Chapter 8

Cardinality Homogeneous Set Systems, Cycles in Matroids, and Associated Polytopes

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Abstract. A subset $\mathcal{C}$ of the power set of a finite set E is called cardinality homogeneous if, whenever $\mathcal{C}$ contains some set F, $\mathcal{C}$ contains all subsets of E of cardinality $|F|$. Examples of such set systems $\mathcal{C}$ are the sets of all even or of all odd cardinality subsets of E, or, for each uniform matroid, its set of circuits and its set of cycles. With each cardinality homogeneous set system $\mathcal{C}$, we associate the polytope $P(\mathcal{C})$, the convex hull of the incidence vectors of all sets in $\mathcal{C}$. We provide a complete and nonredundant linear description of $P(\mathcal{C})$. We show that a greedy algorithm optimizes any linear function over $P(\mathcal{C})$; we construct, by a dual greedy procedure, an explicit optimum solution of the dual linear program; and we describe a polynomial time separation algorithm for the class of polytopes of type $P(\mathcal{C})$.

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8.1 Introduction

Cycles in matroids can be viewed as far-reaching common generalizations of Eulerian subgraphs and cuts of a graph. From an optimization point of view it is of interest to understand the polytopes naturally associated with cycles.

The aim is to develop linear programming techniques for the solution of weighted cycle optimization problems. This chapter contributes to this issue by investigating a class of polytopes, namely, the polytopes associated with cardinality homogeneous set systems,

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which properly contains, e.g., the class of cycle and circuit polytopes associated with uniform matroids.

8.2 Matroids

Good books on matroid theory are [6] and [11]. We follow their notation and terminology to a large extent.

Let E be a finite set. We usually assume that $E = \{1, \ldots, n\}$, $n \geq 1$. A subset $\mathcal{I}$ of the power set 2^E of E is called an *independence system* if $\emptyset \in \mathcal{I}$ and if, whenever $I \in \mathcal{I}$, every subset of I also belongs to $\mathcal{I}$. An independence system $\mathcal{I}$ is called a *matroid* if, whenever $I, J \in \mathcal{I}$ with $|I| < |J|$, there is an element $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$. We also write $M = (E, \mathcal{I})$ to give a matroid a name and stress that we are dealing with a matroid $\mathcal{I}$ on the ground set E.

Every set in $\mathcal{I}$ is called *independent* and every set in $2^E \setminus \mathcal{I}$ is said to be *dependent.* The minimal dependent subsets of E are called *circuits* (such sets do not properly contain other dependent sets). Every subset of E that is the disjoint union of circuits is called a *cycle.* For every set $F \subseteq E$, a set $B \subseteq E$ is called a *basis* of F if $B \subseteq F$, $B \in \mathcal{I}$, and F does not contain an independent set B' properly containing B, i.e., B is a maximal independent subset of F.

If $\mathcal{B}$ is the set of bases of the ground set E of a matroid $M = (E, \mathcal{I})$, then $\mathcal{B}^* := \{E \setminus B | B \in \mathcal{B}\}$ is the set of bases of another matroid, denoted by $M^* = (E, \mathcal{I}^*)$ and called the matroid *dual* to M. By construction we have $M^{**} = M$. It is customary to call the bases, circuits, and cycles of M^* the *cobases, cocircuits,* and *cocycles* of M.

It is well known that, for any graph $G = (V, E)$, the set of edgesets of its forests forms the system of independent sets of a matroid, the so-called graphic matroid, denoted by $M(G)$. The matroid dual to a graphic matroid is called *cographic* and is denoted by $M(G)^*$. The circuits of a graphic matroid are the edgesets of the *circuits* of the underlying graph G. The cycles are the (not necessarily connected) *Eulerian subgraphs* of G, i.e., the edgesets of all subgraphs with nodes of even degree. The cycles of $M(G)^*$ are the cuts of G, i.e., edgesets of the form $\delta(W) = \{ij \in E \mid i \in W, j \in V \setminus W\}$. The circuits of a cographic matroid are the edgesets of minimal cuts.

Another nice class of matroids is composed of *representable* (or matric) matroids. We choose a field F and an $m \times n$ matrix A with entries from F. A set $I \subseteq E = \{1, \ldots, n\}$ is called independent if the submatrix of A consisting of the columns indexed by I has rank $|I|$, i.e., if the column vectors $A_{.j}$, $j \in I$, are linearly independent in the m-dimensional vector space over F. A matroid that is isomorphic to a matroid of this type is called *representable over F.* A matroid representable over the two-element field **GF(2)** is called *binary.* If M is representable over F, then this also holds for its dual matroid M^*.

There are many equivalent characterizations of binary matroids; see [11], Chapter 10. For instance, we have the following theorem.

Theorem 8.1. *The following statements about a matroid M are equivalent.*

(i) *M is binary.*
(ii) *For any circuit C and any cocircuit C^*, $|C \cap C^*|$ is even.*
(iii) *Every cycle of M is the symmetric difference of distinct circuits of M.*

Graphic matroids (and therefore also cographic matroids) are representable over any field and, hence, they are binary.

One, in many respects, very simple class of matroids comprises the uniform matroids. They are defined as follows. We are given integers $1 \le k \le n$. The ground set is $E = \{1, \dots, n\}$ and every subset with at most k elements is declared to be independent. This matroid is called the *uniform matroid on n elements of rank k* and is denoted by $U_{k,n}$. It has $\binom{n}{k}$ bases (the sets of size k) and $\binom{n}{k+1}$ circuits (the sets of size $k+1$). The cycles of $U_{k,n}$ are the sets of cardinality $i(k+1)$, $0 \le i \le \lfloor \frac{n}{k+1} \rfloor$.

8.3 Cycle Polytopes

Polyhedral combinatorics deals with the geometric description of combinatorial problems. Instead of solving a combinatorial problem directly, one associates a polytope with the problem and tries to solve the combinatorial problem as a linear program over this polytope. Two prominent examples are the Chinese postman and the max-cut problems. With respect to these problems, the approach works as follows.

Given a graph $G = (V, E)$ with weights c_e on the edges $e \in E$, we wish to find an Eulerian subgraph of maximum weight. To do this we define the polytope

$$\mathrm{CP}(G) := conv\{\chi^C \in \mathbb{R}^E \mid C \subseteq E \text{ Eulerian subgraph}\},$$

where $\chi^C = (\chi_e^C)_{e \in E}$ denotes the incidence vector of C with $\chi_e^C = 1$ if $e \in C$ and $\chi_e^C = 0$ otherwise. CP(G) is called the *Chinese postman polytope*. Solving the Chinese postman problem is equivalent to solving the linear programming problem

$$\max\ c^T x, x \in \mathrm{CP}(G).$$

Similarly, given a graph $G = (V, E)$ with weights c_e for all $e \in E$, finding a cut of G with maximum weight is equivalent to maximizing the linear function $c^T x$ over the *cut polytope*

$$\mathrm{CUT}(G) := conv\{\chi^{\delta(W)} \in \mathbb{R}^E \mid W \subseteq V\}.$$

Cut problems have a wide range of applications and arise in various, sometimes disguised, forms. One such different looking but equivalent appearance is quadratic 0/1-programming. The polyhedron arising here is the Boolean quadratic polytope investigated, e.g., in [7].

Recall that Eulerian subgraphs and cuts are cycles of the corresponding graphic and cographic matroids, respectively; i.e., the Chinese postman and the cut polytope are special instances of a *cycle polytope*

$$P(M) := conv\{\chi^C \in \mathbb{R}^E \mid C \text{ is a cycle of } M\},$$

which is the convex hull of the incidence vectors of all cycles of a matroid M on a ground set E.

Guided by the complete characterization of the Chinese postman polytope for all graphs by Edmonds and Johnson [3] and of the cut polytope for graphs not contractible to the complete graph K_5 by Barahona [1] and based on a deep theorem of Seymour [9] characterizing matroids with the "sum of circuits property," Barahona and Grötschel [2] characterized polytopes of certain binary matroids as follows.

