*Europ. J. Combinatorics* (1996) **17**, 39–52

# Packing Steiner Trees: Further Facets

M. GRÖTSCHEL, A. MARTIN AND R. WEISMANTEL

In this paper we continue the investigations in [3] for the Steiner tree packing polyhedron. We present several new classes of valid inequalities and give sufficient (and necessary) conditions for these inequalities to be facet-definining. It is intended to incorporate these inequalities into an existing cutting plane algorithm that is applicable to practical problems arising in the design of electronic curcuits.



## 1. INTRODUCTION

Given a graph $G = (V, E)$ and a node set $T \subseteq V$, we call an edge set $S \subseteq E$ a *Steiner tree for* $T$ if, for each pair of nodes $u, v \in T$, $S$ contains a $[u, v]$-path. In this paper we investigate the following problem, that we call the *Steiner tree packing problem*:

Given an undirected graph $G = (V, E)$ with edge capacities $c_e \in \mathbb{N}$ for all $e \in E$ and a list of node sets $\mathcal{N} = \{T_1, \ldots, T_N\}$, $N \in \mathbb{N}$, find Steiner trees $S_k$ for $T_k$, $k = 1, \ldots, N$ such that each edge $e \in E$ is contained in at most $c_e$ of the edge sets $S_1, \ldots, S_N$.

Every collection of Steiner trees $S_1, \ldots, S_N$ with this property is called a Steiner tree packing. If a weighting of the edges is given in addition and a (with respect to this weighting) minimal Steiner tree packing must be found, we call this the weighted Steiner tree packing problem.

The Steiner tree packing problem has important applications in the layout of electronic circuits. One of the major tasks in VLSI design is the so-called routing problem. Here, given sets of contact points (also called terminals) have to be connected by wires such that certain technical side constraints are taken into account and an objective function such as the total wiring length is minimized. The routing problem in general is too complex to be solved in one step. Depending on the user's choice of decomposing the chip design problem into a hierarchy of stages, on the underlying technology, and on the given design rules, various subproblems arise. Many of the routing problems that come up this way can be formulated as Steiner tree packing problems (for details, see for instance [6] or [8]).

The Steiner tree packing problem is not only interesting because of its important applications. Special cases of it have been the focal point of deep theoretical work in graph theory. For instance, the problem of packing edge-disjoint paths (i.e., the Steiner tree packing problem in which all node sets have cardinality 2) has been intensively studied in the literature (surveys are [1] and [10]).

To our knowledge, most published work on that topic (either theoretical or practical) concerns the task of finding feasible solutions. We have found almost no paper (one exception is [2]) where optimal solutions or at least good lower bounds for the Steiner tree packing problem are investigated.

In [3] and [4] we considered the Steiner tree packing problem from a polyhedral point of view and developed a branch and cut algorithm. We tested our algorithm on an important subclass of routing problems, namely on so-called switchbox routing problems. Here, the underlying graph is a complete rectangular grid graph and the



0195-6698/96/010039 + 14 $12.00/0 

node sets are located on the outer face of the grid. The results we obtained are encouraging. We could solve most of the problems discussed in the literature to optimality. Unfortunately, the inequalities described in [4] are not sufficient to yield integer solutions of these practical problem instances (without using the branching phase of our algorithm). This fact results either from the lack of exact separation algorithms for the known classes of inequalities or from the lack of a sufficient knowledge of the facet structure of the Steiner tree packing polyhedron. In this paper we concentrate on the second aspect and present new classes of (facet-defining) inequalities. These inequalities will form the backbone of our cutting plane algorithm in order to further improve the lower bounds of certain (weighted) Steiner tree packing problems and in order to apply our algorithm to problem instances of large scale.

The paper is organized as follows. In Section 2 we define the Steiner tree packing polyhedron and outline some results known for this polyhedron. Sections 3 and 4 present new classes of facet-defining inequalities. The first two classes, the matching and matching-tree inequalities, involve two different node sets. We give sufficient and necessary conditions for these inequalities to be facet-defining. Section 4 describes inequalities that combine more than two node sets. The first inequalities with three node sets are called 2-eared alternating cycle inequalities and the second class applies to an arbitrary number of node sets. It is obtained by composition of alternating cycle inequalities.

## 2. The Steiner Tree Packing Polyhedron

In this section we introduce a polyhedron associated with the Steiner tree packing problem. We assume the reader to be familiar with polyhedral theory (see, for instance, [9]).

First, we sketch some graph-theoretic notation. Let $G=(V,E)$ be an undirected graph. For a given edge set $F \subseteq E$, we denote by $V(F)$ all nodes that are incident to an edge in $F$. Given two node sets $U, W \subseteq V$, we denote by $[U:W]$ the set of edges in $G$ with one endnode in $U$ and the other in $W$. For a node set $W$, we also use $E(W)$ instead of $[W:W]$, and, if $\emptyset \neq W \neq V$, we write $\delta(W)$ for $[W: V\backslash W]$. If $W=\{v\}$, we abbreviate $\delta(\{v\})$ to $\delta(v)$.

Suppose we are given a graph $G=(V,E)$ with capacities $c_e \in \mathbb{N}$ for all $e \in E$ and a list of node sets $\mathcal{N}=\{T_1,\dots,T_N\}$, $N \geq 1$. Each set $T_k$ in $\mathcal{N}$ is called a *terminal set* or a *net*, each node in $T_k$ a *terminal*, and the list of node sets $\mathcal{N}$ a *net list*. We will denote an *instance of the Steiner tree packing problem* by the triple $(G,\mathcal{N},c)$. If a collection of Steiner trees $S_1,\dots,S_N$ defines a Steiner tree packing for $(G,\mathcal{N},c)$, it is convenient to order the sets $S_k$ and denote the Steiner tree packing by the $N$-tuple $(S_1,\dots,S_N)$. Moreover, we introduce the following technically useful operations on $N$-tuples of edge sets. For an $N$-tuple of edge sets $P=(F_1,\dots,F_N)$ and an edge set $F \subseteq E$, we define $P\backslash_k F := (F_1,\dots,F_k\backslash F,\dots,F_N)$ and $P \cup_k F := (F_1,\dots,F_k \cup F,\dots,F_N)$. We abbreviate $P\backslash_k\{e\}$ to $P\backslash_k e$ and $P\cup_k\{e\}$ to $P \cup_k e$.

Our definition of a Steiner tree (see the beginning of the Introduction) differs from the terminology most frequently used in the literature. A Steiner tree is usually supposed to be a tree. However, our definition simplifies notation and is more convenient for the polyhedral investigations in the following. A Steiner tree that is a tree and the leaves of which are terminals is called *edge-minimal*. Accordingly, a Steiner tree packing $P=(S_1,\dots,S_n)$ is *edge-minimal* if each Steiner tree $S_k$ is edge-minimal.

Let $\mathbb{R}^{\mathcal{N}\times E}$ denote the $N\cdot|E|$-dimensional vector space $\mathbb{R}^E \times \cdots \times \mathbb{R}^E$, where the components of each vector $x \in \mathbb{R}^{\mathcal{N}\times E}$ are indexed by $x_e^k$ for $k \in \{1,\dots,N\}$, $e \in E$.

Moreover, for a vector $x \in \mathbb{R}^{\mathcal{N} \times E}$ and $k \in \{1, \ldots, N\}$, we denote by $x^k \in \mathbb{R}^E$ the vector $(x_e^k)_{e \in E}$, and we simply write $x = (x^1, \ldots, x^N)$ instead of $x = ((x^1)^T, \ldots, (x^N)^T)^T$. For an edge set $F \subseteq E$, $\chi^F \in \mathbb{R}^E$ denotes the *incidence vector of F*, i.e., $\chi_e^F := 1$, if $e \in F$, and $\chi_e^F := 0$, otherwise. The *incidence vector of a Steiner tree packing* $P = (S_1, \ldots, S_N)$ is denoted by $(\chi^{S_1}, \ldots, \chi^{S_N})$ or, for short, $\chi^P$. If $a^T x \geqslant \alpha$ is some inequality with $a \in \mathbb{R}^{\mathcal{N} \times E}$ and $P$ is a Steiner tree packing with $a^T \chi^P = \alpha$, we call $P$ a *root* (*with respect to* $a^T x \geqslant \alpha$).

