is a 2-matching. For any $W\subset V$ and $F\subset E$ we use the following abbreviations:

$$V(F):=\{i\in V\colon i \text{ is contained in an edge of } F\}=\textstyle\bigcup_{e\in F} e,$$
$$E(W):=\{\{i,j\}\in E\colon i\in W,\ j\in W\}.$$

If $G=[V,E]$ is a graph and $\{x_e\colon e\in E\}$ is a set of variables indexed by E, and if $F\subset E$, $W\subset V$, then we write

$$x(F):=\sum_{e\in F} x_e$$

and

$$x(W) := x(E(W)).$$

3. Polytopes related to the symmetric travelling salesman problem

The (symmetric) travelling salesman problem can be stated as follows: Given a complete graph $K_n = [V, E]$ and edge lengths $c_e \in \mathbf{R}$ for all $e \in E$, find the shortest tour in K_n, i.e. a tour T such that $\sum_{e \in T} c_e$ is minimal. This combinatorial optimization problem can be formulated in algebraical terms in the following way: To each edge $e \in E$ we associate a variable x_e, and to each tour $T \subset E$ we associate an *incidence vector* x^T, i.e. a vector such that

$$x_e^T = \begin{cases} 1 & \text{if } e \in T, \\ 0 & \text{otherwise.} \end{cases}$$

As $|E| = \frac{1}{2}n(n-1) =: m$ we have $x^T \in \mathbf{R}^m$. The convex hull Q_T^n of all incidence vectors of tours is called (symmetric) *travelling salesman polytope*, i.e.

$$Q_T^n := \operatorname{conv}\{x^T \in \mathbf{R}^m : T \text{ is a tour in } K_n\}. \tag{3.1}$$

Hence, to each vertex of Q_T^n corresponds a tour in K_n and vice versa. If a complete characterization of Q_T^n by means of linear equations and inequalities were known, then the TSP could be solved (theoretically) through the linear program min cx, $x \in Q_T^n$.

By definition Q_T^n is contained in the unit hypercube $\{x \in \mathbf{R}^m : 0 \le x_e \le 1$ for all $e \in E\}$, and a tour (also a 2-matching) T has the property that every node is contained in exactly two edges of T, hence the system of equations

$$Ax = 2e_n \tag{3.2}$$

where A is the node-edge incidence matrix of K_n, e_n is an n-vector of ones, must be satisfied by all incidence vectors of tours and 2-matchings. This implies that $Q_T^n \subset \bar{\bar{Q}}_{2M}^n$ where

$$\bar{\bar{Q}}_{2M}^n := \{x \in \mathbf{R}^m : Ax = 2e_n,\ 0 \le x_e \le 1 \text{ for all } e \in E\}. \tag{3.3}$$

It is not difficult to characterize the vertices of $\bar{\bar{Q}}_{2M}^n$: they are incidence vectors of tours, incidence vectors of 2-matchings which are not tours (so called subtour vertices), or simply-structured fractional vertices, i.e. $0 < x_e < 1$ for some $e \in E$ (cf. [7]). To get a polytope closer to Q_T^n both the subtour vertices and the fractional vertices of $\bar{\bar{Q}}_{2M}^n$ have to be chopped off.

In order to cut off the subtour vertices Dantzig et al. [3] introduced the following *subtour-elimination constraints*

$$x(W) \le |W| - 1 \quad \text{for all } W \subset V,\ 2 \le |W| \le n-1. \tag{3.4}$$

Defining

$$Q_S^n := \{x \in \bar{\bar{Q}}_{2M}^n \colon x \text{ satisfies (3.4)}\}, \tag{3.5}$$

it is obvious that Q_S^n has no other integer vertices apart from incidence vectors of tours, hence $Q_T^n = \operatorname{conv}\{x \in Q_S^n\colon x \text{ integer}\}$. The subtour-elimination constraints, however, do not suffice to eliminate all fractional vertices from $\bar{\bar{Q}}_{2M}^n$. Inequalities doing this were found by Edmonds [5]. He introduced the so called *2-matching constraints*

$$\sum_{i=0}^{k} x(W_i) \le |W_0| + \tfrac{1}{2}(k-1), \tag{3.6}$$

where the node sets $W_0, W_1, \ldots, W_k \subset V$ satisfy

$$|W_0 \cap W_i| = 1, \quad i = 1, \ldots, k, \tag{3.6.1}$$

$$|W_i| = 2, \quad i = 1, \ldots, k, \tag{3.6.2}$$

$$k \ge 1 \text{ and odd.} \tag{3.6.3}$$

Letting

$$\begin{aligned} Q_{2M}^n &:= \operatorname{conv}\{x^M \in \mathbf{R}^m\colon M \text{ is a 2-matching in } K_n\} \\ &= \operatorname{conv}\{x \in \bar{\bar{Q}}_{2M}^n\colon x \text{ integer}\}, \end{aligned} \tag{3.7}$$

Edmonds [5] proved:

$$Q_{2M}^n = \{x \in \bar{\bar{Q}}_{2M}^n\colon x \text{ satisfies all inequalities (3.6)}\}.$$

This result shows that the 2-matching constraints cut off all fractional vertices of $\bar{\bar{Q}}_{2M}^n$ without creating new ones. By construction we have $Q_T^n \subset Q_S^n \cap Q_{2M}^n$, but unfortunately equality does not hold. This means that although all fractional and subtour vertices of $\bar{\bar{Q}}_{2M}^n$ are chopped off, new fractional vertices which are "more complicated" than those of $\bar{\bar{Q}}_{2M}^n$ are created by the intersection of the halfspaces defined by (3.4) with those given by (3.6).

