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Polyhedral Approaches to Network Survivability

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ABSTRACT. In designing communication networks using fiber optic technology, survivability from node and/or link failure has become an important issue. We present a model that captures the notion of survivability by means of certain node or edge connectivity requirements. This model includes a number of network design problems occurring in practice. Furthermore, it has been accepted as a reasonable abstraction of real world situations by practitioners and is being used as a basis for generating solutions for the network design problems. This model is analyzed using methods of polyhedral combinatorics; in particular, we describe classes of valid and facet-defining inequalities for the associated integer polyhedra. It is then shown how these results can be utilized in a cutting plane procedure. We also present some computational results with data from real telecommunication networks.

1. Introduction

The currently prevailing network technology based on copper cables leads to highly connected communication networks due to the fact that copper cables have relatively small capacity. In such a case, failure of a single link or node affects only a small percentage of the traffic. By contrast, fiber optic cables have extremely high capacity and thus the desire to design cost-effective networks could lead to very sparse tree-like network designs. In such a case, the loss of a single link or node would result in a large percentage of traffic being lost without the capability of rerouting. In fact, a number of catastrophic failures of this type have occurred due to fire, storm, flooding, or construction damages; see [NY1], [NY2], [SL1], [SL2], [SL3], [WSJ] for instance. These disasters make it clear that fiber-based communication networks have to be designed in such a way that recovery from node or link failures is possible by rerouting the traffic around the damage. This implies that the network has to be supplied with extra connectivity. The level of connectivity will depend on the needs of each individual office. Such considerations lead to the concepts of network survivability introduced in the next section. In fact, the model to be presented in Section 2 has been accepted by practitioners and is being used

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in the area of telecommunication network design; see e.g., [CMW], [BC].

In Section 3 we will focus on network design problems that have relatively low connectivity requirements. This model applies in particular to Intra-LATA fiber network design problems faced by the regional Bell telephone companies. In particular, we will describe large classes of facet-defining inequalities for the associated integer polyhedra, we will discuss separation routines, outline a cutting plane algorithm, and present some computational results with real networks. Section 4 treats network design problems with relatively high levels of required connectivity in a similar way.

For the low-connectivity case the approach we describe has reached a somewhat mature state and has proven to be very effective in practice. However, for the high-connectivity case, theory and algorithm design are still in an early stage of development. Nevertheless, the success of the previous effort gives hope for similar results in this case.

The results presented in this paper are to a large extent based on our papers [GM], [GMS1], [GMS2], and two further papers currently in preparation [GMS4], [GMS5].

2. The network survivability models

In this section, we formalize the network design problems that are being considered in this paper. A set V of **nodes** is given that represents the locations of the switches (offices) that must be interconnected into a network in order to provide the desired services. A collection E of **edges** is also specified which represents the possible pairs of nodes between which a direct transmission link can be placed. We let $G = (V, E)$ be the (undirected) graph of possible direct link connections. Each edge $e \in E$ has a nonnegative **fixed cost** c_e of establishing the direct link connection. The graph G may have parallel edges but contains no loops. (Thus we assume throughout this paper that all graphs considered are loopless. But they may have parallel edges. Graphs without parallel edges are called **simple**). The cost of establishing a network consisting of a subset $F \subseteq E$ of edges is the sum of the costs of the individual links contained in F. The goal is to build a minimum-cost network so that the required survivability conditions (which we describe below) are satisfied. We note that the cost here represents setting up the topology for the communication network and includes placing conduits in which to lay the fiber cables, placing the cables into service, and other related costs. We do not consider costs that depend on how the network is implemented such as routing, multiplexing, and repeater costs. Although these costs are also important, it is usually the case that a topology is first designed and then these other costs are considered in a second stage of optimization; see e.g., [WC], [WKC], [CMW].