Let M be a matroid on E. Consider the systems of inequalities

$$0 \leq x_e \leq 1 \quad \text{for all } e \in E \tag{8.1}$$

and

$$x(F) - x(C \backslash F) \leq |F| - 1 \quad \text{for all cocircuits } C \subseteq E \text{ and all } F \subseteq C, |F| \text{ odd}, \tag{8.2}$$

and define

$$Q(M) := \{x \in \mathbb{R}^E | x \text{ satisfies (8.1) and (8.2)}\}.$$

Because of Theorem 8.1(ii), every incidence vector of a cycle of a binary matroid satisfies (8.1) and (8.2). And if $J \subseteq E$ is not a cycle, there must be, by Theorem 8.1(ii) and (iii), a cocircuit C and an odd subset F of C such that χ^J violates the corresponding inequality of (8.2). Thus, all integral points of $Q(M)$ are incidence vectors of cycles—provided M is binary. The main theorem of [2] is as follows.

Theorem 8.2. *For a binary matroid M, $P(M) = Q(M)$ if and only if M has no F_7^*, R_{10}, and $M(K_5)^*$ minor.*

Here, $M(K_5)^*$ is the cographic matroid of the complete graph on five nodes, F_7^* is the matroid dual to the Fano matroid, and R_{10} is the binary matroid associated with the 5×10 matrix whose columns are the ten 0/1-vectors with three ones and two zeros. A *minor* of a matroid $M = (E, \mathcal{I})$ is a matroid that can be obtained from M by deleting and contracting some elements of E.

A precise description of all the facets of $P(M)$ is given in [2], i.e., a complete and nonredundant characterization of $P(M)$ for this class of binary matroids M. This yields, in particular, complete and nonredundant characterizations of the Chinese postman polytope for any graph [3] and for the cut polytope of all graphs not contractible to K_5 [1].

Grötschel and Truemper [5] have shown, among other things, that one can solve the separation problem for $Q(M)$ for the class of matroids not containing F_7^*; hence by [4], for this class of matroids, one can maximize any linear function over $Q(M)$. This implies that one can maximize over $P(M)$ if M has no F_7^*, R_{10}, $M(K_5)^*$ minor; thus, for this class of binary matroids, the weighted cycle problem can be solved in polynomial time.

It turns out that knowledge about cycles in matroids and the associated polytopes is rather poor for matroids not in the class considered in Theorem 8.2. There is, e.g., a characterization of so-called master polytopes for cycles in binary matroids; see [5]. For another example, the facets of $P(F_7^*)$ are known; but—in contrast to Theorem 8.2—none of the inequalities defining $Q(F_7^*)$ defines a facet of $P(F_7^*)$; see [2]. The situation is even worse in the nonbinary case. Not even a decent integer programming formulation, such as max $c^T x, x \in Q(M) \cap \{0, 1\}^E$ for binary matroids M, is known in this case.

Just as it was worthwhile to investigate a joint generalization of the Chinese postman and the max-cut problems yielding, e.g., a unified description of the associated polytopes, it may be rewarding to better understand cycles of those matroids that are more general than the matroids of Theorem 8.2, in particular, cycles of nonbinary matroids.

Strangely enough, it is not even completely obvious how to generalize the concept of cycle to the nonbinary case. Looking at the proofs, e.g., in [2], it becomes clear that, although cycles are usually defined as disjoint unions of circuits, the (in the binary case) equivalent definition that a cycle is a set that can be obtained from the set of circuits by taking symmetric differences (see Theorem 8.1) is of much greater help in proofs. It turns out that, for nonbinary matroids, this second definition does not lead to anything interesting in general. It is also worth noting that condition (ii) of Theorem 8.1 is the one that yields the so-called cocircuit inequalities (8.2), which provide an integer programming formulation and enable Theorem 8.2. This condition is not available in the nonbinary case. Is there a condition that can replace it?

To leave the class of binary matroids, there is a wonderful excluded minor theorem of Tutte [10] that, as one might hope, could lead the way.

Theorem 8.3. *A matroid is binary if and only if it has no minor isomorphic to* $U_{2,4}$.

This result shows that all uniform matroids are nonbinary except for $U_{1,n}$, $n \geq 1$, and $U_{2,3}$. It also suggests that investigating the cycles of uniform matroids may provide some polyhedral insight.

The cycles of $U_{2,4}$ are its circuits, which are the four sets of size three, and the empty set. The convex hull of the corresponding five points $(0,0,0,0)$, $(0,1,1,1)$, $(1,0,1,1)$, $(1,1,0,1)$, $(1,1,1,0)$ in $\mathbb{R}^4$ is a simplex defined by the inequalities

$$
\begin{aligned}
-x_1 - x_2 - x_3 + 2x_4 &\leq 0,\\
-x_1 - x_2 + 2x_3 - x_4 &\leq 0,\\
-x_1 + 2x_2 - x_3 - x_4 &\leq 0,\\
+2x_1 - x_2 - x_3 - x_4 &\leq 0,\\
+x_1 + x_2 + x_3 + x_4 &\leq 3.
\end{aligned}
$$

Unfortunately, there is not much one can learn from this observation.

8.4 Cardinality Homogeneous Set Systems

The initial proof of a linear characterization of the class of cycle polytopes of uniform matroids became easier by generalizing this result to a more abstract setting. This will be presented here.

Let $E = \{1, \ldots, n\}$ be a finite set. We will assume throughout the paper that $E \neq \emptyset$, i.e., $n \geq 1$. We call a subset $\mathcal{C} \subseteq 2^E$ *cardinality homogeneous* if, whenever $\mathcal{C}$ contains some subset of cardinality k, $0 \leq k \leq n$, then $\mathcal{C}$ contains all subsets of cardinality k.

Example 8.4. The following set systems are cardinality homogeneous.

(i) $\mathcal{C} = 2^E$, the set of all subsets of E;
(ii) $\mathcal{C} = \{F \subseteq E \mid |F| \text{ is even}\}$;
(iii) $\mathcal{C} = \{F \subseteq E \mid |F| \text{ is odd}\}$;
(iv) $\mathcal{C}$ = set of circuits of $U_{k,n}$;
(v) $\mathcal{C}$ = set of cycles of $U_{k,n}$.

To simplify statements and proofs we introduce the following notation. Let $E = \{1, \ldots, n\}$ be given. From now on, $a = (a_1, \ldots, a_m)$ denotes a nonempty sequence of integers such that $a_i \in \{0, 1, \ldots, n\}$ and $0 \le a_1 < a_2 < \ldots < a_m \le n$ holds. We call such a sequence a *cardinality sequence.* We set

$$C(n; a_i) := \{C \in E \mid |C| = a_i\}, i = 1, \ldots, m,$$

$$C(n; a) := C(n; a_1, \ldots, a_m) := \bigcup_{i=1}^{m} C(n; a_i).$$

Clearly, each cardinality homogeneous set system $\mathcal{C}$ is of the form $C(n; a)$ for some ground set $E = \{1, \ldots, n\}$ and some cardinality sequence $a = (a_1, \ldots, a_m)$; thus

$$P(n; a) := P(n; a_1, \ldots, a_m) := conv\{\chi^C \in \mathbb{R}^E \mid C \in C(n; a)\}$$

is a generic member of the class of polytopes associated with cardinality homogeneous set systems. We want to find a system of linear inequalities and equations describing the members of the class of polytopes $P(n; a)$ completely and nonredundantly.

There are some inequalities that are obviously valid for $P(n; a)$:
the *trivial inequalities*

$$0 \le x_j \le 1, \ j = 1, \ldots, n, \tag{8.3}$$

and the *cardinality bounds*

$$a_1 \le x(E) \le a_m, \tag{8.4}$$

where $x(E)$ denotes the sum $\sum_{e \in E} x_e = x_1 + \ldots + x_n$.

We introduce now a new class of inequalities that we call *cardinality-forcing inequalities* (or briefly CF-inequalities). For a given cardinality sequence $a = (a_1, \ldots, a_m)$ set

$$\mathcal{F} := \mathcal{F}(a_1, \ldots, a_m) := \{F \subseteq E \mid a_1 < |F| < a_m, |F| \ne a_j, j = 2, \ldots, m-1\},$$

$$f := f(F) := \max\{j \in \{1, \ldots, m-1\} \mid a_j < |F|\} \text{ for all } F \in \mathcal{F},$$

where $\mathcal{F}$ consists of all sets that are not in $C(n; a)$ and have a number of elements that is between a_1 and a_m. For $F \in \mathcal{F}$, $f(F)$ denotes the index $f \in \{1, \ldots, m\}$ with $a_f < |F| < a_{f+1}$.