Several new types of inequalities which are valid with respect to Q_T^n, i.e. Q_T^n is contained in the half spaces defined by these inequalities or equivalently all incidence vectors of tours satisfy these inequalities, were proposed in the literature [2, 7, 10, 11, 13, 15], the largest and best studied class of these are the *comb inequalities* (cf. [2, 7, 10, 11]):
Given $W_0, W_1, \ldots, W_k \subset V$ such that

$$|W_0 \cap W_i| \ge 1, \quad i = 1, \ldots, k, \tag{3.8.1}$$

$$|W_i - W_0| \ge 1, \quad i = 1, \ldots, k, \tag{3.8.2}$$

$$W_i \cap W_j = \emptyset, \quad i \le i < j \le k, \tag{3.8.3}$$

$$k \ge 3 \text{ and odd} \tag{3.8.4}$$

then one can show (cf. [10]) that

$$\sum_{i=0}^{k} x(W_i) \le |W_0| + \sum_{i=1}^{k} (|W_i| - 1) - \frac{k+1}{2} =: s(C) \tag{3.8}$$

is a valid inequality with respect to Q_T^n.

Validity, however, is not a proper criterion for checking the "sharpness" of inequalities, i.e. their suitability as cutting planes. The concept of facets allows one to definitely judge the goodness of cutting planes. An inequality $ax \le a_0$ valid with respect to Q_T^n is called a *facet* of Q_T^n if $\dim(Q_T^n \cap \{x \in \mathbf{R}^M: ax = a_0\}) = \dim Q_T^n - 1$, and two facets $ax \le a_0$ and $bx \le b_0$ are called *equivalent* if $Q_T^n \cap \{x \in \mathbf{R}^m: ax = a_0\} = Q_T^n \cap \{x \in \mathbf{R}^m: bx = b_0\}$. As Q_T^n is not a fully-dimensional polytope ($\dim Q_T^n = m - n = |E| - |V|$, cf. [10]) many different inequalities turn out to be equivalent with respect to Q_T^n (cf. [7]). If K is the number of different classes of equivalent facets of Q_T^n, and if we choose from each of these classes exactly one inequality $a^i x \le a_0^i$, $i = 1, \ldots, K$, then it is well-known that

$$Q_T^n = \{x \in \mathbf{R}^m: Ax = 2e_n,\ a^i x \le a_0^i,\ i = 1, \ldots, K\} \tag{3.9}$$

holds.

Furthermore this characterization of Q_T^n is non-redundant, i.e. if we drop any of the equations of $Ax = 2e_n$ or any of the inequalities $a^i x \le a_0^i$ the polytope on the right-hand side of (3.9) is no longer equal to Q_T^n. These properties establish in a precise sense that facets are best cutting planes because only the knowledge of at least one element of all classes of facets of Q_T^n renders a complete and non-redundant characterization of Q_T^n possible. This also shows that in cutting plane algorithms only facetial inequalities (if such are known) should be used, all other (non-facet) inequalities do not suffice to fully establish the polytope considered—although in some practical applications they may suffice to prove optimality.

With respect to the travelling salesman polytope the following results using quite involved proof techniques were obtained (cf. [7, 10, 11]).

Theorem 1. *Let* $n \ge 6$.

(a) *The trivial inequalities* $x_e \ge 0$, $x_e \le 1$ *are facets of* Q_T^n *for all* $e \in E$.

(b) *The subtour elimination constraints* $x(W) \le |W| - 1$ *are facets of* Q_T^n *for all* $W \subset V$, $3 \le |W| \le n - 3$.

(c) *Two different subtour elimination constraints* $x(W) \le |W| - 1$, $x(W') \le |W'| - 1$ *are equivalent with respect to* Q_T^n *if and only if* $W = V - W'$.

(d) *All comb inequalities* (3.8) *are facets of* Q_T^n.

(e) *Two different comb inequalities*

$$\sum_{i=0}^{k} x(W_i) \le s(C), \qquad \sum_{i=0}^{h} x(W_i') \le s(C')$$

are equivalent with respect to Q_T^n *if and only if* $k = h$, $W_0 = V - W_0'$, $W_i = W_i'$, $i = 1, \ldots, k$.

(f) *Trivial inequalities, comb inequalities and subtour-elimination constraints are pairwise non-equivalent.*

One can also show (cf. [7]) that the comb inequalities (3.8) contain as a special case all 2-matching inequalities (3.6) and all Chvátal-comb inequalities (cf. [2]) which define facets of Q_T^n, hence, comb inequalities are a fairly general class of facets of Q_T^n.

Letting

$$Q_C^n := \{x \in \bar{\bar{Q}}_{2M}^n \colon x \text{ satisfies all inequalities (3.4) and (3.8)}\} \tag{3.10}$$

then obviously $Q_T^n \subset Q_C^n \subset Q_S^n \cap Q_{2M}^n$ and for larger n these inclusions are proper. By Theorem 1 we know all the facets of Q_C^n, and furthermore, Theorem 1 says that all facets of Q_C^n are also facets of Q_T^n. Although Theorem 1 does not characterize the travelling salesman polytope completely, the polytope Q_C^n seems to be quite a good approximation of Q_T^n, and it seems reasonable to use this polytope as a relaxation of the travelling salesman problem, in particular because the combinatorial structure of the facets of Q_C^n is quite simple. Theorem 1 makes it possible to design a cutting plane algorithm for the TSP using facets only and avoiding the use of inequalities which are equivalent to others or do not define facets.

4. Solution of a 120-city problem by linear programming

Having claimed that a good knowledge of the polytope Q_T^n is of high value for the solution of travelling salesman problems, we are going to demonstrate this by solving a real-world symmetric 120-city problem using the facets found as cutting planes. Our method is the same interactive procedure carried out by Dantzig et al. [3]. We did not try to mechanize the generation of cutting planes but rather found cutting planes by inspecting the solutions of relaxed linear programs and added them in an interactive way. For a discussion of the possibilities of mechanically identifying violated facets and activating them in order to cut off non-tour solutions, the reader should consult [15] where several methods of automatizing these procedures are described.

The data of our problem were taken from [4], the edge lengths are the road distances between every two of 120 cities of the Federal Republic of Germany (including some cities bordering Germany). For a complete listing of the cities see the Appendix.

Before using cutting planes we tried to solve the problem with all branch-and-bound algorithms available to us, but none of them terminated with an optimal tour. To get a "good" estimate of the order of magnitude of the length of the optimal tour and to obtain a starting basis for our simplex procedure, we generated tours using several heuristical methods. The shortest of the heuristically found tours had length 7091 km; by visual analysis of this tour on a map of Germany we were able to construct a tour of length 7011 km, but no other hand- or machine improvements could be obtained. As every tour is a 2-matching the shortest 2-matching (which can be obtained by a good algorithm) gives a lower bound for the length of the optimal tour. The shortest 2-matching of our 120-city problem has length 6694 km, hence we know that the shortest tour will be in the interval 6694 km to 7011 km.