The *Steiner tree packing polyhedron* STP$(G, \mathcal{N}, c)$ is defined as the convex hull of all incidence vectors of Steiner tree packings. It is easy to see that the following holds:

STP$(G, \mathcal{N}, c) = \text{conv}\{x \in \mathbb{R}^{\mathcal{N} \times E} |$

$$\begin{array}{lll}
\text{(i)} \sum_{e \in \delta(W)} x_e^k \geqslant 1, & \text{for all } W \subset V,\ W \cap T_k \neq \emptyset, & \\
 & (V \backslash W) \cap T_k \neq \emptyset,\ k = 1, \ldots, N; & \\
\text{(ii)} \sum_{k=1}^{N} x_e^k \leqslant c_e, & \text{for all } e \in E; & (2.1) \\
\text{(iii)} 0 \leqslant x_e^k \leqslant 1, & \text{for all } e \in E,\ k = 1, \ldots, N; & \\
\text{(iv)} x_e^k \in \{0, 1\}, & \text{for all } e \in E,\ k = 1, \ldots, N\}. &
\end{array}$$

The inequalities (2.1)(i) are called *Steiner cut inequalities,* inequalities (2.1)(ii) are called *capacity inequalities* and the ones in (2.1)(iii) *trivial inequalities.* In case $N = 1$, the Steiner tree packing polyhedron is also called the *Steiner tree polyhedron.* The weighted Steiner tree packing problem can be solved—in principle—via the following linear program:

$$\min \sum_{k=1}^{N} w^T x^k, \qquad x \in \text{STP}(G, \mathcal{N}, c), \tag{2.2}$$

where $w_e \in \mathbb{R}_+$ denotes the nonnegative weight of edge $e \in E$. In order to apply linear programming techniques, a 'good' description of the Steiner tree packing polyhedron by means of equations and inequalities is indispensable. The aim of our paper is to present several new valid and facet-defining inequalities for STP$(G, \mathcal{N}, c)$.

To this end we must determine the dimension of STP$(G, \mathcal{N}, c)$. Unfortunately, the problem of deciding whether, for some given $l \in \mathbb{N}$, the dimension of the Steiner tree packing polyhedron is at least $l$ is $\mathcal{NP}$-complete. This follows from the fact that the Steiner tree packing problem itself is $\mathcal{NP}$-complete (see, for instance, [5, 11]).

Due to this fact, we decided to study the facial structure of instances in which the underlying graph is complete and the net list $\mathcal{N} = \{T_1, \ldots, T_N\}$ is *disjoint* (i.e., $T_i \cap T_j = \emptyset$ for all $i, j \in \{1, \ldots, N\}$, $i \neq j$). In [3] it is verified that the corresponding Steiner tree packing polyhedron STP$(G, \mathcal{N}, c)$ is fulldimensional in this case. We also show how validity results for the Steiner tree packing polyhedron for some graph can be transformed to validity results for the Steiner tree packing polyhedron for the graph obtained by deleting some edge or splitting some node and thus, by repeated application, how validity results for the complete graph can be transformed to the general case.

Let us now summarize some results for the case that $G$ is complete and the net list is disjoint. The reader interested in the proofs of these results is referred to [3].

First, the trivial inequalities $x_e^k \geqslant 0$ of (2.1)(iii) are facet-defining iff $|V| \geqslant 5$ or $e \notin E(T_k)$, whereas the trivial inequalities $x_e^k \leqslant 1$ of (2.1)(iii) are facet-defining iff $c_e \geqslant 2$. Moreover, the capacitiy constraints (2.1)(ii) are facet-defining iff $c_e \leqslant N - 1$.

We have also shown that each nontrivial facet-defining inequality of the Steiner tree polyhedron can be lifted to yield a facet-defining inequality of the Steiner tree packing polyhedron. More precisely, if $\hat{a}^T x \geq \alpha$ defines a facet of the Steiner tree polyhedron $\mathrm{STP}(G, \{T_k\}, c)$ for some $k \in \{1, \ldots, N\}$, then $a^T x \geq \alpha$ defines a facet of $\mathrm{STP}(G, \mathcal{N}, c)$, where $a_e^l = 0$ for $l \neq k$ and $a_e^k = \hat{a}_e$ for all $e \in E$. This theorem implies that, in order to obtain a complete description of some Steiner tree packing polyhedron $\mathrm{STP}(G, \mathcal{N}, c)$, at least all 'individual' Steiner tree polyhedra $\mathrm{STP}(G, \{T\}, c)$, $T \in \mathcal{N}$, must be known completely. Of course, this knowledge will hardly do. There are many classes of inequalities that combine at least two nets. We call such inequalities *joint.*

In [3] several classes of joint inequalities are described. Among them are the alternating cycle inequalities, the grid inequalities and the critical cut inequalities. All these inequalities and all joint inequalities that we are going to present in this paper are of the form $a^T x \geq \alpha$, $a \geq 0$. The coefficients of some of the edges turn out to be zero for all nets. We call these edges *zero edges* and the graph induced by the zero edges the *zero graph.* We will use the structure of the zero graph to name the inequalities. This has the following reasons. The zero graph is structured in such a way that there exists no Steiner tree packing for the nets involved in this graph. Therefore, each feasible solution must use edges the coefficients of which are different from zero. This means that each inequality is in some sense (but not necessarily uniquely) determined by the zero graph. In addition, edges obtain value zero for some single nets (we typically denote these sets by $F_1, \ldots, F_N$). We will always define the inequalities for an arbitrary instance without guaranteeing that the inequality is also valid for the corresponding polyhedron. In the succeeding theorem we characterize the instances for which the inequality defines a facet of the corresponding polyhedron. We will see that the edge sets $F_1, \ldots, F_N$ must usually satisfy very technical restrictions. All our results apply to the case when the edge capacities are equal to one, i.e., $c_e = 1$, $e \in E$; for short, $c = \mathbb{1}$.

## 3. Matching and Matching-tree Inequalities

For the first class of inequalities the edge set of the zero graph defines a matching.

Definition 3.1. We are given a graph $G = (V, E)$ and a net list $\mathcal{N} = \{T_1, T_2\}$. Let $M \subseteq [T_1 : T_2]$ be a matching and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. The inequality

$$(\chi^{E\setminus(M \cup F_1)}, \chi^{E\setminus(M \cup F_2)})^T x \geq |M|$$

is called a matching inequality.

An interesting question is: For which choices of $F_1$ and $F_2$ is the matching inequality valid or facet-defining for the Steiner tree packing polyhedron? If $F_1 = F_2 = \emptyset$, the inequality is obviously valid (in fact, the right-hand side can be increased to $2|M| - 2$). On the other hand, if one of both edge sets, $F_1$, say, contains a spanning tree for $T_2$ that is not a star, the inequality is no longer valid. In fact, if both sets are stars the inequality is valid and, in general, also facet-definining. However, are these the only choices for $F_1$ and $F_2$? The following theorem gives the answer.

Theorem 3.2. *We are given the complete graph $G = (V, E)$ with node set $V$, and a disjoint net list $\mathcal{N} = \{T_1, T_2\}$ with $T_1 \cup T_2 = V$ and $|T_1| = |T_2| \geq 4$. Let $M$ be a perfect matching in $(V, [T_1 : T_2])$ and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. Then, the matching inequality*

$$(\chi^{E\setminus(M \cup F_1)}, \chi^{E\setminus(M \cup F_2)})^T x \geq |M|$$

*defines a facet of* $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$ *iff there exist $\tau_1 \in T_1$ and $\tau_2 \in T_2$ such that one of the following three conditions holds:*

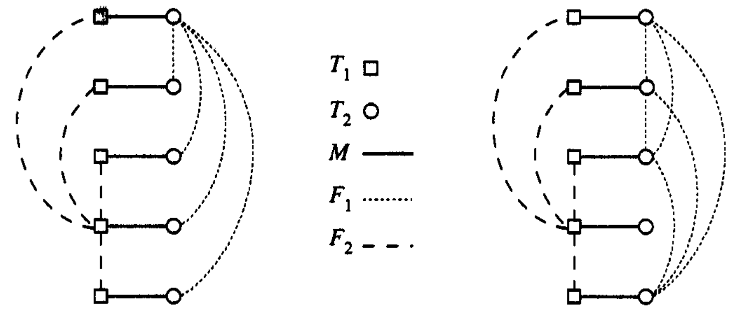


FIGURE 1.

(i) $F_1 = [\tau_2 \colon T_2]$, $F_2 = [\tau_1 \colon T_1]$ *and* $\tau_1\tau_2 \notin M$ (*see* Figure 1(a)).
(ii) $F_1 = E(T_2 \backslash \{\tau_2\})$, $F_2 = [\tau_1 \colon T_1]$ *and* $\tau_1\tau_2 \in M$ (*see* Figure 1(b)).
(iii) $F_1 = [\tau_2 \colon T_2]$, $F_2 = E(T_1 \backslash \{\tau_1\}$ *and* $\tau_1\tau_2 \in M$.