For each $F \in \mathcal{F}$, its corresponding CF-inequality, where $f = f(F)$, is the following:

$$\text{CF}_F(x) := \sum_{j \in F}(a_{f+1} - |F|)x_j - \sum_{j \in E \setminus F} (|F| - a_f)x_j \le (a_{f+1} - |F|)a_f =: s(F). \tag{8.5}$$

Proposition 8.5.

(i) *Every CF-inequality is valid for $P(n; a)$.*
(ii) *For every 0/1-vector $y \in \mathbb{R}^E \setminus P(n; a)$ with $a_1 < y(E) < a_m$ there is at least one CF-inequality separating y from $P(n; a)$.*
(iii) *There are $\sum_{i=1}^{m-1} \sum_{k=a_i+1}^{a_{i+1}-1} \binom{n}{k}$ CF-inequalities; i.e., the number of CF-inequalities is, in general, not bounded by a polynomial in n.*
(iv) *CF-inequalities are completely dense; i.e., all coefficients are different from zero.*

Proof.

(iv) The coefficient of a variable x_j, $j \in E$, in a CF-inequality is either $a_{f+1} - |F|$ or $|F| - a_f$. These values are different from zero by definition.

(iii) This follows from simple counting.

(i) Let $F \in \mathcal{F}$, $f = f(F)$, and $S \in C(n; a)$. Substituting the incidence vector χ^S into the left-hand side of the CF-inequality $\text{CF}_F(x) \leq s(F)$ results in

$$\sum_{j \in F}(a_{f+1} - |F|)\chi_j^S - \sum_{j \in E \setminus F}(|F| - a_f)\chi_j^S = (a_{f+1} - |F|)\,|F \cap S| \\ -(|F| - a_f)\,|(E \setminus F) \cap S|.$$

If $|S| \leq a_f$, then $|F \cap S| \leq a_f$ and χ^S obviously does not violate (8.5).
If $|S| > a_f$, then $|S| \geq a_{f+1}$ and hence $|(E \setminus F) \cap S| = |S \setminus F| \geq a_{f+1} - |F|$. Trivially, $|F \cap S| \leq |F| = a_f + |F| - a_f$ and we obtain

$$\begin{aligned}&(a_{f+1} - |F|)\,|F \cap S| - (|F| - a_f)\,|(E \setminus F) \cap S| \\ &\quad \leq (a_{f+1} - |F|)a_f + (a_{f+1} - |F|)(|F| - a_f) - (|F| - a_f)(a_{f+1} - |F|) \\ &\quad = (a_{f+1} - |F|)a_f,\end{aligned}$$

which shows that the incidence vectors of all sets in $C(n; a)$ satisfy (8.5).

(ii) Let $y \in \{0, 1\}^E \setminus P(n; a)$, $a_1 < y(E) < a_m$, be given and let F be the subset of E with $\chi^F = y$. By our choice $F \in \mathcal{F}$. Substituting y into the CF-inequality associated with F yields the value $(a_{f+1} - |F|)\,|F|$ on the left-hand side. This is larger than the right-hand side since $|F| > a_f$; hence y violates the CF-inequality $\text{CF}_F(x) \leq s(F)$ associated with F. □

Given a cardinality sequence $a = (a_1, \ldots, a_m)$, we introduce the polyhedron

$$Q(n; a) := Q(n; a_1, \ldots, a_m) := \{x \in \mathbb{R}^E | x \text{ satisfies (8.3), (8.4), (8.5)}\}.$$

Proposition 8.5(i) yields

$$P(n; a) \subseteq Q(n; a)$$

and Proposition 8.5(ii) together with the cardinality bounds yields

$$P(n; a) = conv\{x \in \{0, 1\}^E | x \in Q(n; a)\}.$$

In other words,

$$\max c^T x, x \in Q(n; a),$$

is a linear programming relaxation of

$$\max c^T x, x \in P(n; a).$$

Our main result is the following.

Theorem 8.6. *For all $E = \{1, \ldots, n\}$ and all cardinality sequences $a = (a_1, \ldots, a_m)$, $P(n; a) = Q(n; a)$.*

We will prove this in several steps and give, moreover, a characterization of all facets of $P(n; a)$.

8.5 A Primal and a Dual Greedy Algorithm

The proof of Theorem 8.6 consists of two algorithms and their analysis. We first state a greedy algorithm that finds, for every objective function c, a feasible solution for max $c^T x$, $x \in P(n; a)$. Then we describe an algorithm that produces a feasible solution of the LP dual to max $c^T x, x \in Q(n; a)$. We then show that the objective function values of the primal and the dual solution are identical. This yields, by a standard argument, that $P(n; a) = Q(n; a)$.

We are given a ground set $E = \{1, \ldots, n\}$, a cardinality sequence $a = (a_1, \ldots, a_m)$, and weights c_j, $j \in E$. We want to find a cardinality homogeneous set of largest weight. We do this with the following heuristic.

Algorithm 8.7 (Primal Greedy Algorithm).

1. Sort the elements of E such that $c_1 \geq c_2 \geq \ldots \geq c_n$.
2. If $c_{a_m} \geq 0$, set $C_g := \{1, \ldots, a_m\}$ and go to 6.
3. If $c_{a_1} \leq 0$, set $C_g := \{1, \ldots, a_1\}$ and go to 6.
4. Otherwise (i.e., $c_{a_m} < 0 < c_{a_1}$), let us define the following integers:
    - p is the largest integer in $\{1, \ldots, n\}$ such that $c_p > 0 \geq c_{p+1}$,
    - q is the index in $\{1, \ldots, m\}$ such that $a_q \leq p < a_{q+1}$,
    - $h := \sum_{j=a_q+1}^{a_{q+1}} c_j$.
5. If $h > 0$, set $C_g := \{1, \ldots, a_{q+1}\}$,
   else $C_g := \{1, \ldots, a_q\}$.
6. Output C_g.

We call C_g the *greedy solution*; χ^{C_g} is a vertex of $P(n; a)$, so its objective function value $c^T \chi^{C_g}$ is a lower bound for max $c^T x, x \in P(n; a)$, which in turn is not larger than the value of its linear programming relaxation, i.e., of the corresponding LP over $Q(n; a)$:

$$\max c^T x$$

$$\begin{aligned} x_j &\leq 1, \ j = 1, \ldots, n, \\ -x(E) &\leq -a_1, \\ x(E) &\leq a_m, \end{aligned}$$

$$\sum_{j \in F} (a_{f+1} - |F|) x_j - \sum_{j \in E \setminus F} (|F| - a_f) x_j \leq (a_{f+1} - |F|) a_f \quad \text{for all } F \in \mathcal{F},$$

$$x_j \geq 0, j = 1, \ldots, n.$$

We denote this LP by $L(n; a; c)$. Let us state the LP dual to $L(n; a; c)$, for which we assume, without loss of generality, that the elements of E are ordered such that $c_1 \geq c_2 \geq \ldots \geq c_n$:

$$\min \sum_{j=1}^{n} u_j - a_1 v + a_m w + \sum_{F \in \mathcal{F}} (a_{f+1} - |F|) a_f y_F$$

$$u_j - v + w + \sum_{\substack{F \in \mathcal{F} \\ j \in F}} (a_{f+1} - |F|)\, y_F - \sum_{\substack{F \in \mathcal{F} \\ j \notin F}} (|F| - a_f)\, y_F \geq c_j, \; j = 1, \ldots, n, \tag{8.6}$$

$$v, w \geq 0,$$
$$u_j \geq 0, \; j = 1, \ldots, n,$$
$$y_F \geq 0, \; F \in \mathcal{F}.$$

We denote this dual LP by $D(n; a; c)$. We call the inequalities (8.6) above *dual CF inequalities.*

If the objective function c satisfies $c_{a_m} \geq 0$ or $c_{a_1} \leq 0$, the optimality of the greedy solution is easy to see.