We now used the linear programming cutting plane technique suggested in [3] which works as follows: Relax the TSP as far as possible, i.e. choose a polytope Q_1 that has few facets only and contains Q_T^{120}, and solve the linear program $\min cx,\ x \in Q_1$. If the optimal solution x_1 of Q_1 is a tour one is done, if not choose from the facets given in Theorem 1 a set of inequalities $Bx \leq b$ which are violated by x_1 and add these to Q_1, thus cutting off x_1. Then solve the linear program $\min cx,\ x \in Q_2 = Q_1 \cap \{x\colon Bx \leq b\}$ and proceed in the same manner.

In our case we started the procedure with the polytope $\bar{\bar{Q}}_{2M}^{120}$ (3.3) which we consider to be the coarsest meaningful relaxation of the symmetric travelling salesman problem.

After every LP-run we represented the optimal solution graphically by hand on a map. In the beginning a plotter was used, but as the number of different fractional components of the solutions increased there were not enough symbols to distinguish them and the plottings became too cluttered. Using the graphical representation of the optimal solution we looked for subtour elimination constraints and comb inequalities to cut off the present solution and added them to the present constraint system. Drawing and searching took from 30 man-minutes in the beginning up to 3 man-hours after the last runs.

Altogether 13 LP-runs were needed and a total number of 96 additional inequalities had to be added to $\bar{\bar{Q}}_{2M}^{120}$. Among these 96 cutting planes we used 36 subtour-elimination constraints (3.4) and 60 comb inequalities (3.8). The 60 (general) comb inequalities were composed of 25 2-matching inequalities, 14 Chvátal-comb inequalities and 21 other comb inequalities. Thus the polytope defined by the intersection of $\bar{\bar{Q}}_{2M}^{120}$ with these 96 half spaces contains Q_T^{120} and has the same optimal solution for the given 120-city problem as the LP over Q_T^{120}.

In the following we list all the 96 facetial inequalities we used and the value of the objective function after each run. We subdivide the inequalities into subtour-elimination constraints $x(W) \leq |W| - 1$ which we give by $W = \{v_1, v_2, \dots, v_k\}$, $RS = |W| - 1 = k - 1$, and comb inequalities

$$\sum_{i=0}^{k} x(W_i) \leq |W_0| + \sum_{i=1}^{k} (|W_i| - 1) - \tfrac{1}{2}(k+1) =: s(C)$$

which we represent by $W_0, W_1, \dots, W_k$ and $s(C)$.

Run 1:

$$\bar{\bar{Q}}^{120}_{2M} = \Big\{x \in \mathbf{R}^{7140}: \sum_{j<i} x_{ji} + \sum_{i<j} x_{ij} = 2 \text{ for } i = 1, \dots, 120,$$

$$0 \le x_{ij} \le 1 \text{ for } 1 \le i < j \le 120\Big\}.$$

Minimum: 6662.5.

Run 2: Subtour-elimination constraints:

(1) $W = \{100, 52, 33\}$, $RS = 2$;
(2) $W = \{100, 91, 79, 68, 58, 52, 33\}$, $RS = 6$;
(3) $W = \{117, 66, 31\}$, $RS = 2$;
(4) $W = \{118, 49, 13\}$, $RS = 2$;
(5) $W = \{118, 49, 17, 13\}$, $RS = 3$;
(6) $W = \{116, 70, 8\}$, $RS = 2$;
(7) $W = \{34, 26, 4\}$, $RS = 2$;
(8) $W = \{119, 115, 103, 82, 51, 23, 11, 9, 3, 2\}$, $RS = 9$;
(9) $W = \{97, 95, 12\}$, $RS = 2$;
(10) $W = \{67, 62, 37\}$, $RS = 2$;
(11) $W = \{104, 99, 84, 36, 35, 10\}$, $RS = 5$;
(12) $W = \{104, 99, 84, 36, 35, 10, 6\}$, $RS = 6$;
(13) $W = \{119, 116, 115, 114, 112, 110, 109, 106, 104, 103, 102, 101, 99, 97, 95, 93, 89, 88, 84, 83, 82, 80, 77, 73, 71, 70, 67, 64, 63, 62, 57, 55, 54, 53, 51, 48, 47, 39, 37, 36, 35, 34, 27, 26, 23, 21, 12, 11, 10, 9, 8, 6, 5, 4, 3, 2\}$, $RS = 55$.

Minimum 6883.5.

Run 3: Subtour-elimination constraints:

(14) $W = \{114, 112, 110, 106, 104, 102, 101, 99, 89, 84, 83, 73, 67, 62, 57, 55, 48, 47, 37, 36, 35, 10, 6\}$, $RS = 22$;
(15) $W = \{109, 97, 95, 93, 88, 77, 64, 63, 53, 39, 27, 21, 12, 5\}$, $RS = 13$;
(16) $W = \{109, 97, 95, 93, 88, 77, 64, 63, 53, 39, 21, 12, 5\}$, $RS = 12$;
(17) $W = \{120, 92, 32, 30, 29, 28\}$, $RS = 5$;
(18) $W = \{120, 92, 32, 30, 29\}$, $RS = 4$;
(19) $W = \{105, 74, 72, 40\}$, $RS = 3$;
(20) $W = \{105, 72, 40\}$, $RS = 2$;
(21) $W = \{117, 85, 66, 31, 22\}$, $RS = 4$;
(22) $W = \{117, 85, 66, 31, 22, 18\}$, $RS = 5$;
(23) $W = \{100, 91, 79, 68, 58, 52, 43, 33\}$, $RS = 7$.