PROOF. We start by showing that Property (i) is sufficient. Set $a := (\chi^{E\backslash(M \cup F_1)}, \chi^{E\backslash(M \cup F_2)})$. First, we prove that $a^{\mathrm{T}}x \geqslant |M|$ is valid. Suppose $P = (S_1, S_2)$ is an arbitrary Steiner tree packing. W.l.o.g., $S_1$ and $S_2$ are edge-minimal. Set $s_k := |M \cap S_k|$ for $k = 1, 2$. We distinguish two cases, as follows:

(a) $s_1 = 0$ or $s_2 = 0$. Suppose that $s_1 = 0$ (the case $s_2 = 0$ can be shown analogously). Then, $(a^1)^{\mathrm{T}}\chi^{S_1} \geqslant |M| - 1$, since no edge of $M \cup F_1$ is incident to $T_1$ and $S_1$ spans $T_1$, with $|T_1| = |M|$. If $(a^2)^{\mathrm{T}}\chi^{S_2} > 0$, we are done. Otherwise, $S_2 = M \cup F_2$ and we know that $V(S_1) \cap T_2 \neq \emptyset$, since $[\tau_1 \colon T_1] = F_2 \subset S_2$. This, however, implies that $(a^1)^{\mathrm{T}}\chi^{S_1} \geqslant |M|$.

(b) $s_1 > 0$ and $s_2 > 0$. It is easy to see that in this case $(a^k)^{\mathrm{T}}\chi^{S_k} \geqslant |M| - |M \cap S_k| = |M| - s_k$ for $k = 1, 2$. This implies that $a^{\mathrm{T}}\chi^P \geqslant 2\,|M| - (s_1 + s_2) \geqslant |M|$, since $s_1 + s_2 \leqslant |M|$.

Now suppose that $b^{\mathrm{T}}x \geqslant \beta$ is a facet-defining inequality of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$ with $I_a := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^{\mathrm{T}}x = |M|\} \subseteq I_b := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^{\mathrm{T}}x = \beta\}$. We show in the following that $b$ is a multiple of $a$. For ease of exposition, set $\bar{k} := 1$ if $k = 2$, and $\bar{k} := 2$ if $k = 1$.

(1) $b^k_e = 0$ for all $e \in F_k$, $k = 1, 2$. Let $A$ be a spanning tree in $(T_k \backslash \{\tau_k\}, E(T_k \backslash \{\tau_k\}))$. Let $u \in T_k \backslash \{\tau_k\}$, $v \in T_{\bar{k}}$ with $uv \notin M$ and $\tau_k v \notin M$ (these nodes exist for $|M| \geqslant 3$). Set $S_k := A \cup \{uv, \tau_k v\}$ and $S_{\bar{k}} = M \cup F_{\bar{k}}$. Then, $P := (S_1, S_2)$ and $P' := P \cup_k e$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$. Thus, $\chi^P, \chi^{P'} \in I_b$, and we have that $0 = \chi^{P'} - \chi^P = b^k_e$.

(2) $b^k_e = 0$ for all $e \in M$, $k = 1, 2$. Let $e = uv \in M$, $u \in T_1$, $v \in T_2$. Due to (i) $uv \neq \tau_1\tau_2$. Suppose, w.l.o.g., that $v \neq \tau_2$. Let $e' \in [u \colon T_1]$. Choose $S_1 := M \cup F_1 \cup \{e'\} \backslash \{e, \tau_2 v\}$ and $S_2 := [v \colon T_2]$. Then, $P := (S_1, S_2)$ and $P' := P \cup_k e$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we conclude that $0 = \chi^{P'} - \chi^P = b^k_e$.

(3) $b^k_e = b^k_{e'}$ for all $e, e' \in E(T_k)$, $k = 1, 2$. Let $u \in T_k$ with $u\tau_{\bar{k}} \notin M$. Let $e_1, e_2 \in [u \colon T_k]$, $e_1 \neq e_2$ and $v \in T_{\bar{k}}$ with $uv \in M$. Choose $S_k := M \cup F_k \cup \{e_1\} \backslash \{uv, \tau_{\bar{k}} v\}$, $S_{\bar{k}} := [v \colon T_{\bar{k}}]$ and $S'_k := S_k \backslash \{e_1\} \cup \{e_2\}$. Then $P := (S_1, S_2)$ and $P' := (P \backslash_k S_k) \cup_k S'_k$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we obtain that $0 = \chi^{P'} - \chi^P = b^k_{e_2} - b^k_{e_1}$. This holds for all $e_1, e_2 \in [u \colon T_k]$ and $u \in T_k$ with $u\tau_{\bar{k}} \notin M$. This implies (3).

(4) $b^k_e = b^k_{e'}$ for all $e \in E(T_{\bar{k}} \backslash \tau_{\bar{k}})$, $e' \in E(T_k)$, $k = 1, 2$. Let $e = uv$ with $u, v \in T_{\bar{k}} \backslash \{\tau_{\bar{k}}\}$. Let $w, x \in T_k$ with $uw, vx \in M$. Choose $S_k := M \cup F_k \cup \{e\} \backslash \{u\tau_{\bar{k}}\}$ and $S'_k := S_k \backslash \{e\} \cup \{wx\}$. Furthermore, let $S_{\bar{k}}$ be a spanning tree in $(T_{\bar{k}}, E(T_{\bar{k}}) \backslash S_k)$ (such a tree exists, since $|M| \geqslant 4$). By construction, $P := (S_1, S_2)$ and $P' := (P \backslash_k S_k) \cup_k S'_k$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we obtain $0 = \chi^{P'} - \chi^P = b^k_{wx} = b^k_e$. This, together with (3), yields the statement.

(5) $b^k_e = b^k_{e'}$ for all $e \in [T_{\bar{k}} \colon T_k] \backslash M$, $e' \in E(T_k)$, $k = 1, 2$. Let $e = uv$, $u \in T_{\bar{k}}$, $v \in T_k$. Let $w \in T_k$, $x \in T_{\bar{k}}$ with $uw, vx \in M$. If $u = \tau_{\bar{k}}$, set $t := x$, otherwise set $t := u$. Choose

$S_k := M \cup F_k \cup \{e\} \setminus \{\tau_{\bar{k}} t\}$, $S_{\bar{k}} := [t: T_{\bar{k}}]$ and $S'_k := S_k \setminus \{e\} \cup \{vw\}$. Then, $P := (S_1, S_2)$ and $P' := (P \setminus_k S_k) \cup_k S'_k$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we have that $0 = \chi^{P'} - \chi^P = b^k_{vw} - b^k_e$. This together with (3), implies (5).

(6) $b^1_e = b^2_{e'}$ for all $e \in E(T_1)$, $e' \in E(T_2)$. Let $A_k$ be a spanning tree in $(T_k \setminus \{\tau_k\}, E(T_k \setminus \{\tau_k\}))$. Let $u_k \in T_k \setminus \{\tau_k\}$, $v_k \in T_{\bar{k}}$ with $u_k v_k \notin M$ and $\tau_k v_k \notin M$ (these nodes exist for $|M| \geq 3$). Set $S_k := A_k \cup \{u_k v_k, \tau_k v_k\}$ and $S'_k := M \cup F_k$ for $k = 1, 2$. Then, $P := (S'_1, S_2)$ and $P' := (S_1, S'_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and it follows, together with (3) and (5), that $0 = \chi^{P'} - \chi^P = |M| \cdot b^1_{u_1\tau_1} - |M| \cdot b^2_{u_2\tau_2}$. This shows (6).

(1)–(6) imply that $b$ is a multiple of $a$. Hence, we have proved that $a^T x \geq |M|$ defines a facet for STP$(G, \mathcal{N}, \mathbb{1})$, if (i) holds. In a very similar way it can be shown that Properties (ii) and (iii) are sufficient as well. So we omit the proofs.

Next, we show that (i)–(iii) of Theorem 3.2 indeed describe all possible cases for $F_1$ and $F_2$ such that the corresponding matching inequality is facet-defining. Suppose that $(\chi^{E \setminus (M \cup F_1)}, \chi^{E \setminus (M \cup F_2)})^T x \geq |M|$ defines a facet for STP$(G, \mathcal{N}, \mathbb{1})$. Set $a := (\chi^{E \setminus (M \cup F_1)}, \chi^{E \setminus (M \cup F_2)})$ and let $l_k$ denote the number of (connected) components of $(V, M \cup F_k)$, for $k = 1, 2$. We assume, w.l.o.g., that $l_1 \leq l_2$.

Suppose that $l_k = 1$ for $k = 1$, say, and $F_1$ is not a star. Then, there exist two pairwise edge-disjoint spanning trees $A_1$ and $A_2$ in $(T_2, E(T_2))$ with $A_1 \subseteq F_1$. Thus, $P = (S_1, S_2)$, where $S_1 := M \cup A_1$ and $S_2 := A_2$ is a Steiner tree packing with $a^T \chi^P = |M| - 1$, a contradiction to the validity of $a^T x \geq |M|$.