For any pair of distinct nodes $s, t \in V$, an **$[s, t]$-path** P is a sequence of nodes and edges $(v_0, e_1, v_1, e_2, \dots, v_{l-1}, e_l, v_l)$, where each edge e_i is incident with the nodes v_{i-1} and v_i $(i = 1, \dots, l)$, where $v_0 = s$ and

$v_l = t$, and where no node or edge appears more than once in P. A collection $P_1, P_2, \ldots, P_k$ of $[s, t]$-paths is called **edge-disjoint** if no edge appears in more than one path, and is called **node-disjoint** if no node (except for s and t) appears in more than one path. (Remark: In order to be consistent with standard graph theory we do not consider two parallel edges as two node-disjoint paths.) Two nodes $s, t \in V$ are called (**locally**) k**-node connected** or (**locally**) k**-edge connected** if G contains k $[s, t]$-paths that are node-disjoint or edge-disjoint, respectively. G is called k**-edge** (resp. k**-node**) **connected** if all node pairs are locally k-edge (resp. k-node) connected. A graph is called **connected** if it is 1-node connected. A **component** of a graph is a subgraph that is connected and maximal with respect to this property.

Let us fix some further graph notation here. Let $G = (V, E)$ be a graph. If $F \subseteq E$ then $G - F$ denotes the graph obtained by removing the edges in F; we write $G - e$ for $G - \{e\}$. If $W \subseteq V$ then $G - W$ is the subgraph obtained from G by removing all nodes in W; $G[W] = G - (V \setminus W)$ is the subgraph induced by W. Given disjoint node sets $U, W \subseteq V$, we denote by $[U : W]$ the set of edges in G with one endnode in U and the other in W, $\delta(W) := [W : V \setminus W]$ is the **cut** induced by W; and $E(W)$ is the set of edges of G with both endnodes in W. Instead of $\delta(\{v\})$ we will write simply $\delta(v)$. G/W denotes the graph obtained by contracting node set W to a single node.

In [GM] we described a rather general notion of survivability. However, it turned out that the data needed for implementing the model were often not available in practice. A specialized version, to be described below, proved to be acceptable from the point of view of data acquisition and was still considered a reasonable model of reality by practitioners.

To model survivability we introduce the concept of node types. For each node $s \in V$ a nonnegative integer r_s, called the **type** of s, is specified. We say that the network $N = (V, F)$ to be designed satisfies the **node survivability conditions** if, for each pair $s, t \in V$ of distinct nodes, N contains at least

$$r_{st} := \min\{r_s, r_t\}$$

node disjoint $[s, t]$-paths. Similarly, we say that $N = (V, F)$ satisfies the **edge survivability conditions** if, for each pair $s, t \in V$ of distinct nodes, N contains r_{st} edge disjoint $[s, t]$-paths. These conditions ensure that some communication path between s and t will survive a prespecified level of node or link failures.

Of course one could combine node and edge survivability conditions into a single model as we did in [GM]; but in practice, it turned out that the focus was either on edge or node survivability and mixed models were never considered. That is why we introduce only one node type here; it will be clear from the context whether we are dealing with the edge or node survivability case.

To be able to name the different problem types efficiently we introduce the following notation and conventions. Given a graph $G = (V, E)$ and a vector of node types $r = (r_s)_{s \in V}$, we assume—without loss of generality—that there are at least two nodes of the largest type. If we say that we consider the kNCON problem (for G and r) then we mean that we are looking for a minimum-cost network that satisfies the node survivability conditions and where $k = \max\{r_s | s \in V\}$. Similarly, we speak of the kECON problem (for G and r). To shorten some notation we extend the type function r to a function operating on sets by setting

$$\begin{aligned} r(W) &:= \max\{r_s | s \in W\} \quad \text{for all } W \subseteq V \text{ and} \\ \operatorname{con}(W) &:= \max\{r_{st} | s \in W, t \in V \setminus W\} \\ &= \min\{r(W), r(V \setminus W)\} \quad \text{for all } W \subseteq V, \emptyset \neq W \neq V. \end{aligned}$$

Let us now introduce, for each edge $e \in E$, a variable x_e and consider the vector space $\mathbf{R}^E$. Every subset $F \subseteq E$ induces an **incidence vector** $\chi^F = (\chi_e^F)_{e \in E} \in \mathbf{R}^E$ by setting $\chi_e^F := 1$ if $e \in F$, $\chi_e^F := 0$ otherwise, and vice versa, each 0/1-vector $x \in \mathbf{R}^E$ induces a subset $F^x := \{e \in E | x_e = 1\}$ of the edge set E of G. (If we speak of the incidence vector of a path in the sequel we mean the incidence vector of the edges of the path.) For any subset of edges $F \subseteq E$, we let $x(F)$ stand for the sum $\Sigma_{e \in F} x_e$. We consider the following integer linear program.