Comb inequalities:

(24) $W_0 = \{85, 22, 18\}$, $W_1 = \{117, 85\}$, $W_2 = \{66, 22\}$, $W_3 = \{19, 18\}$, $s(C) = 4$;
(25) $W_0 = \{120, 92, 28\}$, $W_1 = \{120, 29\}$, $W_2 = \{92, 32\}$, $W_3 = \{45, 28\}$, $s(C) = 4$;

(26) $W_0 = \{104, 99, 10\}$, $W_1 = \{104, 36\}$, $W_2 = \{99, 62\}$, $W_3 = \{35, 10\}$, $s(C) = 4$;
(27) $W_0 = \{53, 27, 5\}$, $W_1 = \{80, 27\}$, $W_2 = \{64, 53\}$, $W_3 = \{63, 5\}$, $s(C) = 4$;
(28) $W_0 = \{89, 55, 48, 47\}$, $W_1 = \{89, 55, 6\}$, $W_2 = \{102, 48\}$, $W_3 = \{71, 47\}$, $s(C) = 6$.

Minimum: 6912.5.

Run 4: Subtour-elimination constraints:

(29) $W = \{109, 97, 95, 93, 88, 77, 64, 53, 21, 12\}$, $RS = 9$;
(30) $W = \{119, 115, 114, 112, 110, 109, 106, 104, 103, 102, 101, 99, 97, 95, 93, 89, 88, 84, 83, 82, 80, 77, 73, 71, 67, 64, 63, 62, 57, 55, 53, 51, 48, 47, 39, 37, 36, 35, 27, 26, 23, 21, 12, 11, 10, 9, 6, 5, 4, 3, 2\}$, $RS = 50$;
(31) $W = \{113, 107, 69\}$, $RS = 2$.

Comb inequalities:

(32) $W_0 = \{73, 62, 57\}$, $W_1 = \{114, 73\}$, $W_2 = \{83, 57\}$, $W_3 = \{62, 37\}$, $s(C) = 4$;
(33) $W_0 = \{118, 113, 107, 98, 69, 65, 50, 49, 46, 44, 20, 17, 13\}$, $W_1 = \{98, 42\}$, $W_2 = \{68, 65\}$, $W_3 = \{75, 44\}$, $s(C) = 14$;
(34) $W_0 = \{118, 113, 107, 69, 65, 49, 20, 17, 13\}$, $W_1 = \{98, 17\}$, $W_2 = \{68, 65\}$, $W_3 = \{46, 20\}$, $s(C) = 10$;
(35) $W_0 = \{119, 72, 40, 38, 34, 4\}$, $W_1 = \{105, 72, 40\}$, $W_2 = \{119, 103\}$, $W_3 = \{116, 34\}$, $W_4 = \{38, 7\}$, $W_5 = \{26, 4\}$, $s(C) = 9$.

Minimum: 6918.75.

Run 5: Subtour-elimination constraints:

(36) $W = \{93, 64, 53\}$, $RS = 2$;
(37) $W = \{118, 98, 49, 42, 17, 13\}$, $RS = 5$;
(38) $W = \{104, 99, 89, 84, 55, 36, 35, 10, 6\}$, $RS = 8$;
(39) $W = \{114, 112, 110, 106, 104, 99, 89, 84, 83, 73, 67, 62, 57, 55, 37, 36, 35, 10, 6\}$, $RS = 18$.

Comb inequalities:

(40) $W_0 = \{118, 98, 50, 49, 46, 42, 41, 20, 17, 13\}$, $W_1 = \{50, 46, 44\}$, $W_2 = \{56, 41\}$, $W_3 = \{107, 20\}$, $s(C) = 12$;
(41) $W_0 = \{118, 113, 107, 98, 69, 65, 50, 49, 46, 44, 42, 41, 20, 17, 13\}$, $W_1 = \{75, 44\}$, $W_2 = \{68, 65\}$, $W_3 = \{56, 41\}$, $s(C) = 16$;
(42) $W_0 = \{119, 116, 103, 72, 71, 70, 54, 40, 38, 34, 26, 8, 4\}$, $W_1 = \{119, 115, 103, 82, 51, 23, 11, 9, 3, 2\}$, $W_2 = \{105, 72, 40\}$, $W_3 = \{71, 47\}$, $W_4 = \{90, 54\}$, $W_5 = \{38, 7\}$, $s(C) = 24$;
(43) $W_0 = \{119, 116, 105, 103, 72, 71, 70, 54, 40, 38, 34, 26, 8, 4\}$, $W_1 = \{119, 115, 109, 103, 97, 95, 93, 88, 82, 80, 77, 64, 63, 53, 51, 39, 27, 23, 21, 12, 11, 9, 5, 3, 2\}$, $W_2 = \{105, 74\}$, $W_3 = \{71, 47\}$, $W_4 = \{90, 54\}$, $W_5 = \{38, 7\}$, $s(C) = 39$;
(44) $W_0 = \{71, 54, 8\}$, $W_1 = \{116, 70, 8\}$, $W_2 = \{71, 47\}$, $W_3 = \{90, 54\}$, $s(C) = 5$.

Minimum: 6928.

Run 6: Subtour-elimination constraints:

(45) $W = \{118, 113, 107, 105, 98, 87, 75, 74, 72, 69, 65, 56, 50, 49, 46, 44, 42, 41, 40, 38, 20, 17, 14, 13, 7\}$, $RS = 24$;

(46) $W = \{120, 118, 117, 113, 108, 107, 105, 100, 98, 94, 92, 91, 87, 86, 85, 81, 79, 78, 76, 75, 74, 72, 69, 68, 66, 65, 61, 59, 58, 56, 52, 50, 49, 46, 45, 44, 43, 42, 41, 40, 38, 33, 32, 31, 30, 29, 28, 25, 22, 20, 19, 18, 17, 16, 15, 14, 13, 7, 1\}$, $RS = 58$.

Comb inequalities:

(47) $W_0 = \{68, 65, 13\}$, $W_1 = \{91, 68\}$, $W_2 = \{69, 65\}$, $W_3 = \{49, 13\}$, $s(C) = 4$;

(48) $W_0 = \{90, 71, 54, 8\}$, $W_1 = \{96, 90, 54\}$, $W_2 = \{116, 70, 8\}$, $W_3 = \{71, 47\}$, $s(C) = 7$;

(49) $W_0 = \{89, 55, 48, 47\}$, $W_1 = \{104, 99, 89, 84, 55, 36, 35, 10, 6\}$, $W_2 = \{102, 48\}$, $W_3 = \{71, 47\}$, $s(C) = 12$;

(50) $W_0 = \{118, 113, 108, 107, 100, 91, 79, 69, 68, 65, 58, 52, 49, 43, 33, 20, 17, 13\}$, $W_1 = \{108, 25\}$, $W_2 = \{98, 17\}$, $W_3 = \{46, 20\}$, $s(C) = 19$.