Since $a^T \geq$ |M| defines a facet, we know that, for every edge $e \in M$, there exists a root $P = (S_1, S_2)$ with $e \notin P$; otherwise $I_a \subseteq \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid x^1_e + x^2_e = 1\}$, a contradiction. Moreover, we know that, for a root $P = (S_1, S_2)$ with $e \notin P$, $e \in M$, either $M \cap S_1 = \emptyset$ or $M \cap S_2 = \emptyset$; otherwise $a^T \chi^P \geq (|M| - |M \cap S_1|) + (|M| - |M \cap S_2|) = 2|M| - (|M \cap S_1| + |M \cap S_2|) \geq |M| + 1$. In the following we show that, for all possible remaining choices of $F_1$ and $F_2$, we can find an edge $e \in M$ such that there does not exist a root $P = (S_1, S_2)$ with $e \notin P$ and $M \cap S_1 = \emptyset$ or $M \cap S_2 = \emptyset$. This proves the statement.

First, suppose that both $F_1 = [\tau_2: T_2]$, $\tau_2 \in T_2$, and $F_2 = [\tau_1: T_1]$, $\tau_1 \in T_1$, are stars, but $\tau_1 \tau_2 \in M$. Suppose that there exists a root $P = (S_1, S_2)$ with $\tau_1 \tau_2 \notin P$ with, w.l.o.g., $M \cap S_1 = \emptyset$. Then, we know that $(a^2)^T \chi^{S_2} = 1$, since $(a^1)^T \chi^{S_1} \geq |M| - 1$. Since $\tau_1 \tau_2 \notin S_2$, we conclude that $F_2 \subset S_2$. This, however, implies that $(a^1)^T \chi^{S_1} \geq |M|$, since $F_2 = [\tau_1: T_1]$, a contradiction.

Now, we know that $l_2 \geq 2$. Suppose still that $F_1 = [\tau_2: T_2]$. Then, since (iii) does not apply, we conclude that the node $t \in T_1$ with $t\tau_2 \in M$ is incident to an edge in $F_2$. Suppose that there exists a root $P = (S_1, S_2)$ with $t\tau_2 \notin P$ and $M \cap S_1 = \emptyset$ or $M \cap S_2 = \emptyset$. If $M \cap S_1 = \emptyset$, we know that $(a^1)^T \chi^{S_1} \geq |M| - 1$ and, since the number of (connected) components of $(V, (M \cup F_2) \setminus \{t\tau_2\})$ is at least three (note that $t \in V(F_2)$ and $l_2 \geq 2$), that $(a^2)^T \chi^{S_2} \geq 2$, a contradiction. If $M \cap S_2 = \emptyset$, we have $(a^1)^T \chi^{S_1} = 1$. This implies that $F_1 \subset S_1$, since $t\tau_2 \neq S_1$. However, since $F_1$ is a star, $(a^2)^T \chi^{S_2} \geq |M|$, a contradiction.

We also conclude that $l_1 \geq 2$. Then, at least one of the following cases applies:

(1) $l_k \geq 3$ for $k = 1, 2$. In this case, we immediately obtain a contradiction, since $(a^k)^T \chi^{S_k} \geq 2$ and $(a^k)^T \chi^{S_k} \geq |M| - 1$, if $M \cap S_k = \emptyset$, for $k = 1, 2$.

(2) There exists an edge $uv \in M$ with $u \in V(F_1)$ and $v \in V(F_2)$. Suppose there exists a root $P = (S_1, S_2)$ with $uv \notin P$ and w.l.o.g. $M \cap S_1 = \emptyset$. Since $uv \notin S_2$ and the number of (connected) components of $(V, (M \cup F_2) \setminus \{uv\})$ is at least three (note that $v \in V(F_2)$ and $l_2 \geq 2$), we have that $a^T \chi^P \geq (|M| - 1) + 2 = |M| + 1$, a contradiction.

(3) $F_2 = \emptyset$. In this case we know that, for every Steiner tree packing $P = (S_1, S_2)$, $(a^2)^T \chi^{S_2} \geq |M| - 1$ and $(a^1)^T \chi^{S_1} \geq 1$, since $l_1 \geq 2$. Thus, every root $P = (S_1, S_2)$ satisfies $(a^2)^T \chi^{S_2} = |M| - 1$. This implies that $I_a \subseteq \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid \frac{1}{2} \sum_{uv \in M} x(\delta(\{u, v\})) = |M| - 1\}$, a contradiction.

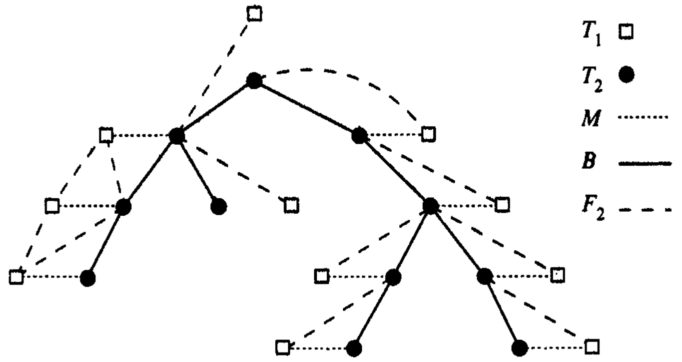


FIGURE 2.

Summing up, we conclude that the only choices for $F_1$ and $F_2$ are those described in (i)–(iii) of Theorem 3.2. □

The last theorem gives necessary and sufficient conditions for the matching inequality to be facet-defining under the assumptions that the zero graph is a matching and all edges in $F_1$ and $F_2$, are incident to nodes in $T_2$ and $T_1$, respectively. What happens if we relax one of these assumptions? In the following we give a partial answer to this question and extend the zero graph by a tree.

DEFINITION 3.3. We are given a graph $G = (V, E)$ and a net list $\mathcal{N} = \{T_1, T_2\}$. Let $M \subseteq [T_1 : T_2]$ be a matching and let $B$ be a spanning tree in $(V(M) \cap T_2, E(V(M) \cap T_2))$ (see Figure 2). Moreover, let $F_1, F_2 \subseteq E \setminus (M \cup B)$. Then, the ineaquality

$$(\chi^{E\setminus(M\cup B\cup F_1)}, \chi^{E\setminus(M\cup B\cup F_2)})x \geq |B|$$

is called *matching-tree inequality*.

It is easy to see that the basic form of a matching-tree inequality, i.e., $F_1 = F_2 = \emptyset$, is valid for STP$(G, \mathcal{N}, \mathbb{1})$, but in general it is not facet-defining. In the next theorem we present necessary and sufficient conditions for $F_1$, $F_2$ such that the matching-tree inequality is facet-defining.

THEOREM 3.4. *Let $G = (V, E)$ be a complete graph on node set $V$ and let $\mathcal{N} = \{T_1, T_2\}$ be a disjoint net list with $T_1 \cup T_2 = V$, $|T_1| = |T_2| \geq 2$. Suppose that $M$ is a perfect matching in $(V, [T_1 : T_2])$, $B$ is a spanning tree in $(T_2, E(T_2))$ and $F_1, F_2 \subseteq E \setminus (M \cup B)$. For two nodes $u, v \in V$, let $bd(u, v)$ denote the number of edges in $B$ contained in the unique path $P$ from $u$ to $v$ in $(V, M \cup B)$, i.e., $bd(u, v) := |P \cap B|$. Then, the matching-tree inequality*

$$(\chi^{E\setminus(M\cup B\cup F_1)}, \chi^{E\setminus(M\cup B\cup F_2)}) \geq |B|$$

*defines a facet of* STP$(G, \mathcal{N}, \mathbb{1})$ *iff $F_1$ and $F_2$ satisfy the following properties:*
(i) *$F_1 = \emptyset$.*
(ii) *$(V, M \cup F_2)$ is connected.*
(iii) *For $r, s = 1, 2$, all pairs of nodes $u \in T_s$ and $v \in T_r$ with $bd(u, v) \geq 5 - r - s$ are not connected in $(V(F_2), F_2)$.*
(iv) *$F_1$ and $F_2$ are maximal with respect to Properties* (i)–(iii).

PROOF. For ease of notation we assume that $T_k = \{t_1^k, \dots, t_l^k\}$, $k = 1, 2$ with

$l := |T_1| = |T_2|$ such that $M = \{t_i^1 t_i^2 \mid i = 1, \ldots, l\}$. We begin by showing that Properties (i)–(iv) are sufficient. Let $a := (\chi^{E\setminus(M\cup B\cup F_1)}, \chi^{E\setminus(M\cup B\cup F_2)})$.

First, we prove that $a^T x \geq |B|$ is valid. Let $P = (S_1, S_2)$ be any Steiner tree packing. We assume w.l.o.g. that $S_1$ and $S_2$ are edge-minimal. We show that there always exists a Steiner tree packing $P' = (S_1', S_2')$ with $a^T\chi^{P'} \leq a^T\chi^P$ that satisfies the following two properties:

(A) If $\delta(t_i^2) \cap S_1' \cap B \neq \emptyset$, then $t_i^2 t_i^1 \in S_1'$ (for $i = 1, \ldots, l$).