Remark 8.8. If $c_{a_m} \geq 0$, set $w := c_{a_m}$, $u_j := c_j - c_{a_m}$ for $j = 1, \ldots, a_m$, and set all other variables to zero.

If $c_{a_1} \leq 0$, set $v := -c_{a_1}$, $u_j := c_j - c_{a_1}$ for $j = 1, \ldots, a_1$, and set all other variables to zero.

In both cases, the solution is feasible for $D(n; a; c)$ and the objective function value is equal to the value of the greedy solution C_g.

Let us now assume that the primal greedy algorithm has to enter step 4 and thus that the index q is defined. We will handle this case by discussing three different possibilities: $h = 0$, $h < 0$, and $h > 0$.

Before entering the case distinction, we define a set $\mathcal{F}_0$ that consists of the following subsets of $\mathcal{F}$:

$$F_k := \{1, 2, \ldots, k\}, \; k = a_q + 1, a_q + 2, \ldots, a_{q+1} - 1.$$

We claim that an optimal solution of $L(n; a; c)$ can be found by solving the relaxed LP $L_{\mathcal{F}_0}(n; a; c)$ that is obtained by dropping the cardinality constraints and all CF-inequalities but those coming from the sets $F \in \mathcal{F}_0$. This means that $L_{\mathcal{F}_0}(n; a; c)$ has the following form:

$$\max c^T x$$
$$x_j \leq 1, \; j = 1, \ldots, n,$$
$$\sum_{j=1}^{k} (a_{q+1} - k)\, x_j - \sum_{j=k+1}^{n} (k - a_q)\, x_j \leq (a_{q+1} - k)\, a_q, \; k = a_q + 1, \ldots, a_{q+1} - 1,$$
$$x \geq 0.$$

We point out that the incidence vector χ^{C_g} of the greedy solution C_g satisfies all CF-inequalities associated with sets $F \in \mathcal{F}_0$ with equality.

The dual to this relaxed LP, denoted by $D_{\mathcal{F}_0}(n; a; c)$, is

$$\min \sum_{j=1}^{n} u_j + \sum_{k=a_q+1}^{a_{q+1}-1} (a_{q+1} - k)\, a_q y_{F_k}$$

$$u_j + \sum_{\substack{k=a_q+1\\ k\geq j}}^{a_{q+1}-1} (a_{q+1} - k)\, y_{F_k} - \sum_{\substack{k=a_q+1\\ k<j}}^{a_{q+1}-1} (k - a_q)\, y_{F_k} \geq c_j,\ j = 1, \dots, n,$$

$$\begin{aligned} &u_j \geq 0,\ j = 1, \dots, n,\\ &y_{F_k} \geq 0,\ k = a_q + 1, \dots, a_{q+1} - 1. \end{aligned}$$

We claim that, for objective functions not covered by Remark 8.8 and for which $h = 0$, $D_{\mathcal{F}_0}(n; a; c)$ can be solved as follows.

Algorithm 8.9 (Dual Greedy Algorithm for $h = 0$).

1. For $k = a_q + 1, \dots, a_{q+1} - 1$ set
$$y^*_{F_k} := \frac{c_k - c_{k+1}}{a_{q+1} - a_q}.$$
2. For $j = 1, \dots, a_{q+1}$ set
$$u^*_j := c_j - \sum_{\substack{k=a_q+1\\ k\geq j}}^{a_{q+1}-1} (a_{q+1} - k)\, y^*_{F_k} + \sum_{\substack{k=a_q+1\\ k<j}}^{a_{q+1}-1} (k - a_q) y^*_{F_k}.$$
3. Set all other variables to zero.

We call the solution u^*, y^* defined in Algorithm 8.9 the *dual greedy solution.* Let us state a few observations that follow directly from the definitions.

Remark 8.10.

(a) Since $c_k \geq c_{k+1}$ and $a_{q+1} > a_q$, all values $y^*_{F_k}$ are nonnegative.

(b) Deleting all variables set in step 3 to zero, the dual CF-inequalities for $j = a_{q+1} + 1, \dots, n$ reduce to
$$- \sum_{k=a_q+1}^{a_{q+1}-1} (k - a_q) y_{F_k} \geq c_j.$$
Since $c_j \geq c_{j+1}$, checking whether these inequalities are satisfied by the dual greedy solution, it suffices to prove that
$$- \sum_{k=a_q+1}^{a_{q+1}-1} (k - a_q) y^*_{F_k} \geq c_{a_{q+1}}.$$
This is the case if we can prove that $u^*_{a_{q+1}} = 0$.

(c) Deleting all variables set in step 3 to zero, the dual CF-inequalities for $j = 1, 2, \dots, a_q + 1$ reduce to

$$u_j + \sum_{k=a_q+1}^{a_{q+1}-1} (a_{q+1} - k)\, y_{F_k} \geq c_j.$$

The values u_j^* are set in step 2 of Algorithm 8.9 in such a way that these inequalities are satisfied with equality by the dual greedy solution. Since $c_j \geq c_{j+1}$, to prove that $u_j^* \geq 0$ it remains to show that $u_{a_q+1}^* \geq 0$.

(d) Proving feasibility of the dual greedy solution for $D_{\mathcal{F}_0}(n; a; c)$ reduces to showing that

$$u_j^* \geq 0, \; j = a_q + 1, \dots, a_{q+1}.$$

We will show that, in fact, $u_j^* = 0$, $j = a_q + 1, \dots, a_{q+1}$.

Remark 8.11. If $h = \sum_{j=a_q+1}^{a_{q+1}} c_j = 0$, then

$$u_j^* = \frac{h}{a_{q+1} - a_q} = 0, \; j = a_q + 1, \dots, a_{q+1}.$$

Proof. Let $a_q + 1 \leq j \leq a_{q+1}$.

$$\begin{aligned}
u_j^* &= c_j + \sum_{k=a_q+1}^{j-1} (k - a_q) \frac{c_k - c_{k+1}}{a_{q+1} - a_q} - \sum_{k=j}^{a_{q+1}-1} (a_{q+1} - k) \frac{c_k - c_{k+1}}{a_{q+1} - a_q} \\
&= c_j + \frac{1}{a_{q+1} - a_q} \left(\sum_{k=a_q+1}^{j-1} k c_k - \sum_{k=a_q+2}^{j} (k-1) c_k - \sum_{k=a_q+1}^{j-1} a_q c_k + \sum_{k=a_q+2}^{j} a_q c_k \right. \\
&\qquad \left. + \sum_{k=j}^{a_{q+1}-1} k c_k - \sum_{k=j+1}^{a_{q+1}} (k-1) c_k - \sum_{k=j}^{a_{q+1}-1} a_{q+1} c_k + \sum_{k=j+1}^{a_{q+1}} a_{q+1} c_k \right) \\
&= c_j + \frac{1}{a_{q+1} - a_q} \left(\sum_{k=a_q+1}^{a_{q+1}-1} k c_k - \sum_{k=a_q+2}^{a_{q+1}} (k-1) c_k - a_q c_{a_q+1} + (a_q - a_{q+1}) c_j \right. \\
&\qquad \left. + a_{q+1} c_{a_{q+1}} \right) \\
&= c_j + \frac{1}{a_{q+1} - a_q} \left(\sum_{k=a_q+1}^{a_{q+1}} c_k + a_q c_{a_q+1} - a_{q+1} c_{a_{q+1}} - a_q c_{a_q+1} + a_{q+1} c_{a_{q+1}} \right. \\
&\qquad \left. - (a_{q+1} - a_q) c_j \right) \\
&= \frac{h}{a_{q+1} - a_q}. \qquad \square
\end{aligned}$$

The definitions of the values u_j^* in Algorithm 8.9 and Remark 8.11 imply immediately the following remark.

Remark 8.12. If $h = 0$, then

$$u_j^* = c_j - c_{a_q+1}, \; j = 1, \dots, a_q.$$

Let us now determine the objective function value $\sum_{j=1}^n u_j^* + \sum_{k=a_q+1}^{a_{q+1}-1} s(F_k) y_{F_k}^*$ of the dual greedy solution.