Minimum: 6935.3.

Run 7: Subtour-elimination constraints:

(51) $W = \{71, 47, 26\}$, $RS = 2$;

(52) $W = \{94, 86, 81, 78\}$, $RS = 3$;

(53) $W = \{119, 103, 82, 23, 9, 3\}$, $RS = 5$.

Comb inequalities:

(54) $W_0 = \{69, 68, 65\}$, $W_1 = \{113, 69\}$, $W_2 = \{91, 68\}$, $W_3 = \{65, 13\}$, $s(C) = 4$;

(55) $W_0 = \{69, 68, 65\}$, $W_1 = \{113, 107, 68\}$, $W_2 = \{91, 68\}$, $W_3 = \{118, 65, 49, 13\}$, $s(C) = 7$;

(56) $W_0 = \{114, 112, 106, 104, 99, 84, 83, 73, 67, 62, 57, 37, 36, 35, 10\}$, $W_1 = \{114, 83, 73, 63, 57, 39, 5\}$, $W_2 = \{112, 110\}$, $W_3 = \{84, 6\}$, $s(C) = 21$;

(57) $W_0 = \{118, 113, 107, 98, 69, 68, 65, 50, 49, 46, 44, 42, 20, 17, 13\}$, $W_1 = \{91, 68\}$, $W_2 = \{75, 44\}$, $W_3 = \{42, 41\}$, $s(C) = 16$;

(58) $W_0 = \{119, 116, 115, 114, 112, 111, 110, 109, 106, 104, 103, 102, 101, 99, 97, 96, 95, 93, 90, 89, 88, 84, 83, 82, 80, 77, 73, 72, 71, 70, 67, 64, 63, 62, 60, 57, 55, 54, 53, 51, 48, 47, 40, 39, 38, 37, 36, 35, 34, 27, 26, 24, 23, 21, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$, $W_1 = \{105, 74, 72, 40\}$, $W_2 = \{60, 16\}$, $W_3 = \{56, 7\}$, $s(C) = 68$;

(59) $W_0 = \{115, 93, 21, 2\}$, $W_1 = \{119, 115, 103, 82, 51, 23, 11, 9, 3, 2\}$, $W_2 = \{93, 64, 53\}$, $W_3 = \{109, 21\}$, $s(C) = 14$.

Minimum: 6937.222.

Run 8: Comb inequalities:

(60) $W_0 = \{34, 26, 4\}$, $W_1 = \{119, 4\}$, $W_2 = \{116, 34\}$, $W_3 = \{71, 26\}$, $s(C) = 4$;

(61) $W_0 = \{53, 27, 5\}$, $W_1 = \{93, 64, 53\}$, $W_2 = \{80, 27\}$, $W_3 = \{63, 5\}$, $s(C) = 5$;

(62) $W_0 = \{115, 103, 93, 82, 51, 23, 21, 11, 9, 2\}$, $W_1 = \{119, 103\}$, $W_2 = \{109, 21\}$, $W_3 = \{93, 64, 53\}$, $W_4 = \{82, 3\}$, $W_5 = \{51, 13\}$, $s(C) = 13$;

(63) $W_0 = \{115, 93, 82, 51, 23, 21, 11, 9, 2\}$, $W_1 = \{103, 23, 9\}$, $W_2 = \{109, 21\}$, $W_3 = \{93, 64, 53\}$, $W_4 = \{82, 3\}$, $W_5 = \{51, 13\}$, $s(C) = 13$;

(64) $W_0 = \{103, 51, 23, 11, 9\}$, $W_1 = \{119, 103\}$, $W_2 = \{115, 11\}$, $W_3 = \{51, 13\}$, $s(C) = 6$;

(65) $W_0 = \{89, 55, 48, 47\}$, $W_1 = \{114, 112, 110, 106, 104, 99, 89, 84, 83, 73, 67, 62, 57, 55, 37, 36, 35, 10, 6\}$, $W_2 = \{102, 48\}$, $W_3 = \{71, 47, 26\}$, $s(C) = 23$;

(66) $W_0 = \{112, 106, 104, 99, 84, 67, 62, 37, 36, 35, 10\}$, $W_1 = \{114, 106, 83, 73, 67, 62, 57, 37\}$, $W_2 = \{112, 110\}$, $W_3 = \{84, 6\}$, $s(C) = 18$;

(67) $W_0 = \{106, 104, 99, 67, 62, 37\}$, $W_1 = \{114, 106, 83, 73, 67, 62, 57, 37\}$, $W_2 = \{104, 36\}$, $W_3 = \{99, 10\}$, $s(C) = 13$.

Minimum: 6939.5.

Run 9: Comb inequalities:

(68) $W_0 = \{72, 56, 40, 38, 7\}$, $W_1 = \{105, 74, 72, 40\}$, $W_2 = \{119, 103, 38\}$, $W_3 = \{56, 41\}$, $s(C) = 9$;

(69) $W_0 = \{119, 116, 105, 103, 74, 72, 70, 56, 40, 38, 34, 26, 8, 7, 4\}$, $W_1 = \{119, 115, 103, 82, 51, 23, 11, 9, 3, 2\}$, $W_2 = \{87, 74\}$, $W_3 = \{71, 26\}$, $W_4 = \{56, 41\}$, $W_5 = \{54, 8\}$, $s(C) = 25$;

(70) $W_0 = \{112, 110, 89, 84, 55, 48, 47, 36, 6\}$, $W_1 = \{112, 106\}$, $W_2 = \{104, 36\}$, $W_3 = \{102, 48\}$, $W_4 = \{84, 35\}$, $W_5 = \{71, 47\}$, $s(C) = 11$;

(71) $W_0 = \{112, 110, 106, 104, 99, 89, 84, 67, 62, 55, 48, 47, 37, 36, 35, 10, 6\}$, $W_1 = \{102, 48\}$, $W_2 = \{83, 67\}$, $W_3 = \{71, 47\}$, $s(C) = 18$;

(72) $W_0 = \{118, 117, 113, 108, 107, 100, 98, 91, 85, 79, 75, 69, 68, 66, 65, 58, 52, 50, 49. 46, 44, 43, 42, 41, 33, 31, 25, 22, 20, 19, 18, 17, 13\}$, $W_1 = \{81, 22\}$, $W_2 = \{75, 14\}$, $W_3 = \{56, 41\}$, $s(C) = 34$.