(B) If $t_i^2 t_j^2 \in S_1'$ for some $i, j \in \{1, \ldots, l\}$ with $bd(t_i^2, t_j^2) = 1$ (i.e., $t_i^2 t_j^2 \in B$), then $t_i^2$ and $t_j^2$ are not connected in $(V, (M \cup B \cup F_2)\setminus S_1')$.

Intuitively, Property (A) becomes clear by drawing a picture. A formal proof of this statement is quite technical and we omit it here (for details, see [7]). Property (B) directly follows from (A) and from Property (iii). Thus, in order to prove that $a^T x \geq |B|$ is valid, we can assume that $P$ already satisfies Properties (A) and (B).

Now let $s_1$ denote the number of (connected) components of $(T_2, B \cap S_1)$ and let $s_2$ denote the number of (connected) components of $(T_2, B\setminus S_1)$. Since $B$ is a spanning tree of $T_2$, it is not difficult to see that

$$(*) \qquad s_1 + s_2 = (|B\setminus S_1| + 1) + (|B \cap S_1| + 1) = |B| + 2.$$

Property (B) implies that two components of $(T_2, B\setminus S_1)$ are not connected in $(V, (M \cup B \cup F_2)\setminus S_1)$. Thus, we have that $(a^2)^T\chi^{S_2} \geq s_2 - 1$. Moreover, since $F_1 = \emptyset$, there does not exist a path in $(V, M \cup B \cup F_1)$ connecting two different components of $(T_2, B \cap S_1)$. Thus, $(a^1)^T\chi^{S_1} > s_1 - 1$. Summing up, we conclude that $a^T\chi^P = (a^1)^T\chi^{S_1} + (a^2)^T\chi^{S_2} \geq (s_1 - 1) + (s_2 - 1) = s_1 + s_2 - 2 = |B|$, where the last equality follows from (*).

Now, let $b^T x \geq \beta$ be a facet-defining inequality of STP$(G, \mathcal{N}, \mathbb{1})$ with $I_a := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^T x = |B|\} \subseteq I_b := \{x \in \text{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^T x = \beta\}$. In the following we show that $b$ is a multiple of $a$.

(1) $b_e^k = 0$ for $e \in M$, $k = 1, 2$. Let $S_1 := [t_1^1 : T_1]$ and $S_2 := B$. Then, $P := (S_1, S_2)$ and $P' := P \cup_k e$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$. Thus, $\chi^P, \chi^{P'} \in I_b$ and we have $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(2) $b_e^2 = 0$ for $e \in F_2$. This can be shown as in (1).

(3) $b_e^k = 0$ for $e \in B$, $k = 1, 2$. Let $(V_1, B_1)$ and $(V_2, B_2)$ be the two (connected) components of $(V(B), B\setminus\{e\})$. Property (ii) implies that there exists an edge $e' = t_u^p t_v^q \in F_2$ with $t_u^2 \in V_1$ and $t_v^2 \in V_2$. If $l = 2$, Property (iv) guarantees that $e'$ can be chosen such that $p \neq q$, and we set $S_1 := \{t_u^1 t_v^1\}$ and $S_2 := M \cap \{e'\}$. In the other case $(l \geq 3)$, choose an index $i \in \{1, \ldots, l\}$, $i \neq u$, $i \neq v$. We set $S_1 := [t_i^1 : T_1]$ and $S_2 := B_1 \cup B_2 \cup M \cup \{e'\}$. Then, $P := (S_1, S_2)$ and $P' := P \cup_k e$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we obtain $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(4) $b_2^1 = b_{e'}^1$ for $e, e' \in E(T_1)$. Let $e = t_u^1 t_v^1$, where $t_u^1, t_v^1 \in T_1$. Set $S_2 := B$ and $S_1 := [t_v^1 : T_1]$. Moreover, let $e' \in [t_u^1 : T_1]\setminus\{e\}$ and $S_1' := S_1\setminus\{e\} \cup \{e'\}$. Then, $P := (S_1, S_2)$ and $P' := (S_1', S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$. We conclude that $0 = b^T(\chi^{S_1'}, \chi^{S_2}) - b^T(\chi^{S_1}, \chi^{S_2}) = b_{e'}^1 - b_e^1$, for all $e, e \in \delta(t_u^1)$, $t_u^1 \in T_1$.

(5) $b_e^1 = b_{e'}^1$ for $e \in E\setminus(E(T_1) \cup M \cup B)$, $e' \in E(T_1)$. Let $e = t_u^p t_v^q$ with $e \notin E(T_1) \cup M \cup B$. Set $S_2 := B$, $S_1 := [t_u^1 : T_1]$ and $S_1' := S_1\setminus\{t_u^1 t_v^1\} \cup M \cup \{e\}$. Then, $P := (S_1, S_2)$ and $P' := (S_1', S_2)$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and it follows that $0 = \mathrm{b}^T(\chi^{S_1}, \chi^{S_2}) - b^T(\chi^{S_1'}, \chi^{S_2}) = b_{t_u^1 t_v^1}^1 - b_e^1$. This, together with (4), implies (5).

(6) $b_e^2 = b_{e'}^1$ for $e \in E\setminus(M \cup B \cup F_2)$, $e' \in E(T_1)$. Let $e = t_u^p t_v^q$ with $e \notin M \cup B \cup F_2$, where $u, v \in \{1, \ldots, l\}$, $p, q \in \{1, 2\}$. Due to Properties (iii) and (iv) we know that there exist $i, j \in \{1, \ldots, l\}$, $s, r \in \{1, 2\}$ with $bd(t_i^s, t_j^r) \geq 5 - r - s$ such that there exists a path $W$ from $t_i^s$ to $t_j^r$ in $(V(F_2 \cup \{e\}), F_2 \cup \{e\})$ with $e \in W$ (in case $W$ is not unique, choose $W$ such that $|W \cap E(T_1)|$ is minimal). We distinguish three cases:

(a) $s = r = 1$: since $bd(t_i^s, t_j^r) \geqslant 3$, there exists indices $i_0, j_0 \in \{1, \ldots, l\} \setminus \{i, j\}$ with $bd(t_{i_0}^2, t_{j_0}^2) = 1$, $bd(t_{i_0}^2, t_i^s) < bd(t_{j_0}^2, t_i^s)$ and $bd(t_{i_0}^2, t_j^r) > bd(t_{j_0}^2, t_j^r)$. Set $M_2 := \{t_i^1 t_i^2, t_j^1 t_j^2\}$.
(b) $s = 1$, $r = 2$ (the other case $s = 2$; $r = 1$ can be shown analogously): since $bd(t_i^s, t_j^r) \geqslant 2$, there exists an index $j_0 \in \{1, \ldots, l\} \setminus \{i, j\}$ with $bd(t_i^s, t_{j_0}^2) = 1$ and $bd(t_{j_0}^2, t_j^r) < bd(t_i^s, t_j^r)$. Set $i_0 := i$ and $M_2 := \{t_j^1 t_j^2\}$.
(c) $s = 2$, $r = 2$: since $bd(t_i^s, t_j^r) \geqslant 1$, there exists $j_0 \in \{1, \ldots, l\}$ with $bd(t_i^s, t_{j_0}^2) = 1$ and $bd(t_{j_0}^2, t_j^r) \leqslant bd(t_i^s, t_j^r)$. Set $i_0 := i$ and $M_2 := \emptyset$.

Set $U := \{t_{i_0}^1 t_{i_0}^2, t_{i_0}^2 t_{j_0}^2, t_{j_0}^2 t_{j_0}^1\}$ (note that $U \cap M_2 = \emptyset$). Choose $S_2 := (B \setminus U) \cup M_2 \cup W$, $s_1' := [t_{i_0}^1 : T_1]$ and $S_2' := B$. If $l = 2$, set $S_1 := U$. If $l = 3$, there exists an edge $\bar{e} \in E(T_1) \setminus W$ with $\bar{e} \neq t_{i_0}^1 t_{j_0}^1$, since $W$ was chosen such that $|W \cap E(T_1)|$ is minimal. Set $S_1 := \{\bar{e}\} \cup U$. For $l \geqslant 4$, there exists a spanning tree $A$ in $(T_1, E(T_1) \setminus W)$, because $W$ is a path. Let $\hat{e} \in A$ such that $A \setminus \{\hat{e}\} \cup \{t_{i_0}^1 t_{j_0}^1\}$ is a spanning tree as well. Set $S_1 := A \setminus \{\hat{e}\} \cup U$. Then, $P := (S_1, S_2)$ and $P' := (S_1', S_2')$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$, and we have $0 = b^{\mathrm{T}}(\chi^{S_1}, \chi^{S_2}) - b^{\mathrm{T}}(\chi^{S_1'}, \chi^{S_2'}) = b_e^2 - b_{t_{i_0}^1 t_{j_0}^1}^1$. This, together with (4), proves the statement

(1)–(6) imply that $b$ is a multiple of $a$.

It remains to be shown that Properties (i)–(iv) are necessary as well.