By definition and Remark 8.11, $u_j^* = 0$ for $j > a_q$. Taking the values of the other variables from Remark 8.12 and recalling that $h = \sum_{j=a_q+1}^{a_{q+1}} c_j$ we obtain

$$\sum_{j=1}^{n} u_j^* = \sum_{j=1}^{a_q} u_j^* + \sum_{j=a_q+1}^{a_{q+1}} u_j^* = \sum_{j=1}^{a_q} c_j - a_q c_{a_q+1}.$$

The second term in the dual objective function yields

$$\begin{aligned}
\sum_{k=a_q+1}^{a_{q+1}-1} s(F_k) y_{F_k}^* &= \sum_{k=a_q+1}^{a_{q+1}-1} \left(a_{q+1} - k\right) a_q \frac{c_k - c_{k+1}}{a_{q+1} - a_q} \\
&= \frac{a_q}{a_{q+1} - a_q} \sum_{k=a_q+1}^{a_{q+1}-1} \left(a_{q+1} - k\right)\left(c_k - c_{k+1}\right) \\
&= \frac{a_q}{a_{q+1} - a_q} \left(\sum_{k=a_q+1}^{a_{q+1}-1} a_{q+1} c_k - \sum_{k=a_q+1}^{a_{q+1}-1} k c_k - \sum_{k=a_q+2}^{a_{q+1}} a_{q+1} c_k \right. \\
&\qquad \left. + \sum_{k=a_q+2}^{a_{q+1}} (k-1) c_k \right) \\
&= \frac{a_q}{a_{q+1} - a_q} \left(a_{q+1}(c_{a_q+1} - c_{a_{q+1}}) - (a_q + 1) c_{a_q+1} + (a_{q+1} - 1) c_{a_{q+1}} \right. \\
&\qquad \left. - \sum_{k=a_q+2}^{a_{q+1}-1} c_k \right) \\
&= \frac{a_q}{a_{q+1} - a_q} \left((a_{q+1} - a_q) c_{a_q+1} - h \right) \\
&= a_q \left(c_{a_q+1} - \frac{h}{a_{q+1} - a_q} \right) = a_q c_{a_q+1}.
\end{aligned}$$

Adding the two objective function terms we obtain

$$\sum_{j=1}^{n} u_j^* + \sum_{k=a_q+1}^{a_{q+1}-1} (a_{q+1} - k) a_q y_k^* = \sum_{j=1}^{a_q} c_j = c(C_g),$$

which is the value of the primal greedy solution. These calculations prove the following.

Remark 8.13. If $h = 0$, the dual greedy solution u^*, y^* is optimal for the LP $D(n; a; c)$ and has the same value as the primal greedy solution.

We now indicate how the solution of the case $h = 0$ can be utilized to handle the cases $h < 0$ and $h > 0$.

Remark 8.14. If $h < 0$, we increase some of the objective function coefficients c_j, $j = a_q + 1, \ldots, a_{q+1}$, such that, after the increase, the ordering of the variables is still respected and such that $h = 0$. Note that this change of the c_j values does not change the value of the primal greedy solution (in fact, now $\{1, \ldots, a_q\}$ and $\{1, \ldots, a_{q+1}\}$ are both optimal) and that any feasible solution of $D(n; a; c)$ after increase is feasible for the LP without modification. Thus applying Algorithm 8.9 to the modified dual LP $D(n; a; c)$ provides a solution u^*, y^* that is feasible and optimal for the unmodified $D(n; a; c)$ and has the same value as the primal greedy solution.

Remark 8.15. If $h > 0$, we modify the objective function vector c into a vector c' by decreasing some of the coefficients c_j, $j = a_q + 1, \ldots, a_{q+1}$, to values c'_j such that $c'_1 \geq c'_2 \geq \ldots \geq c'_n$ and $h' = \sum_{j=a_q+1}^{a_{q+1}} c'_j = 0$. If I_g and I'_g are the primal greedy solutions with respect to c and c', respectively, then clearly $c(I_g) = c'(I'_g) + h$. If we now use Algorithm 8.9 to solve $D_{\mathcal{F}_0}(n; a; c')$, we obtain an optimal solution u', y' for $D(n; a; c')$ with value $c'(I'_g)$. Setting $u^*_j := u'_j + c_j - c'_j$, $j = 1, \ldots, n$, and $y^* := y'$ yields a solution u^*, y^* with value $c'(I'_g) + h = c(I_g)$ that is feasible for $D(n; a; c)$. This implies the optimality of χ^{I_g} for $L(n; a; c)$ and of u^*, y^* for $D(n; a; c)$.

This finishes the discussion of all cases occurring in the treatment of the dual LP $D(n; a; c)$. Hence, the proof of Theorem 8.6 providing a complete linear description of all polytopes associated with cardinality homogeneous systems is also finished.

We now put together all the pieces of the dual greedy algorithm discussed above to specify the complete greedy algorithm that solves the dual LP.

Algorithm 8.16 (Complete Dual Greedy Algorithm). Let $E = \{1, \ldots, n\}$, a cardinality sequence $a = (a_1, \ldots, a_m)$, and an objective function $c = (c_1, \ldots, c_n)$ be given.

1. Set all variables v, w, u_j, y_F of $D(n; a; c)$ to zero.
2. Sort the elements of E such that $c_1 \geq c_2 \geq \ldots \geq c_n$ holds and set $c' := c$.
3. If $c_{a_m} \geq 0$, set
$$w := c_{a_m}, \qquad u_j := c_j - c_{a_m} \quad \text{for } j = 1, \ldots, a_m.$$
   Go to 11.
4. If $c_{a_1} \leq 0$, set
$$v := -c_{a_1}, \qquad u_j := c_j - c_{a_1} \quad \text{for } j = 1, \ldots, a_1.$$
   Go to 11.

5. Otherwise, let p be the largest integer in $\{1, \ldots, n\}$ such that $c_p > 0 \geq c_{p+1}$, and let q be the index in $\{1, \ldots, m\}$ such that $a_q \leq p < a_{q+1}$. Set

$$h := h' := \sum_{j=a_q+1}^{a_{q+1}} c_j .$$

6. If $h < 0$, modify the objective function values as follows.
For $k = a_q + 1, a_q + 2, \ldots, a_{q+1}$ do

$$\begin{aligned} \Delta &:= \min\{c'_{k-1} - c'_k, -h'\}, \\ c'_k &:= c'_k + \Delta, \\ h' &:= h' + \Delta. \end{aligned}$$

7. If $h > 0$, modify the objective function values as follows.
For $k = p, p - 1, \ldots, a_q + 1$ do

$$\begin{aligned} \Delta &:= \min\{c'_k - c'_{k+1}, h'\}, \\ c'_k &:= c'_k - \Delta, \\ h' &:= h' - \Delta. \end{aligned}$$

8. For $k = a_q + 1, a_q + 2, \ldots, a_{q+1} - 1$ set

$$y^*_{F_k} := \frac{1}{a_{q+1} - a_q}(c'_k - c'_{k+1}) .$$

9. If $h \leq 0$, do the following.
For $j = 1, 2, \ldots, a_q$ set

$$u^*_j := c'_j - c'_{a_q+1} .$$

10. If $h > 0$, do the following.
For $j = 1, 2, \ldots, a_q$ set

$$u^*_j := c_j - c'_{a_q+1} .$$

For $j = a_q + 1, a_q + 2, \ldots, a_{q+1}$ set

$$u^*_j := c_j - c'_j .$$

11. Output the nonzero variables.

As outlined before, the solution u^*, y^* is feasible and optimal for the dual LP $D(n; a; c)$ and has the same value as the primal greedy solution.

Let us remark that the dual solution constructed above is one of typically very many optimal solutions. For instance, any modification of the c_j's in step 6 that makes h equal to zero and maintains the ordering $c_j \geq c_{j+1}$ and that is different from the one chosen in step 6 yields a different optimal dual solution. Even if we assume that all objective function coefficients are integral, the above solution is, in general, fractional. There are cases where all or some optimal dual solutions are integral, but we know examples where, for $c \in \mathbb{Z}^n$, no optimal solution of $D(n; a; c)$ is integral; see Example 8.18 below.