Minimum: 6940.38281.

Run 10. Subtour-elimination constraints:

(73) $W = \{117, 113, 108, 107, 100, 91, 85, 79, 69, 68, 66, 65, 58, 52, 43, 33, 31, 25, 22, 19, 18\}$, $RS = 20$.

Comb inequalities:

(74) $W_0 = \{116, 70, 34, 26, 8, 4\}$, $W_1 = \{116, 70, 54, 8\}$, $W_2 = \{119, 4\}$, $W_3 = \{71, 26\}$, $s(C) = 9$;

(75) $W_0 = \{114, 106, 83, 73, 67, 62, 57, 37\}$, $W_1 = \{112, 106\}$, $W_2 = \{99, 62\}$, $W_3 = \{83, 39\}$, $s(C) = 9$;

(76) $W_0 = \{118, 117, 113, 108, 107, 100, 98, 94, 91, 87, 86, 85, 81, 79, 75, 69, 68, 66, 65, 58, 56, 52, 50, 49, 46, 44, 43, 42, 41, 33, 31, 25, 22, 20, 19, 18, 17, 14, 13\}$, $W_1 = \{120, 94, 92, 86, 81, 78, 45, 32, 30, 29, 28, 15\}$, $W_2 = \{87, 74\}$, $W_3 = \{56, 41, 7\}$, $s(C) = 51$.

Minimum: 6940.81641.

Run 11: Comb inequalities:

(77) $W_0 = \{108, 100, 91, 79, 69, 68, 65, 58, 52, 43, 33\}$, $W_1 = \{113, 107, 69\}$,

$W_2 = \{108, 25\}$, $W_3 = \{65, 13\}$, $s(C) = 13$;
(78) $W_0 = \{120, 92, 76, 59, 32, 30, 29, 15\}$, $W_1 = \{92, 28\}$, $W_2 = \{81, 15\}$, $W_3 = \{76, 1\}$, $s(C) = 9$;
(79) $W_0 = \{94, 87, 86, 78, 75, 44, 14\}$, $W_1 = \{94, 81\}$, $W_2 = \{87, 74\}$, $W_3 = \{78, 45\}$, $s(C) = 8$;
(80) $W_0 = \{116, 70, 34, 26, 8, 4\}$, $W_1 = \{119, 4\}$, $W_2 = \{71, 47, 26\}$, $W_3 = \{54, 8\}$, $s(C) = 8$;
(81) $W_0 = \{114, 112, 106, 104, 99, 84, 83, 73, 67, 62, 57, 37, 36, 35, 10\}$, $W_1 = \{112, 110\}$, $W_2 = \{84, 6\}$, $W_3 = \{83, 39\}$, $s(C) = 16$;
(82) $W_0 = \{108, 100, 91, 79, 69, 68, 65, 58, 52, 43, 33, 13\}$, $W_1 = \{118, 49, 13\}$, $W_2 = \{113, 107, 69\}$, $W_3 = \{108, 25\}$, $s(C) = 15$;
(83) $W_0 = \{120, 92, 76, 59, 45, 32, 30, 29, 28, 15\}$, $W_1 = \{81, 15\}$, $W_2 = \{78, 45\}$, $W_3 = \{76, 1\}$, $s(C) = 11$;
(84) $W_0 = \{94, 87, 86, 81, 78, 75, 44, 41, 22\}$, $W_1 = \{117, 85, 66, 31, 22\}$, $W_2 = \{78, 45\}$, $W_3 = \{87, 74\}$, $s(C) = 13$;
(85) $W_0 = \{120, 92, 45, 32, 28\}$, $W_1 = \{120, 29\}$, $W_2 = \{78, 45\}$, $W_3 = \{32, 30\}$, $s(C) = 6$;
(86) $W_0 = \{116, 105, 72, 71, 70, 60, 54, 47, 40, 38, 34, 26, 24, 8, 7, 4\}$, $W_1 = \{105, 87, 74, 72, 56, 41, 40, 38, 7\}$, $W_2 = \{114, 112, 110, 106, 104, 102, 101, 99, 89, 84, 83, 73, 71, 67, 62, 57, 55, 48, 47, 37, 36, 35, 10, 6\}$, $W_3 = \{96, 90, 54\}$, $W_4 = \{60, 16\}$, $W_5 = \{111, 24\}$, $s(C) = 48$;
(87) $W_0 = \{73, 67, 57\}$, $W_1 = \{67, 62, 37\}$, $W_2 = \{114, 73\}$, $W_3 = \{83, 57\}$, $s(C) = 5$;
(88) $W_0 = \{83, 73, 67, 57\}$, $W_1 = \{83, 39\}$, $W_2 = \{114, 73\}$, $W_3 = \{67, 37\}$, $s(C) = 5$.

Minimum: 6941.18359.