(i) Suppose that $F_1 \neq \emptyset$. Let $e = t_u^p t_v^q \in F_1$, $e \notin M \cup B$. Choose $S_1 := [t_u^1 : T_1 \setminus \{t_v^1\}] \cup M \cup \{e\}$ and $s_2 := B$. Then, $P := (S_1, S_2)$ is a packing of Steiner trees with $a^{\mathrm{T}} \chi^P = |B| - 1$, a contradiction to the validity of $a^{\mathrm{T}} x \geqslant |B|$.

(ii) Suppose that $(V, M \cup F_2)$ is not connected. Then, there exist indices $i, j \in \{1, \ldots, l\}$ with $bd(t_i^2, t_j^2) = 1$ such that there does not exist a path from $t_i^2$ to $t_j^2$ in $(V, M \cup F_2)$. Since $a^{\mathrm{T}} x \geqslant |B|$ defines a facet of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$, there exists a Steiner tree packing $P = (S_1, S_2)$ with $a^{\mathrm{T}} \chi^P = |B|$ and $t_i^2 t_j^2 \notin P$. Otherwise, we will have the contradiction that $I_a \subseteq \{X \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid x_{t_i^2 t_j^2}^1 + x_{t_i^2 t_j^2}^2 = 1\}$. Let $W$ be the unique path from $t_i^2$ to $t_j^2$ in $(V(S_2), S_2)$, where we assume w.l.o.g. that $S_2$ is edge-minimal. Since $t_i^2 t_j^2 \notin S_2$, and since there does not exist a path from $t_i^2$ to $t_j^2$ in $(V, M \cup F_2)$, there is an edge $e \in W$ with $a_e^2 = 1$. Choose $S_2' := S_2' \setminus \{e\} \cup \{t_i^2 t_j^2\}$. Note that $t_i^2 t_j^2 \notin S_1$. Then, $P' := (S_1, S_2')$ is also a Steiner tree packing, and we have that $a^{\mathrm{T}} \chi^{P'} = a^{\mathrm{T}} \chi^P - 1 = |B| - 1$, a contradiction.

(iii) Suppose that there exist indices $i, j \in \{1, \ldots, l\}$, $r, s \in \{1, 2\}$ with $bd(t_i^r, t_j^s) \geqslant 5 - r - s$ such that there is a path $W$ from $t_i^r$ to $t_j^s$ in $(V(F_2), F_2)$. Then, in the same manner as described in (6), we can construct a packing of Steiner trees $P = (S_1, S_2)$ with $a^{\mathrm{T}} \chi^P = |B| - 1$, which yields a contradiction.

(iv) Suppose that $F_1$ and $F_2$ are not maximal with respect to Properties (i)–(iii). Then, choose $F_2' \subset E \setminus (M \cup B)$ such that $F_2 \subset F_2'$, and $F_1$ and $F_2'$ are maximal with respect to Properties (i)–(iii). According to Part 1 of this proof, $(\chi^{E \setminus (M \cup B \cup F_1)}, \chi^{E \setminus (M \cup B \cup F_2')})^{\mathrm{T}} x \geqslant |B|$ defines a facet of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$. Summing up this facet-defining inequality together with the valid inequalities $x_e^2 \geqslant 0$ for all $e \in F_2' \setminus F_2$ we obtain $a^{\mathrm{T}} x \geqslant |B|$. Thus, $a^{\mathrm{T}} x \geqslant |B|$ does not define a facet of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$, a contradiction. □

In this section we have presented two classes of inequalities each combining two nets. The zero graphs have quite simple structures; namely, they either form a matching or a matching and a tree. The fact that a maximum matching or a maximum spanning tree can be determined in polynomial time gives hope to efficiently solve the corresponding separation problems. However, the structure of the additional edge sets $F_1$ and $F_2$, which are the edges the coefficient of which is zero for net $T_1$ and $T_2$, is

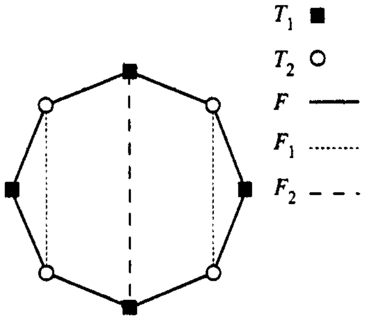


FIGURE 3.

rather complicated and makes it difficult to develop good separation algorithms that take these edge sets into account.

In the next section, the situation becomes even more complicated. When more than two nets are involved not only the edge sets $F_1, \ldots, F_N$ but also the zero graphs may have difficult structures.

## 4. Inequalities Involving More Than Two Nets

The two classes of inequalities we present in this section are extensions of so-called alternating cycle inequalities introduced in [3]. First, we extend the alternating cycle inequality by a third net and add two 'ears' to the alternating cycle. An inequality of the second class is composed of two or more alternating cycle inequalities. We will see that this composition applies to an arbitrary number of terminal sets.

Before describing both inequalities let us give the definition of an alternating cycle inequality and recall a theorem from [3] characterizing conditions under which this inequality is facet-defining.

DEFINITION 4.1. Let $G = (V, E)$ be a graph and $\mathcal{N} = \{T_1, T_2\}$ a net list. We call a cycle $F$ an alternating cycle with respect to $T_1$, $T_2$, if $F \subseteq [T_1 \colon T_2]$ and $V(F) \cap T_1 \cap T_2 = \emptyset$ (see Figure 3). Moreover, let $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ be two sets of diagonals of the alternating cycle $F$ with respect to $T_1$, $T_2$. The inequality

$$(\chi^{E\setminus(F \cup F_1)}, \chi^{E\setminus(F \cup F_2)})^{\mathrm{T}} x > \tfrac{1}{2}|F| - 1$$

is called an *alternating cycle inequality.*

The following theorem gives necessary and sufficient conditions for $F_1$ and $F_2$ so that the alternating cycle inequality is facet-defining. In order to state this result we need some definitions. We say that two diagonals $uv$ and $rs$ of a cycle $F$ *cross* if they appear on $F$ in the sequence $u, r, v, s$ or $u, s, v, r$; otherwise, $uv$ and $rs$ are called *cross free.* For an alternating cycle $F$ with respect to $T_1$, $T_2$, we call two sets of diagonals $F_1 \subseteq E(T_2)$ and $F_2 \subseteq E(T_1)$ *maximal cross free* if $F_1$ and $F_2$ are cross free (that is, each pair of edges $e_1 \in F_1$ and $e_2 \in F_2$ is cross free), each diagonal $e_1 \in E(T_1)\setminus F_2$ crosses $F_1$ and each diagonal $e_2 \in E(T_2)\setminus F_1$ crosses $F_2$.

THEOREM 4.2. *Let $G = (V, E)$ be the complete graph with node set $V$ and let $\mathcal{N} = \{T_1, T_2\}$ be a disjoint net list with $T_1 \cup T_2 = V$ and $|T_1| = |T_2| = l$, $l \geq 2$. Furthermore, let $F$ be an alternating cycle with respect to $T_1$, $T_2$ with* $V(F) = V$ and $F_1 \subseteq E(T_2)$, $F_2 \subseteq E(T_1)$. *Then the alternating cycle inequality*

$$(\chi^{E\setminus(F \cup F_1)}, \chi^{E\setminus(F \cup F_2)})^{\mathrm{T}} x \geq l - 1$$

*defines a facet of* $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$ *iff $F_1$ and $F_2$ are maximal cross free.*

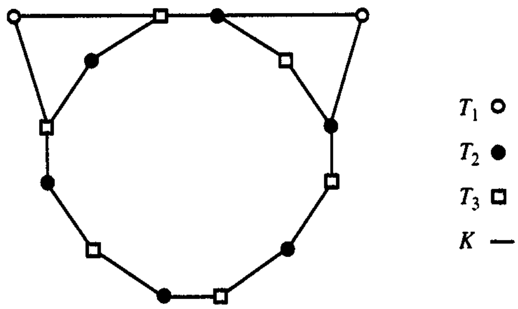


FIGURE 4.

4.1. 2-*Eared Alternating Cycle Inequalities*

DEFINITION 4.3. We are given a graph $G = (V, E)$ and a net list $\mathcal{N} = \{T_1, T_2, T_3\}$. Let $C$ be an alternating cycle with respect to $T_2$, $T_3$ and let $t_1, t_2 \in T_1 \backslash V(C)$. Moreover, choose $e_i, e_j \in [t_1: T_2 \cap V(C)]$, $e_i \neq e_j$, and $e_r, e_s \in [t_2: T_3 \cap V(C)]$, $e_r \neq e_s$. Set $K := C \cup \{e_i, e_j, e_r, e_s\}$ and let $F_1, F_2, F_3 \subseteq E$. The inequality

$$(\chi^{E \backslash (K \cup F_1)}, \chi^{E \backslash (K \cup F_2)}, \chi^{E \backslash (K \cup F_3)})^{\mathrm{T}} x \geq 1$$

is called 2-*eared alternating cycle inequality* (see Figure 4).