Remark 8.17. If the objective function values are sorted, then the Primal Greedy Algorithm 8.7 (steps 2–6) and the Complete Dual Greedy Algorithm 8.16 (steps 3–11) perform a number of arithmetic steps that is linear in n on numbers whose size is linear in the input length. Thus, the running time of the algorithm is dominated by sorting, which requires $\mathcal{O}(n \log n)$ steps.

Recall that a system of linear equations and inequalities is called *totally dual integral* (TDI) if, for any integral objective function, the LP dual to this LP has an integral optimum solution. We now indicate that none of the three linear systems that can be naturally associated with cardinality homogeneous set systems is TDI.

Example 8.18. Consider the ground set $E = \{1, 2, 3, 4\}$, the cardinality vector $a = (a_1, a_2) = (1, 4)$, and the objective function vector $c^T = (2, 2, 1, -3)$. The linear system $Q(4; a)$ gives rise to the LP

$$
\text{(Q)} \qquad \begin{aligned}
&\max\ 2x_1 + 2x_2 + x_3 - 3x_4 \\
&0 \le x_j \le 1, \qquad j = 1, \dots, 4, \\
&1 \le x(E) \le 4, \\
&\mathrm{CF}_F(x) \le s(F) \qquad \text{for all } F \subseteq E \text{ with } |F| \in \{2, 3\}.
\end{aligned}
$$

The linear system consists of 20 inequalities that describe $P(4; 1, 4)$ completely. This system, however, is redundant; see Proposition 8.21. The following LP has only five inequalities, has the same solution set, and is nonredundant:

$$
\text{(NRQ)} \qquad \begin{aligned}
&\max\ 2x_1 + 2x_2 + x_3 - 3x_4 \\
&x(E) \ge 1, \\
&\mathrm{CF}_F(x) \le 1 = s(F) \quad \text{for all } F \subseteq E,\ |F| = 3.
\end{aligned}
$$

In the proof of the Dual Greedy Algorithm we showed that (for this ordered objective function) the LP $L_{\mathcal{F}_0}(4; a; c)$:

$$
\text{(GQ)} \qquad \begin{aligned}
&\max\ 2x_1 + 2x_2 + x_3 - 3x_4 \\
&2x_1 + 2x_2 - x_3 - x_4 \le 2 \qquad (F_2 = \{1, 2\}), \\
&x_1 + x_2 + x_3 - 2x_4 \le 1 \qquad (F_3 = \{1, 2, 3\}), \\
&0 \le x_j \le 1 \qquad j = 1, 2, 3, 4,
\end{aligned}
$$

yields an optimum solution of (Q). Note that the LPs (Q), (NRQ), and (GQ) have three optimum solutions, namely, the incidence vectors of the sets $\{1\}$, $\{2\}$, and $\{1, 2, 3, 4\}$. (Q) and (NRQ) have, as mentioned, the same solution set. However, (GQ) is a strict relaxation. The solution set of (GQ) has some fractional vertices, such as $x' = (0, 1, 1, 1/2)$.

The LP dual to the "greedy LP" (GQ) has a unique optimum solution, which is the one provided by the Dual Greedy Algorithm: $y^*_{\{1,2\}} = 1/3$, $y^*_{\{1,2,3\}} = 4/3$, and all other variables equal to zero. The dual program of (NRQ) also has a unique optimum solution: $y^*_{\{1,2,3\}} = 5/3$, $y^*_{\{1,2,4\}} = 1/3$, and all other variables equal to zero. The dual to (Q) has a face of dimension 1 as the set of optimum solutions. This face is the convex hull of the two vertices just mentioned. It contains no integral point. Thus none of the three linear systems is TDI. (These computations have been carried out by PORTA [8] and were verified by hand.)

8.6 Facets

We now address the nonredundancy issue and determine the inequalities of $Q(n; a)$ that define facets of $P(n; a)$. As before, we assume throughout this section that $E = \{1, \dots, n\}$, $n \geq 1$, and that $a = (a_1, \dots, a_m)$ is a cardinality vector.

We indicate only a few of the relatively simple proofs. They are all based on well-known facts about 0/1-matrices. The fact used most is that, for $0 < k < n$, the 0/1-matrix $M(n; k)$ with n columns and the $\binom{n}{k}$ rows consisting of all 0/1-vectors with k ones and $n - k$ zeros has rank n. In other words, the incidence vectors of the sets in the set system $C(n; k) = \{C \subseteq E \mid |C| = k\}$ (which form the rows of $M(n; k)$) are linearly, and thus affinely, independent. Clearly, if $k = 0$ or $k = n$, there is only one such vector, the zero vector or the all-ones vector. Proving that a certain inequality $c^T x \leq \alpha$ defines a facet of $P(n; a)$ amounts to observing that certain incidence vectors of sets in $C(n; a)$ (with additional properties) satisfy $c^T x \leq \alpha$ with equality and form a set of vectors of affine rank equal to $\dim P(n; a)$.

Using the facts mentioned above we can easily determine the dimension of $P(n; a)$.

Proposition 8.19. *Let $E = \{1, \dots, n\}$ and let $a = (a_1, \dots, a_m)$ be a cardinality vector.*

(a) *If $m = 1$ and $a_1 = 0$ or $a_1 = n$, then* $\dim P(n; a) = 0$.
(b) *If $m = 1$ and $0 < a_1 < n$, then* $\dim P(n; a) = n - 1$.
(c) *If $m = 2$ and $a_1 = 0$, $a_2 = n$, then* $\dim P(n; a) = 1$.
(d) *In all other cases,* $\dim P(n; a) = n$.

The case $m = 1$ is very special and easy to handle.

Proposition 8.20. *Let $m = 1$; i.e., we are only interested in the system of subsets of E with cardinality a_1.*

(a) *If $a_1 = 0$, then $P(n; a_1) = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \dots = x_n = 0\}$.*
(b) *If $a_1 = n$, then $P(n; a_1) = \{x \in \mathbb{R}^n \mid x_1 = x_2 = \dots = x_n = 1\}$.*
(c) *If $a_1 = 1$ and $n \geq 2$, then $P(n; a_1) = \{x \in \mathbb{R}^n \mid x(E) = 1,\ x_j \geq 0,\ j = 1, \dots, n\}$.*
(d) *If $a_1 = n - 1$ and $n \geq 2$, then $P(n; a_1) = \{x \in \mathbb{R}^n \mid x(E) = n - 1,$ $x_j \leq 1,\ j = 1, \dots, n\}$.*
(e) *If $1 < a_1 < n - 1$ and $n \geq 4$, then $P(n; a_1) = \{x \in \mathbb{R}^n \mid x(E) = a_1,$ $0 \leq x_j \leq 1,\ j = 1, \dots, n\}$.*

The linear systems above define $P(n; a_1)$ completely and nonredundantly.

Proposition 8.20 provides a complete investigation of the nonredundancy issue for the case $m = 1$. The term *hypersimplex* is often used to name a polytope of type $P(n; a_1)$. In the terminology of this chapter, a hypersimplex is the circuit polytope of some uniform matroid $U_{k,n}$; i.e., Proposition 8.20 covers the circuit polytopes of uniform matroids.

We also refrain from providing all facet proofs in detail because many special cases have to be considered. Let us just, as one example, discuss the nonnegativity constraints thoroughly.

Given $E = \{1, \dots, n\}$ and a cardinality vector $(a = a_1, \dots, a_m)$, when does $x_j \geq 0$, $j = 1, \dots, n$, define a facet of $P(n; a)$? First of all, because of symmetry, we have to consider just one of the indices, say $j = 1$.

If $m = 1$ and $a_1 = 0$ or $a_1 = n$ (see Propositions 8.19(a) and 8.20(a), (b)), then $P(n; a)$ is an affine space and has no facets at all.