Run 12: Comb inequalities:
(89) $W_0 = \{105, 74, 72, 56, 40, 38, 7\}$, $W_1 = \{119, 103, 38, 23, 9\}$, $W_2 = \{87, 74\}$, $W_3 = \{56, 41\}$, $s(C) = 11$;
(90) $W_0 = \{118, 117, 113, 108, 107, 100, 98, 94, 91, 87, 86, 85, 81, 79, 78, 75, 69, 68, 66, 65, 58, 52, 50, 49, 46, 44, 43, 42, 41, 33, 31, 25, 22, 20, 19, 18, 17, 14, 13\}$, $W_1 = \{87, 74\}$, $W_2 = \{92, 78, 45, 28\}$, $W_3 = \{56, 41\}$, $s(C) = 42$;
(91) $W_0 = \{116, 71, 70, 54, 34, 26, 8, 4\}$, $W_1 = \{119, 4\}$, $W_2 = \{90, 54\}$, $W_3 = \{71, 47\}$, $s(C) = 9$;
(92) $W_0 = \{105, 74, 72, 56, 40, 38, 7\}$, $W_1 = \{87, 74\}$, $W_2 = \{56, 41\}$, $W_3 = \{119, 103, 38, 23, 9\}$, $s(C) = 11$;
(93) $W_0 = \{108, 69, 68, 65, 43, 13\}$, $W_1 = \{118, 49, 13\}$, $W_2 = \{113, 69\}$, $W_3 = \{108, 25\}$, $W_4 = \{91, 68\}$, $W_5 = \{79, 43\}$, $s(C) = 9$.

Minimum: 6941.5.

Run 13: Comb inequalities:
(94) $W_0 = \{109, 93, 21\}$, $W_1 = \{109, 88\}$, $W_2 = \{93, 64, 53\}$, $W_3 = \{115, 21\}$, $s(C) = 5$;

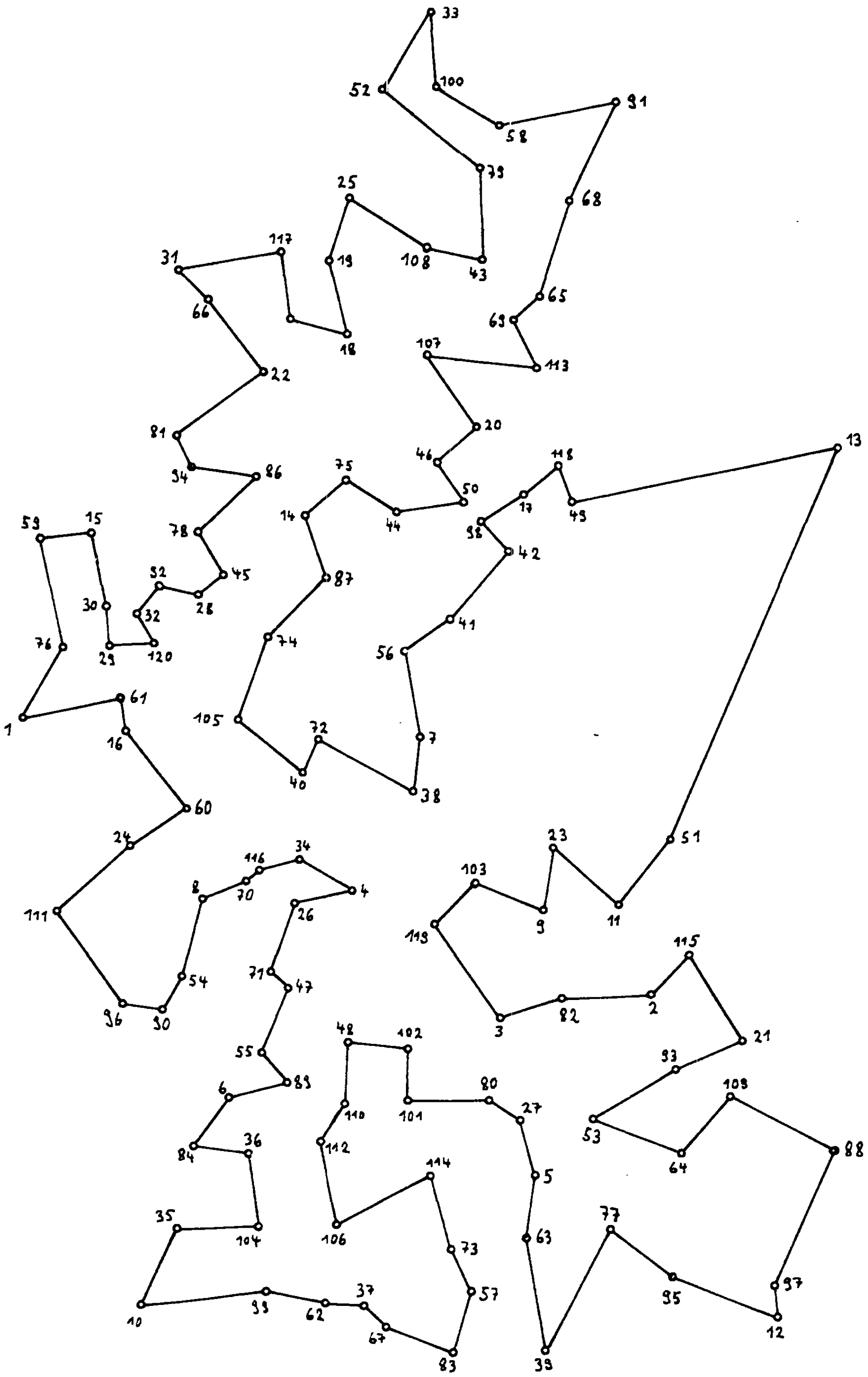

Fig. 1. The shortest roundtrip through 120 German cities. The length of this tour is 6942 km.

(95) $W_0 = \{115, 109, 93, 21, 2\}$, $W_1 = \{119, 115, 103, 82, 51, 23, 11, 9, 3, 2\}$, $W_2 = \{109, 88\}$, $W_3 = \{93, 64, 53\}$, $s(C) = 15$;

(96) $W_0 = \{119, 115, 114, 109, 103, 101, 93, 83, 82, 80, 77, 73, 64, 63, 57, 53, 51, 39, 27, 23, 21, 11, 9, 5, 3, 2\}$, $W_1 = \{114, 112, 110, 106, 104, 99, 89, 84, 83, 73, 67, 62, 57, 55, 37, 36, 35, 10, 6\}$, $W_2 = \{109, 88\}$, $W_3 = \{102, 101\}$, $W_4 = \{95, 77\}$, $W_5 = \{51, 13\}$, $s(C) = 45$.

Minimum: 6942.

The optimal solution of the 13th LP-run was the incidence vector of a tour of length 6942 km, hence this vector represents the shortest roundtrip through the 120 cities of Germany. A graphical representation of this optimal tour is given in Fig. 1.