The following theorem specifies choices for $F_1$, $F_2$ and $F_3$ such that the 2-eared alternating cycle inequality is facet-defining.

THEOREM 4.4. *Let $G = (V, E)$ be the complete graph on node set $V$, $\mathcal{N} = \{T_1, T_2, T_3\}$ a disjoint net list with $T_1 \cup T_2 \cup T_3 = V$ and $|T_2| = |T_3| =: l$, $l \geq 2$. Let $T_1 = \{t_1, t_2\}$, $e_i, e_j \in [t_1: T_2]$, $e_i \neq e_j$ and $e_r, e_s \in [t_2: T_3]$, $e_r \neq e_s$. Moreover, suppose that $C$ is an alternating cycle with respect to $T_2$, $T_3$, where $V(C) = T_2 \cup T_3$. Set $K := C \cup \{e_i, e_j, e_r, e_s\}$, $F_1 := E(T_2) \cup E(T_3)$, $F_2 := E(T_3) \cup ([t_2: T_3] \backslash \{e_r, e_s\})$ and $F_3 := E(T_2) \cup ([t_1: T_2] \backslash \{e_i, e_j\})$. Then, the 2-eared alternating cycle inequality*

$$(\chi^{E \backslash (K \cup F_1)}, \chi^{E \backslash (K \cup F_2)}, \chi^{E \backslash (K \cup F_3)})^{\mathrm{T}} x \geq 1$$

*defines a facet of* $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$.

PROOF. We start by showing that $a^{\mathrm{T}} x \geq 1$ is valid, where $a := (\chi^{E \backslash (K \cup F_1)}, \chi^{E \backslash (K \cup F_2)}, \chi^{E \backslash (K \cup F_3)})$. Let $P = (S_1, S_2, S_3)$ be an arbitrary Steiner tree packing. If $(a^2)^{\mathrm{T}} \chi^{S_2} > 0$ or $(a^3)^{\mathrm{T}} \chi^{S_3} > 0$, the inequality trivially holds. On the other hand, if $(a^2)^{\mathrm{T}} \chi^{S_2} = 0$ and $(a^3)^{\mathrm{T}} \chi^{S_3} = 0$, we have that $C \subset S_2 \cup S_3$. This implies that $S_1 \not\subseteq K \cup F_1$. Thus, $(a^1)^{\mathrm{T}} x \geq 1$ in this case, and we conclude that the inequality is valid.

Let us briefly sketch the proof that $a^{\mathrm{T}} x \geq 1$ is also facet-defining. Again, suppose that $b^{\mathrm{T}} x \geq \beta$ is a facet-defining inequality of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$ that satisfies $I_a := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^{\mathrm{T}} x = 1\} \subseteq I_b := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^{\mathrm{T}} x = \beta\}$, and we show that $b$ is a multiple of $a$.

First of all, one can easily convince oneself that, for each edge $e \in K \cup F_k$, there exists a root $P$ with $e \notin P$, for $k = 1, 2, 3$. This shows that $b_e^k = 0$ for all $e \in K \cup F_k$, $k = 1, 2, 3$.

Moreover, for each edge $e \notin K \cup F_k$, one can find a root $P = (S_1, S_2, S_3)$ with

$S_r \subseteq K \cup F_r$, for $r \neq k$ and $S_k \cap (V \backslash (K \cup F_k)) = \{e\}$, for $k = 1, 2, 3$. This proves that $b_e^k = \beta$, and the result follows.

In fact, finding the appropriate Steiner tree packings as necessary is (somehow) straightforward, but the description of the constructions is quite technical, so we omit the details here. A complete proof of Theorem 4.4 can be found in [7]. □

One of the requirements in Theorem 4.4 is that the net list $\mathcal{N}$ is disjoint. One can drop this assumption and still obtain facet-defining inequalities. In this case, however, the edge sets $F_2$ and $F_3$ must be extended. The following corollary describes one such case. We state this without a proof and refer the interested reader to [7] for more details.

COROLLARY 4.5. *Let $G = (V, E)$ be the complete graph on node set $V$, $\mathcal{N} = \{T_1, T_2, T_3\}$ be a net list with $T_1 \cup T_2 \cup T_3 = V$, $|T_2| = |T_3| =: l + 1$, $l \geq 3$ and $T_1 = \{t_1, t_2\}$ such that $T_1 \cap T_2 = \{t_1\}$ and $T_1 \cap T_3 = \{t_2\}$. Moreover, let $e_i, e_j \in [t_1 : T_2]$, $e_i \neq e_j$, and $e_r, e_s \in [t_2 : T_3]$, $e_r \neq e_s$. Suppose that $C$ is an alternating cycle with respect to $T_2$, $T_3$ with $V(C) = (T_2 \cup T_3) \backslash T_1$. Finally, choose $f_2 \in [t_2 : T_2]$ and $f_3 \in [t_1 : T_3]$. Set $F := C \cup \{e_i, e_j, e_r, e_s\}$, $F_1 := E(T_2) \cup E(T_3)$, $F_2 := (E(T_3) \cup [t_2 : T_3] \cup \{f_2\}) \backslash \{e_r, e_s\}$ and $F_3 := (E(T_2) \cup [t_1 : T_2] \cup \{f_3\}) \backslash \{e_i, e_j\}$. Then, the 2-eared alternating cycle inequality*

$$(\chi^{E \backslash (F \cup F_1)}, \chi^{E \backslash (F \cup F_2)}, \chi^{E \backslash (F \cup F_3)})^{\mathrm{T}} x \geq 1$$

*defines a facet of* STP$(G, \mathcal{N}, \mathbb{1})$.

## 4.2. *Composition of Alternating Cycles*

In this subsection we present a class of inequalities that involves an arbitrary number of nets. The idea behind our construction is to compose several facet-defining alternating cycle inequalities.

THEOREM 4.6. *Let $G = (V, E)$ be a complete graph with node set $V$, and let $\mathcal{N} = \{T_1, \ldots, T_N\}$ be a disjoint net list with $\bigcup_{k=1}^{N} T_k = V$ and $|T_k| =: l$, $l \geq 2$, for $k = 1, \ldots, N$. Moreover, let $C_k$ be an alternating cycle with respect to $T_1$, $T_k$ such that $V(C_k) = T_1 \cup T_k$ for $k = 2, \ldots, N$. Finally, set $F := \bigcup_{k=2}^{N} C_k$ and $F_k := \{[T_p : T_q] \mid p = 1, \ldots, N,\ q = 1, \ldots, N,\ q \neq k,\ p \neq k\}$ for $k = 2, \ldots, N$ (see* Figure 5*). Then, the inequality*

$$(\chi^{E \backslash F}, \chi^{E \backslash (F \cup F_2)}, \ldots, \chi^{E \backslash (F \cup F_N)})^{\mathrm{T}} x \geq l - 1$$

*defines a facet of* STP$(G, \mathcal{N}, \mathbb{1})$.

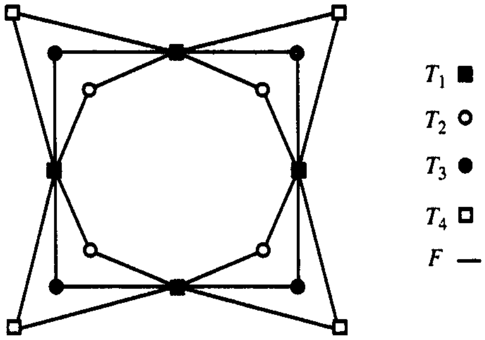


FIGURE 5.

PROOF. Set $a = (\chi^{E\setminus F}, \chi^{E\setminus(F\cup F_2)}, \ldots, \chi^{E\setminus(F\cup F_N)})$. We first show that the inequlity is valid. Let $P = (S_1, \ldots, S_N)$ be any edge-minimal Steiner tree packing. Let $I_k := \{t \in T_k \mid \delta(t) \cap C_k \subset S_1\}$ denote the set of nodes of $T_k$ that are isolated by $S_1$ on the cycle $C_k$. Since $\{e \in E \mid a_e^1 = 0\} = F$, we have that $(a^1)^T\chi^{S_1} \geq l - 1 - \sum_{k=2}^N |I_k|$. Since $S_1$ is edge-minimal, $T_k \setminus I_k \neq \emptyset$, for all $k = 2, \ldots, N$. Moreover, $\{e \in \delta(t) \mid a_e^k = 0\} \subset F$ for all $t \in T_k$, $k = 2, \ldots, N$. These two facts imply that $(a^k)^T x \geq |I_k|$ for $k = 2, \ldots, N$. Summing up, we obtain $a^T\chi^P \geq (l - 1 - \sum_{k=2}^N |I_k|) + \sum_{k=2}^N |I_k| = l - 1$. Thus, the inequality is valid.