Let $m = 1$ and $0 < a_1 < n$. The set of vertices of $P(n; a_1)$ satisfying $x_1 = 0$ is nothing but the set of incidence vectors of $C(n-1; a_1)$ to which a first component with value zero has been added. The matrix $M(n-1; a_1)$ has rank $n-1$ unless $a_1 = n-1$. Adding a first column of zeros to $M(n-1; a_1)$ yields a matrix of affine rank n with one exception. If $a_1 = n-1$, the affine rank is 1 only. Thus we conclude that $x_j \geq 0$ defines a facet of $P(n; a_1)$ if $m = 1$ and $1 \leq a_1 \leq n-2$ but not if $a_1 = n-1$; see Proposition 8.20(c), (d), (e).

Suppose now that $m = 2$. If $a_1 = 0$ and $a_2 = n$ (see Proposition 8.19(c)), then $P(n; a)$ is just the piece of line from the zero vector to the all-ones vector. In this case, all nonnegativity constraints $x_j \geq 0$, $j = 1, \ldots, n$, define one and the same facet of $P(n; a)$, which consists of the zero vector only. If $a_1 = 0$ and $a_2 = n-1$, then $x_j \geq 0$ does not define a facet of $P(n; a)$ except when $n = 2$ (and in this case $x_j \geq 0$ appears as a degenerate case of a CF-inequality; see Proposition 8.21(c)). If $a_1 = 0$ and $1 \leq a_2 \leq n-2$, then $x_j \geq 0$ defines a facet of $P(n; a_1, a_2)$.

Because of symmetry all observations about $x_j \geq 0$ can be easily translated into corresponding observations about $x_j \leq 1$.

If $a_1 = 0$, then CF-inequalities exist for all F with $a_1 < |F| < a_2$. A moment's thought reveals that these inequalities are redundant unless $|F| = 1$. In this case the CF-inequality $(a_2 - 1)x_k - \sum_{j \neq k} x_j \leq 0$ defines a facet for all $k \in \{1, \ldots, n\}$. This observation immediately translates into an equivalent observation for the case $a_2 = n$.

We summarize the situation for $m = 2$, except for the case $1 \leq a_1 < a_2 \leq n-1$, in the following.

Proposition 8.21. *Suppose $m = 2$.*

(a) *If $a_1 = 0$ and $a_2 = 1$, then*
$P(n; 0, 1) = \{x \in \mathbb{R}^n \mid x(E) \leq 1,\ x_j \geq 0,\ j = 1, \ldots, n\}.$

(b) *If $a_1 = 0$ and $1 < a_2 < n-1$, then*
$P(n; 0, a_2) = \{x \in \mathbb{R}^n \mid x(E) \leq a_2,$
$(a_2 - 1)x_k - \sum_{j \neq k} x_j \leq 0,\ k = 1, \ldots, n,\ x_j \geq 0,\ j = 1, \ldots, n\}.$

(c) *If $a_1 = 0$ and $a_2 = n-1$, then*
$P(n; 0, n-1) = \{x \in \mathbb{R}^n \mid x(E) \leq n-1,$
$(n-2)x_k - \sum_{j \neq k} x_j \leq 0,\ k = 1, \ldots, n\}.$

(d) *If $a_1 = 0$ and $a_2 = n$, then*
$P(n; 0, n) = \{x \in \mathbb{R}^n \mid x_i - x_{i+1} = 0,\ i = 1, \ldots, n-1,\ 0 \leq x_1 \leq 1\}.$

(e) *If $a_1 = 1$ and $a_2 = n$, then*
$P(n; 1, n) = \{x \in \mathbb{R}^n \mid x(E) \geq 1, \sum_{j \neq k} x_j - (n-2)x_k \leq 1,\ k = 1, \ldots, n\}.$

(f) *If $1 < a_1 < n-1$ and $a_2 = n$, then*
$P(n; a_1, n) = \{x \in \mathbb{R}^n \mid x(E) \geq a_1, x_j \leq 1,\ j = 1, \ldots, n,$
$\sum_{j \neq k} x_j - (n - 1 - a_1)\, x_k \leq a_1,\ k = 1, \ldots, n\}.$

(g) *If $a_1 = n-1$ and $a_2 = n$, then*
$P(n; n-1, n) = \{x \in \mathbb{R}^n \mid x(E) \geq n-1,\ x_j \leq 1,\ j = 1, \ldots, n\}.$

All linear systems above are complete and nonredundant.

To finish the discussion of the nonnegativity constraints we observe that, whenever there is an index i such that $0 < a_i < a_{i+1} < n$, then $x_j \geq 0$ (and for symmetry $x_j \leq 1$) defines a facet of $P(n; a)$.

The cardinality constraints are, of course, equations if $m = 1$. They define facets in the following cases.

Proposition 8.22. *Let $m \geq 2$.*

(a) *If $a_1 \geq 1$, then $x(E) \geq a_1$ defines a facet of $P(n; a)$.*
(b) *If $a_m \leq n - 1$, then $x(E) \leq a_m$ defines a facet of $P(n; a)$.*

Let us finish the discussion with the CF constraints. We already considered the special cases when $a_1 = 0$ or $a_m = n$. The general case is as follows.

Proposition 8.23. *Let $m \geq 2$ and $1 < a_i < a_{i+1} < n$. Then for all $F \subseteq E$ with $a_i < |F| < a_{i+1}$ the corresponding CF-inequality*

$$\mathrm{CF}_F(x) = \sum_{j \in F}(a_{i+1} - |F|)\, x_j - \sum_{j \in E \setminus F} (|F| - a_i)\, x_j \leq (a_{i+1} - |F|)\, a_i = s(F)$$

defines a facet of $P(n; a)$.

The proof of Proposition 8.23 is based on the fact that the incidence vectors of sets in $C(n; a)$ satisfying the CF-inequality are the subsets of F of cardinality a_i and the subsets of E of cardinality a_{i+1} containing F. A simple calculation shows that these incidence vectors form a set of affine (in fact linear) rank n.

With this observation we can finish the discussion of the case $m = 2$.

Proposition 8.24. *Let $m = 2$ and $1 \leq a_1 < a_2 \leq n-1$. Then the inequality system defining $Q(n; a)$ provides a complete and nonredundant description of $P(n; a)$.*

The remarks above also immediately yield a full characterization of the case $m \geq 3$.

Theorem 8.25. *Let $E = \{1, \ldots, n\}$, $n \geq 2$, and let $a = (a_1, \ldots, a_m)$, $m \geq 3$, be a cardinality vector. Then the following system of inequalities provides a complete and nonredundant description of $P(n; a)$.*

(a) *$x_j \geq 0$ for all $j \in E$ unless $m = 3$ and $a = (0, n-1, n)$.*
(b) *$x_j \leq 1$ for all $j \in E$ unless $m = 3$ and $a = (0, 1, n)$.*
(c) *$x(E) \geq a_1$ unless $a_1 = 0$.*
(d) *$x(E) \leq a_m$ unless $a_m = n$.*
(e) *$\sum_{j \in F}(a_{f+1} - |F|)\, x_j - \sum_{j \in E \setminus F}(|F| - a_f)\, x_j \leq (a_{f+1} - |F|)\, a_f$ for all $F \in \mathcal{F}$ unless $a_1 = 0$ and $2 \leq |F| < a_2$ or $a_m = n$ and $a_{m-1} < |F| \leq n - 2$.*

Summarizing the results above, we can state that the linear system defining $Q(n; a)$ is not only a complete description of $P(n; a)$ but also is nonredundant, with a few exceptions for $m \leq 3$ and whenever $a_1 = 0$ and $a_m = n$.

Theorem 8.25 (and the discussion of the cases $m = 2$ and $m = 3$) yields, for every uniform matroid $U_{k,n}$, a complete and nonredundant description of its cycle polytope and its circuit polytope. As a byproduct we obtain the well-known characterization of the convex hull of all 0/1-vectors with an even or odd number of ones.

A consequence of Theorem 8.25 is that, among the polytopes associated with cardinality homogeneous set systems, the polytope $P(n; 1, n-1)$, which has $2n$ vertices where any pair of vertices is adjacent, has the largest number of facets, namely, 2^n.