We have calculated the number of non-equivalent facets of Q_T^{120} which are given by Theorem 1. This number is exactly

26792549076063489375554618994821987399578869037768
70780484651943295772470308627340156321170880759399
86913459296483643418942533445648036828825541887362
42799920969079258554704177287.

Considering the fact that the trivial inequalities, the subtour-elimination constraints, and the comb inequalities are by far not all facets of Q_T^{120}, it is quite surprising that only the trivial inequalities and an additional 96 inequalities out of these 10^{179} inequalities and no other were needed to find an optimal tour and prove optimality.

It can be seen from the sequence of minimum values of the 13 linear programs that the increase of these values is considerably large during the first runs (221 km after the second run, a total of 273 km after the first six runs) but it took a further seven runs to beat the last 6 km. This fact was observed in several other experiments of this kind. Due to limited time for the solution procedure or due to possible incorrectness of the data a near optimal solution is often good enough for practical purposes. If for instance a tour at most 2% off optimality would be considered as satisfactory we could have stopped after the second LP-run having a lower bound of 6883.5 km and a "good" (heuristically found) tour of length 7011 km.

For LP-problems of our size (7140 variables, 120 equations, 7140 upper and lower bounds, 96 inequalities) advanced LP-codes and large computers are indispensable. We have used the LP-program of the MPSX-package of IBM, and all runs were executed on the IBM-computer 370/168 of the Rechenzentrum der Universität Bonn. To simplify the data input several auxiliary routines were written: one program that generated the equation system $Ax = 2e_{120}$ and the upper and lower bounds in the input format required by MPSX, another that after each run saved the whole constraint system on tape, a third program that generated comb inequalities and subtour-elimination constraints in MPSX input format if these were given as in our cutting plane list above and which added these inequalities to the present constraint system.

In each run the heuristically obtained tour of length 7011 km was given as partial LP-basis which was then completed by MPSX to a full basis and used as starting basis for the linear program. We found that such a device can reduce the computation times considerably.

The CPU-times needed for the solution of the thirteen programs ranged between 30 seconds and 2 minutes, the number of pivot-operations was between 100 and 1000, both CPU-times and pivot-operations increased slightly but not monotonically with the number of additional inequalities. The last run for instance was executed in 1.76 CPU-minutes, and 714 pivot operations were necessary to obtain the optimal solution.

Considering these moderately sized CPU-times it seems possible that even larger travelling salesman problems can be solved using ordinary linear programming codes provided that the user is capable of identifying violated inequalities and has the patience to solve the problem interactively. Clearly, the author does not suggest the method presented here as a standard method for solving travelling salesman problems. The main purpose of this note is to show the practical usefulness of the theoretical research done in polyhedral combinatorics and to give an example showing that we already have tools to solve very large real world problems optimally. As is to be expected an optimal solution requires a lot of effort but the chances of finding the shortest tour are quite good.

Further research in applying the results on the facetial structure of polytopes associated with hard (NP-complete) combinatorial optimization problems should go in the direction of automatizing the interactive procedure used above. A first step with respect to the TSP was done with considerable success by Padberg and Hong [15]. If good methods for solving some of the problems encountered by them can be found, it seems likely that we will be able to attack truly large problems by sophisticated combinations of heuristics, cutting plane methods and branch and bound techniques.

Appendix

List of the 120 German cities contained in the distance table in [4]:

1	Aachen	2	Amberg	3	Ansbach	4	Aschaffenburg
5	Augsburg	6	Baden-Baden	7	Bad Hersfeld	8	Bad Kreuznach
9	Bamberg	10	Basel	11	Bayreuth	12	Berchtesgaden
13	Berlin	14	Bielefeld	15	Bocholt	16	Bonn
17	Braunschweig	18	Bremen	19	Bremerhaven	20	Celle
21	Cham	22	Cloppenburg	23	Coburg	24	Cochem
25	Cuxhaven	26	Darmstadt	27	Donauwörth	28	Dortmund
29	Düsseldorf	30	Duisburg	31	Emden	32	Essen
33	Flensburg	34	Frankfurt	35	Freiburg	36	Freudenstadt
37	Friedrichshafen	38	Fulda	39	Garm.-Partenk.	40	Gießen

Appendix. *(Continued)*

41	Göttingen	42	Goslar	43	Hamburg	44	Hameln
45	Hamm	46	Hannover	47	Heidelberg	48	Heilbronn
49	Helmstedt	50	Hildesheim	51	Hof	52	Husum
53	Ingolstadt	54	Kaiserslautern	55	Karlsruhe	56	Kassel
57	Kempten	58	Kiel	59	Kleve	60	Koblenz
61	Köln	62	Konstanz	63	Landsberg	64	Landshut
65	Lauenburg	66	Leer	67	Lindau	68	Lübeck
69	Lüneburg	70	Mainz	71	Mannheim	72	Marburg
73	Memmingen	74	Meschede	75	Minden	76	Mönchengladb.
77	München	78	Münster	79	Neumünster	80	Nördlingen
81	Nordhorn	82	Nürnberg	83	Oberstdorf	84	Offenburg
85	Oldenburg	86	Osnabrück	87	Paderborn	88	Passau
89	Pforzheim	90	Pirmasens	91	Puttgarden	92	Recklinghausen
93	Regensburg	94	Rheine	95	Rosenheim	96	Saarbrücken
97	Salzburg	98	Salzgitter, Bad	99	Schaffhausen	100	Schleswig
101	Schwäb. Gmünd	102	Schwäb. Hall	103	Schweinfurt	104	Schwenningen
105	Siegen	106	Sigmaringen	107	Soltau	108	Stade
109	Straubing	110	Stuttgart	111	Trier	112	Tübingen
113	Uelzen	114	Ulm	115	Weiden	116	Wiesbaden
117	Wilhelmshaven	118	Wolfsburg	119	Würzburg	120	Wuppertal

The numbers in Fig. 1 correspond to the numbering of the cities in the list above. Readers interested in trying their TSP-code or -heuristic on the 120-city problem solved in this paper can obtain a card deck with the road distances between the above listed cities from the author.

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