$$\min c^T x \tag{2.1}$$

subject to

(i)	$x(\delta(W)) \geq \operatorname{con}(W)$	for all $W \subseteq V, \emptyset \neq W \neq V$,
(ii)	$x(\delta_{G-Z}(W)) \geq 1$	for all pairs $s, t \in V$, $s \neq t$ and for all $Z \subseteq V \setminus \{s, t\}$ with $\lvert Z \rvert = r_{st} - 1$ and for all $W \subseteq V \setminus Z$ with $s \in W$, $t \notin W$,
(iii)	$0 \leq x_{ij} \leq 1$	for all $ij \in E$,
(iv)	x_{ij} integral	for all $ij \in E$.

It follows from Menger's theorem that the feasible solutions of (2.1) are the incidence vectors of edge sets F such that $N = (V, F)$ satisfies all node survivability conditions; i.e., (2.1) is an integer programming formulation of the kNCON problem. Deleting inequalities (ii) from (2.1) we obtain, again from Menger's theorem, an integer programming formulation for the kECON problem. The inequalities of type (i) will be called **cut inequalities** and those of type (ii) **node cut inequalities.**

The polyhedral approach to the solution of the kNCON (and similarly the kECON) problem consists of studying the polyhedron obtained by taking the convex hull of the feasible solutions of (2.1). We set

$$\begin{aligned} \operatorname{NCON}(G; r) &:= \operatorname{conv}\{x \in \mathbf{R}^E | x \text{ satisfies } (2.1), (i), \dots, (iv)\} \\ \operatorname{ECON}(G; r) &:= \operatorname{conv}\{x \in \mathbf{R}^E | x \text{ satisfies } (2.1)(i), (iii), (iv)\}, \end{aligned}$$

where conv denotes the convex hull operator. It will sometimes be convenient to denote these polyhedra by $k\,\mathrm{NCON}(G; r)$ and $k\,\mathrm{ECON}(G; r)$, where $k = \max\{r_s | s \in V\}$, since this implicitly provides a notation for the maximum node type.

It follows from the definitions that given $G = (V, E)$, a type vector $r \in \mathbf{Z}_+^V$, and an objective function $c \in \mathbf{R}^E$, the NCON and the ECON problem can be solved via the linear programs

$$\begin{array}{ccc} \min c^T x & \text{or} & \min c^T x \\ x \in \mathrm{NCON}(G; r) & & x \in \mathrm{ECON}(G; r). \end{array}$$

What we need for the application of linear programming technology is a complete description or "good partial" description of the polytopes $\mathrm{NCON}(G; r)$ and $\mathrm{ECON}(G; r)$ by means of equations and inequalities. The aim of our paper is to describe some of the classes of valid and facet-defining inequalities that are known for these polytopes and to outline their use in a cutting plane framework.

Let us mention at this point some well-studied special cases of the concepts introduced above.

If $r_s = 1$ for all $s \in V$ then $\mathrm{ECON}(G; r) = \mathrm{NCON}(G; r)$, and this polytope is nothing but the convex hull of all incidence vectors of edge sets that contain a spanning tree. This polytope is usually called the **connected subgraph polytope**. A complete and nonredundant description of this polytope can be easily derived from Edmonds' [E] results on matroid polytopes. The details have been worked out in [GM], for instance.

If $r_s \in \{0, 1\}$ for all $s \in V$ then again $\mathrm{ECON}(G; r) = \mathrm{NCON}(G; r)$. This polytope is the convex hull of all incidence vectors of edge sets that contain a Steiner tree, where $S := \{s \in V | r_s = 1\}$ is the set of terminal nodes. In the general case, where r is any nonnegative integral vector, a complete linear characterization of $\mathrm{NCON}(G; r)$ is unknown. But reasonable partial descriptions have been found; see, for instance, [GM] for more information. A well-solved case arises when $|S| = 2$, i.e., exactly two nodes are of type 1. Here $\mathrm{NCON}(G; r)$ is the convex hull of the incidence vectors of all edge sets containing a $[s, t]$-path, where $S = \{s, t\}$. It is known that in this case, the cut inequalities (2.1) (i) and the trivial constraints $0 \le x_e \le 1$ for all $e \in E$ suffice to describe $\mathrm{NCON}(G; r)$.