Now suppose that $b^T x \geq \beta$ is a facet-defining inequality of $\mathrm{STP}(G, \mathcal{N}, \mathbb{1})$ such that $I_a := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid a^T x = l - 1\} \subseteq I_b := \{x \in \mathrm{STP}(G, \mathcal{N}, \mathbb{1}) \mid b^T x = \beta\}$. We show that $b$ is a multiple of $a$.

First, we observe the following. Consider two terminal sets $T_1$ and $T_k$ for some $k \in \{2, \ldots, N\}$. Let $V' := V(C_k)$ and $E' = E(V')$, and set $F_1' := \emptyset$ and $F_k' := E' \cap F_k = E'(T_1)$. Obviously, $C_k$ is an alternating cycle in the complete graph $G' = (V', E')$, and $F_1'$ and $F_k'$ are maximal cross free with respect to $C_k$. Thus, due to Theorem 4.2, $a' := (\chi^{E'\setminus(C_k\cup F_1')}, \chi^{E'\setminus(C_k\cup F_k')})^T x \geq l - 1$ defines a facet for $\mathrm{STP}(G', \{T_1, T_k\}, \mathbb{1})$. Every root $P' = (S_1', S_k')$ of $(a')^T x \geq l - 1$ can easily be extended to a root $P = (S_1, \ldots, S_N)$ of $a^T x \geq l - 1$ by setting $S_1 := S_1'$, $S_k := S_k'$ and $S_r := C_r$ for all $r \in \{2, \ldots, N\}$, $r \neq k$. Therefore, from Theorem 4.2 we can conclude that:

(1) $b_e^k = 0$ for all $e \in F$, $k = 1, \ldots, N$;
(2) $b_e^k = 0$ for all $e \in F_k'$, $k = 2, \ldots, N$;
(3) $b_e^k = b_{e'}^k$ for all $e, e' \in E(T_k)$, $k = 1, \ldots, N$;
(4) $b_e^1 = b_{e'}^k$ for all $e \in E(T_1)$, $e' \in E(T_k)$, $k = 2, \ldots, N$;
(5) $b_e^1 = b_{e'}^1$ for all $e' \in E(T_1)$, $e \in [T_1 : T_k]$, $k = 2, \ldots, N$;
(6) $b_e^1 = b_{e'}^1$ for all $e' \in E(T_1)$, $e \in E(T_k)$, $k = 2, \ldots, N$;
(7) $b_e^k = b_{e'}^k$ for all $e \in E(T_k)$, $e' \in [T_1 : T_k]$, $k = 2, \ldots, N$.

In the following we fix the remaining coefficients.

(8) $b_e^k = 0$ for all $e \in F_k \setminus F_k'$, $k = 2, \ldots, N$. Let $e \in F_k \setminus F_k'$. Choose $S_1 := [t_1 : T_1]$ for some $t_1 \in T_1$, and set $S_k := C_k$ for $k = 2, \ldots, N$. Then, $P := (S_1, \ldots, S_N)$ and $P' := P \cup_k e$ are Steiner tree packings with $\chi^P, \chi^{P'} \in I_a$. So we obtain $0 = b^T\chi^{P'} - b^T\chi^P = b_e^k$.

(9) $b_e^1 = b_{e'}^1$ for all $e' \in E(T_1)$, $e \in [T_p : T_q]$, $p, q \geq 2$, $p \neq q$. Let $e = t_p t_q$ with $t_p \in T_p$, $t_q \in T_q$. Let $t_1, t_1' \in T_1$, $t_1 \neq t_1'$ such that $t_1 t_p \in C_p$ and $t_1' t_q \in C_q$. Choose $S_1 := [t_1 : T_1]$, $S_p := C_p \setminus \{t_1 t_p\}$, $S_q := C_q \setminus \{t_1' t_q\}$ and $S_i := C_i$ for all $i \in \{2, \ldots, N\} \setminus \{p, q\}$. Furthermore, let $S_1' := S_2 \setminus \{t_1 t_1'\} \cup \{t_1 t_p, t_p t_q, t_q t_1'\}$. Then, $P := (S_1, \ldots, S_N)$ and $P' := (S_1', S_2, \ldots, S_N)$ are Steiner tree packings with $\chi^P$, $\chi^{P'} \in I_a$, and we obtain that $0 = b^T\chi^{P'} - b^T\chi^P = b_{t_p t_q}^1 - b_{t_1 t_1'}^1$. This together with (3) proves the statement.

(10) $b_e^k = b_{e'}^k$ for all $e \in [T_k : T_p]$, $p \geq 2$, $p \neq k$ and $e' \in E(T_k)$. Let $e = t_k t_p$ for some $t_k \in T_k$, $t_p \in T_p$. Let $t_k' \in T_k \setminus \{t_k\}$ and $t_1 \in T_1$ such that $t_1 t_k' \in C_k$ and $t_1 t_p \in C_p$. Choose $S_1 := C_k \setminus \{t_1 t_k'\}$, $S_k := [t_k : T_k]$, $S_p := C_p \setminus \{t_1 t_p\}$ and $S_i := C_i$ for all $i \in \{2, \ldots, N\} \setminus \{p, k\}$. Furthermore, set $S_k' := S_k \setminus \{t_k t_k'\} \cup \{t_k t_p, t_p t_1, t_1 t_k'\}$. Then, $P := (S_1, \ldots, S_N)$ and $P' := (P \setminus_k S_k) \cup_k S_k'$ are Steiner tree packings with $a^T\chi^P, a^T\chi^{P'} \in I_a$. Thus, we have that $0 = \mathrm{b}^T\chi^{P'} - b^T\chi^P = b_{t_k t_p}^k - b_{t_k t_k'}^k$, and the result follows with (3).

(1)–(10) imply that $b$ is a multiple of $a$, which completes the proof. □

Note that, in Theorem 4.6, we generalize only one special case of Theorem 4.2 to an arbitrary number of nets; namely, where $F_1 = \emptyset$. We believe that there also exist similar generalizations for $F_1 \neq \emptyset$. However, the condition 'maximal cross free' is not sufficient anymore in this case. Up to now, we do not know a good characterization for the general case.

## 5. Conclusions

In this paper we have presented several new classes of inequalities for the Steiner tree packing polyhedron. It has turned out that the conditions under which the inequalities define facets are quite complicated. However, the zero graphs have mostly nice (sub-)structures such as cycles, matchings or trees that are more easily tractable. This gives hope to find good and efficient (not necessarily exact) separation algorithms and successfully to incorporate these inequalities in our cutting plane algorithm.

## Acknowledgement

This research was partially supported by Science Program SC1-CT91-620 of the EEC.

## References

1. A. Frank, Packing paths, circuits, and cuts—a survey, in: *Paths, Flows, and VLSI-Layout* B. Korte, L. Lovász, H. J. Prömel and A. Schriver (eds), Springer-Verlag, Berlin, 1990, pp. 47–100.
2. M. Formann, D. Wagner and F. Wagner, Routing through a dense channel with minimum total wire length, *J. Algorithms*, **15** (1993), 267–283.
3. M. Grötschel, A. Martin and R. Weismantel, Packing Steiner trees: polyhedral investigations, Konrad-Zuse-Zentrum for Informationstechnik Berlin, Preprint SC 92-8, 1992.
4. M. Grötschel, A. Martin and R. Weismantel, Packing Steiner trees: a cutting plane algorithm and computational results, Konrad-Zuse-Zentrum for Informationstechnik Berlin, Preprint SC 92-9, 1992.
5. M. R. Kramer and J. van Leeuwen, The complexity of wire-routing and finding minimum area layouts for arbitrary VLSI circuits, in: *Advances in Computing Research*, F. P. Preparata (ed.), Vol. 2: VLSI theory, Jai Press, London, 1984, pp. 129–146.
6. T. Lengauer, *Combinatorial Algorithms for Integrated Circuit Layout*, John Wiley, Chichester, 1990.
7. A. Martin, Packen von Steinerbäumen: Polyedrische Studien und Anwendung, Ph.D. Thesis, Technische Universität Berlin, 1992.
8. A. Martin and R. Weismantel, Packing paths and Steiner trees: routing of electronic circuits, *CWI Q.*, **6** (1993), 185–204.
9. A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley, Chichester, 1986.
10. A. Schrijver, Homotopic routing methods, in: *Paths, Flows, and VLSI-Layout*, B. Korte, L. Lovász, H. J. Prömel and A. Schrijver (eds), Springer-Verlag, Berlin, 1990, pp. 329–371.
11. M. Sarrafzadeh, Channel-routing problem in the knock-knee mode is $\mathcal{NP}$-complete, *IEEE Trans. Computer-Aided-Design*, **CAD-6** (1987), 503–506.

*Received 28 September 1994 and accepted in revised form 3 April 1995*

M. Grötschel, A. Martin and R. Weismantel
*Konrad-Zuse-Zentrum Berlin*
*Heilbronner Str. 10*
*D-10711 Berlin*
*Germany*