Example 8.26. To finish the facet discussion and give another example for the execution of the Dual Greedy Algorithm we consider the uniform matroid $U_{3,9}$. The circuits of $U_{3,9}$ are all subsets of $E = \{1, \dots, 9\}$ of cardinality 4; the cycles of $U_{3,9}$ consist of its circuits together with the empty set and all subsets of E of cardinality 8. In the notation of this chapter the set of cycles of $U_{3,9}$ is the cardinality homogeneous set system $C(9; 0, 4, 8)$. The cycle polytope $P(U_{3,9}) = P(9; 0, 4, 8)$ has $1 + \binom{9}{4} + \binom{9}{8} = 136$ vertices. The system describing the polytope $Q(n; 0, 4, 8)$ has the form

$$0 \le x_j \le 1, \qquad j = 1, \dots, 9,$$

$$0 \le x(E) \le 8,$$

$$\sum_{j \in F} (a_{f+1} - |F|)\, x_j - \sum_{j \in E \setminus F} (|F| - a_f)\, x_j \le (a_{f+1} - |F|)\, a_f$$

$$\text{for all } F \subseteq E \text{ with } |F| \in \{1, 2, 3, 5, 6, 7\}.$$

This system has 395 inequalities. By Theorem 8.25(c) the lower cardinality bound and by (e) the CF-inequalities for $|F| \in \{2, 3\}$ do not define facets. It follows that $P(9; 0, 4, 8)$ has exactly 274 facets.

Let us now maximize the objective function $c^T = (15, 12, 11, 10, 8, 6, -2, -5, -8)$ over $P(9; 0, 4, 8)$. The Primal Greedy Algorithm yields $C_g = \{1, 2, \dots, 8\}$ with $c(C_g) = 55$ and determines $p = 6$, $a_q = a_2 = 4$, $a_{q+1} = a_3 = 8$, and $h = c_5 + \dots + c_8 = 7$. The Complete Dual Greedy Algorithm 8.16 first modifies in step 7 the objective function to $c' = (15, 12, 11, 10, 8, -1, -2, -5, -8)$ so that $h' = c'_5 + \dots + c'_8 = 0$. We have shown in Section 8.5 that we can replace the LP with 274 facet-defining inequalities with the system $L_{\mathcal{F}_0}(9; a; c')$ consisting of 18 upper and lower bounds and only 3 additional CF-inequalities corresponding to the sets $F_k = \{1, \dots, k\}, k \in \{5, 6, 7\}$:

$$\max 15x_1 + 12x_2 + 11x_3 + 10x_4 + 8x_5 - x_6 - 2x_7 - 5x_8 - 8x_9$$

$$0 \le x_j \le 1 \quad j = 1, \dots, 9,$$

$$3x_1 + 3x_2 + 3x_3 + 3x_4 + 3x_5 - x_6 - x_7 - x_8 - x_9 \le 12 \ (F_5),$$

$$2x_1 + 2x_2 + 2x_3 + 2x_4 + 2x_5 + 2x_6 - 2x_7 - 2x_8 - 2x_9 \le 8 \quad (F_6),$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 - 3x_8 - 3x_9 \le 4 \quad (F_7).$$

The dual LP $D_{\mathcal{F}_0}(9; a; c')$ has the following form (where $y_k = y_{F_k}$):

$$\min u_1 + \ldots + u_9 + 12y_5 + 8y_6 + 4y_7$$

$$\begin{aligned}
u_1 + 3y_5 + 2y_6 + y_7 &\geq 15,\\
u_2 + 3y_5 + 2y_6 + y_7 &\geq 12,\\
u_3 + 3y_5 + 2y_6 + y_7 &\geq 11,\\
u_4 + 3y_5 + 2y_6 + y_7 &\geq 10,\\
u_5 + 3y_5 + 2y_6 + y_7 &\geq 8,\\
u_6 - y_5 + 2y_6 + y_7 &\geq -1,\\
u_7 - y_5 - 2y_6 + y_7 &\geq -2,\\
u_8 - y_5 - 2y_6 - 3y_7 &\geq -5,\\
u_9 - y_5 - 2y_6 - 3y_7 &\geq -8,\\
u_1, \ldots, u_9, y_5, y_6, y_7 &\geq 0.
\end{aligned}$$

The Dual Greedy Algorithm 8.9, which is step 8 of Algorithm 8.16, yields the following c'-optimal solution:

$$\begin{aligned}
y_5^* &:= \frac{1}{4}(c_5' - c_6') = \frac{9}{4},\\
y_6^* &:= \frac{1}{4}(c_6' - c_7') = \frac{1}{4},\\
y_7^* &:= \frac{1}{4}(c_7' - c_8') = \frac{3}{4},\\
u_1' &:= 7, u_2' = 4, u_3' = 3, u_4' = 2, u_5' = \ldots = u_9' = 0,
\end{aligned}$$

of $D_{\mathcal{F}_0}(9; a; c')$. To turn this solution u', y' into a solution u^*, y^* of $D_{\mathcal{F}_0}(9; a; c)$ we have to modify the values u_j' belonging to indices j where the objective function c was changed. In our case we only have to modify $u_6' = 0$ to $u_6^* = u_6' + c_j - c_j' = 7$ (this is step 10 of Algorithm 8.16), i.e., $y^* := y'$, $u_j^* := u_j'$ with the exception that $u_6^* = 7$ is an optimal solution of the dual LP $D_{\mathcal{F}_0}(9; a; c)$ and, as we have shown, also of the LP $D_{\mathcal{F}_0}(9; a; c)$. The value of this solution is $55 = c(I_g)$.

The optimum solution of $D_{\mathcal{F}_0}(9; a; c)$ is by no means unique. The face of the optimal solution of this LP has, in fact, 10 vertices.

8.7 Separation

Since we can optimize over $P(n; a)$ in polynomial time we can also solve the separation problem for $P(n; a)$ in polynomial time by the general results described in [4]. There is, however, a much simpler separation algorithm.

Let a vector $y \in Q^n$ be given. It is, of course, trivial to check the bounds $0 \leq x_j \leq 1$, $j = 1, \ldots, n$, and the cardinality constraints $a_1 \leq x(E) \leq a_m$ by substituting y into these inequalities. We may, thus, assume that y satisfies them.

Suppose now that y violates, for some $F \in \mathcal{F}$ of cardinality k, the corresponding CF-inequality, i.e.,

$$\sum_{j \in F} (a_{f+1} - k)\, y_j - \sum_{j \in E \setminus F} (k - a_f)\, y_j > (a_{f+1} - k) a_f.$$

Let F^* be a set in $\mathcal{F}$ of cardinality k such that $\sum_{j \in F^*} y_j$ is maximum. Then, clearly, y violates the corresponding CF-inequality as well. In fact, the CF-inequality associated with F^* is a "most violated" inequality among all CF-inequalities coming from sets in $\mathcal{F}$ of cardinality k. Finding such a set F^* is easy. We sort the components of y such that $y_1 \geq y_2 \geq \ldots \geq y_n$. We set $F^* := \{1, \ldots, k\}$. Then y satisfies the CF-inequality associated with F^* if and only if y satisfies all CF-inequalities associated with sets in $\mathcal{F}$ of cardinality k. This observation gives the following very simple polynomial-time separation algorithm for $P(n; a)$, which, in its major step, can also be viewed as a greedy algorithm.

Algorithm 8.27 (Greedy Separation Algorithm for $P(n; a)$). Let $E = \{1, \ldots, n\}$, a cardinality vector $a = (a_1, \ldots, a_n)$, and a vector $y \in Q^n$ be given.

1. If y has a component smaller than zero or larger than one, report that a bound is violated by y and stop.
2. If $y(E) < a_1$ or $y(E) > a_m$, report that a cardinality constraint is violated by y and stop.
3. Sort the components of y such that $y_1 \geq y_2 \geq \cdots \geq y_n$.
4. For $k = a_1 + 1$ to $a_m - 1$ and $k \neq a_i$, $i = 2, \ldots, m - 1$ do

   If $\sum_{j=1}^{k} (a_{f+1} - k)\, y_j - \sum_{j \in E \setminus F} (k - a_f)\, y_j > (a_{f+1} - k)\, a_f$, where $f = f(\{1, \ldots, k\})$,

   then output that y violates the CF-inequality corresponding to $\{1, \ldots, k\}$.

If the greedy separation algorithm produces no violated inequality, then y is in $P(n; a)$.

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