As remarked above, the $k\,\mathrm{NCON}$ (and $k\,\mathrm{ECON}$) problem has been intensively studied for $k = 1$; a rich variety of results on the associated integer polytopes exists. In the sequel, we will concentrate on the case $k \ge 2$. Section 3 deals with $k = 2$, while the case $k \ge 3$ is treated in Section 4.

3. Low connectivity

In this section we consider the $k\,\mathrm{NCON}$ and $k\,\mathrm{ECON}$ problems for the special case where all nodes are of type $0, 1, 2$. As mentioned before, this case arises in the design of fiber optic communication networks. The regional

Bell telephone companies are using this model for their Intra-LATA network planning. We will assume throughout this chapter that, for every graph G with node type vector r, at least two nodes are of type 2.

3.1. The polytopes $2\,\mathrm{NCON}(G; r)$ **and** $2\,\mathrm{ECON}(G; r)$. This section summarizes results of **[GM]**, **[GMS1]** and **[GMS2]** for the 2 NCON and 2 ECON problems. To state the results precisely we need to introduce further notation.

Given a graph $G = (V, E)$, a node type vector $r \in \mathbf{Z}_+^V$, a node set $W \subseteq V$ with $|W| \geq 2$, we set

$V_i := \{v \in V | r_v \geq i\}$ for $i = 0, 1, 2$,

$\lambda(G, W)$:= minimum cardinality of a subset of E whose removal from G disconnects two nodes of W, and

$\kappa(G, W)$:= minimum cardinality of a set $S \subseteq V \cup E'$ whose removal from G' disconnects two nodes of W in G', where $G' = (V, E')$ is the simple graph underlying G.

If $|W| < 2$ then $\lambda(G, W)$ and $\kappa(G, W)$ are defined as ∞. There are some cases where $W \supseteq V$. Then we write $\lambda(G, W)$ instead of $\lambda(G, W \cap V)$.

We assume throughout this section that G and r satisfy $\kappa(G, V_1) \geq 2$ and $\lambda(G, V_2) \geq 3$ when we deal with the 2 ECON case, and that G and r satisfy $\kappa(G, V_1) \geq 2$ and $\kappa(G, V_2) \geq 3$ when we deal with the 2 NCON case. If this is not so then the problem can be decomposed (in polynomial time) into independent smaller problems that are trivially solvable or satisfy these conditions. The exact decomposition procedure involves many technical details and is described in **[GMS2]** and, to some extent, in Section 3.2. A side benefit of this assumption is that the polyhedra $2\,\mathrm{ECON}(G; r)$ and $2\,\mathrm{NCON}(G; r)$ are full dimensional (see **[GM]**), i.e., they contain $|E| + 1$ affinely independent vectors.

An inequality $a^T x \leq \alpha$ is **valid** with respect to a polyhedron P if $P \subseteq \{x | a^T x \leq \alpha\}$; the set $F_a := \{x \in P | a^T x = \alpha\}$ is called the **face** of P defined by $a^T x \leq \alpha$. If the dimension of F_a is one less than the dimension of P and $F_a \neq 0$ then F_a is a **facet** of P and $a^T x \leq \alpha$ is called **facet-defining** or **facet-inducing**.

We first characterize the trivial inequalities (2.1)(iii) that are facet-defining.

(3.1) THEOREM. *Let* $G = (V, E)$ *be a graph and* $r \in \{0, 1, 2\}^V$.

(a) $x_e \leq 1$ *defines a facet of* $2\,\mathrm{NCON}(G; r)$ *and* $2\,\mathrm{ECON}(G; r)$ *for all* $e \in E$,

(b) $x_e \geq 0$ *defines a facet of* $2\,\mathrm{NCON}(G; r)$ *(resp.* $2\,\mathrm{ECON}(G; r)$*) if and only if for every edge* $f \in E \setminus \{e\}$ *the polytope* $2\,\mathrm{NCON}(G - \{e, f\}; r)$ *(resp.* $2\,\mathrm{ECON}(G - \{e, f\}; r)$*) is nonempty.*

The next theorem describes the facet-defining inequalities among the cut inequalities (2.1) (i).

(3.2) THEOREM. *Let* $G = (V, E)$ *be a graph,* $r \in \{0, 1, 2\}^V$ *and* $W \subseteq V$ *with* $\emptyset \neq W \neq V$.

(i) *If* $\text{con}(W) = 2$ *then* $x(\delta(W)) \geq 2 = \text{con}(W)$ *defines a facet of* $2\text{ECON}(G; r)$ *only if* $\lambda(G[W], V_1) \geq 2$, $\lambda(G[V \setminus W], V_1) \geq 2$, $G[W]$ *and* $G[V \setminus W]$ *are connected.*

(ii) *If* $\text{con}(W) = 1$ *then* $x(\delta(W)) \geq 1 = \text{con}(W)$ *defines a facet of* $2\text{ECON}(G; r)$ *if and only if* $G[W]$ *and* $G[V \setminus W]$ *are connected,* $\lambda(G[W], V_i) \geq i + 1$ *for* $i = 1, 2$, *and* $\lambda(G[V \setminus W], V_i) \geq i + 1$ *for* $i = 1, 2$.

(iii) *If* $\text{con}(W) = 0$ *then* $x(\delta(W)) \geq 0 = \text{con}(W)$ *does not define a facet of* $2\text{ECON}(G; r)$ *or* $2\text{NCON}(G; r)$.

(iv) *If* $\text{con}(W) = 2$ *then* $x(\delta(W)) \geq 2$ *defines a facet of* $2\text{NCON}(G; r)$ *only if all conditions of* (i) *are satisfied and* $\kappa(G[W], V_2) \geq 2$ *and* $\kappa(G[V \setminus W], V_2) \geq 2$.

(v) *If* $\text{con}(W) = 1$ *then* $x(\delta(W)) \geq 1$ *defines a facet of* $2\text{NCON}(G; r)$ *only if all conditions of* (ii) *are satisfied and* $\kappa(G[V \setminus W] - e, V_2) \geq 2$ *for all* $e \in E(V \setminus W)$.

If we look at the uniform case $r_v = 2$ for all $v \in V$ then (3.2) (i) implies that in a 3-edge connected graph a cut inequality $x(\delta(W)) \geq 2$ defines a facet of $2\text{ECON}(G; r)$ if and only if $G[W]$ and $G[V \setminus W]$ are both 2-edge connected. This result was independently obtained by Mahjoub [M]. In fact, [M] investigates this uniform case in more detail. In particular, Mahjoub introduces a new class of facet-defining inequalities for $2\text{ECON}(G; r)$ called odd wheel inequalities. Furthermore he shows that the convex hull of the incidence vectors of all spanning 2-edge connected subgraphs of a series-parallel graph is described completely by the trivial constraints (2.1) (iii) and the cut constraints (2.1) (i).

Results (iv) and (v) of (3.2) above can be turned into "if and only if statements" by adding further (complicated) technical conditions. (See [GMS1].) The following necessary conditions for node cut inequalities (2.1) (ii) to define facets of $2\text{NCON}(G; r)$ can be similarly extended to a full characterization. Observe that in the low-connectivity case considered here we have $r_{st} \leq 2$, so only sets Z with $|Z| = 1$ are to be considered.

(3.3) THEOREM. *Let* $G = (V, E)$ *be a graph,* $r \in \{0, 1, 2\}^V$, $z \in V$, $W \subseteq V \setminus \{z\}$, $\emptyset \neq W \neq V \setminus \{z\}$, *with* $r(W) = 2$ *and* $r(V \setminus (W \cup \{z\})) = 2$. *The node cut inequality* $x(\delta_{G-z}(W)) \geq 1$, *see* (2.1) (ii), *defines a facet of* $2\text{NCON}(G; r)$ *only if*

(a) $G[W]$ *is connected,*

(b) $\lambda(G[W \cup \{z\}], V_1) \geq 2$,

(c) $\lambda(G[W], V_2) \geq 2$, *and*

(d) *Conditions* (a)–(c) *also hold for* $\overline{W} := V \setminus (W \cup \{z\})$ *instead of* W.

We now introduce a new class of inequalities generalizing the so-called partition inequalities that suffice (together with the trivial constraints) to describe the connected subgraph polytope completely. Consider a graph $G = (V, E)$ together with a requirement vector $r \in \{0, 1, 2\}^V$, and a partition of V into nodes disjoint nonempty sets $W_1, W_2, \dots, W_p$, $p \geq 2$, each containing at least one node of type ≥ 1. $r(W_i)$ is supposed to be 2 for at least two of these sets W_i. The **partition inequality** (induced by $W_1, \dots, W_p$) is given by

$$\frac{1}{2}\sum_{i=1}^{p} x(\delta(W_i)) \geq p. \tag{3.4}$$

(3.5) THEOREM. *The partition inequality* (3.4) *induced by* $W_1, \dots, W_p$ *is valid for* $2\,\mathrm{ECON}(G; r)$ *(resp.* $2\,\mathrm{NCON}(G; r)$*)*.

We know necessary and sufficient criteria for a partition inequality to define a facet only for certain special cases. The next theorem gives some necessary and some sufficient conditions that are also used in the heuristic separation routines for the partition inequalities to be described in Section 3.2.

(3.6) THEOREM. *Consider a partition inequality* (3.4) *valid for* $2\,\mathrm{ECON}(G; r)$ *(resp.* $2\,\mathrm{NCON}(G; r)$*). Let* $\hat{G}$ *be the graph* $G/W_1/\cdots/W_p$, *where, for* $i = 1, \dots, p$, *the node set* W_i *is shrunk to a node* w_i *of type* $\hat{r}(w_i) := r(W_i)$.

(a) *The partition inequality defines a facet of* $2\,\mathrm{ECON}(G; r)$ *(resp.* $2\,\mathrm{NCON}(G; r)$*) only if the following conditions are satisfied.*
  - *Every node of type* 2 *in* $\hat{G}$ *is adjacent to some node of type* 1 *in* $\hat{G}$.
  - $\hat{G}$ *has a cycle containing all nodes of type* 2.
  - $G[W_i]$ *is connected for* $i = 1, \dots, p$.
  - *For* $i = 1, \dots, p$, $G[W_i]$ *does not contain an edge* e *such that two nodes of type at least* 1 *are in different components of* $G[W_i] - e$.

(b) *The partition inequality defines a facet of* $2\,\mathrm{ECON}(G; r)$ *if the following conditions hold.*
  - *All conditions* (a) *are satisfied.*
  - *The subgraph of* $\hat{G}$ *induced by the nodes of type* 1 *is connected.*
  - *The subgraph of* $\hat{G}$ *induced by the nodes of type* 2 *is hamiltonian.*
  - *For* $i = 1, \dots, p$ *and for all* $e \in E(W_i)$ *either* $\lambda(G[W_i] - e, V_2) \geq 2$.

A class of valid inequalities for $2\,\mathrm{NCON}(G; r)$ that generalizes the node cut inequalities (2.1) (ii) in a similar manner as partition inequalities (3.4) generalize cut inequalities (2.1) (i) is the following.

Let $G = (V, E)$ be a graph and $r \in \{0, 1, 2\}^V$. Let $z \in V$ and let $W_1, \dots, W_p$ be a partition of $V \setminus \{z\}$ into disjoint nonempty sets. We assume that $r(W_i) \geq 1$ for $i = 1, \dots, p$, and that $r(W_i) = 2$ for at least two sets. Let I_k be $\{i \in \{1, \dots, p\} | r(W_i) = k\}$ for $k = 1, 2$. It is obvious that the following **node partition inequality** (induced by $z, W_1, \dots, W_p$)

$$(3.7) \quad \frac{1}{2}\left(\sum_{i\in I_2} x(\delta_{G-z}(W_i)) + \sum_{i\in I_1} x(\delta(W_i)) + x([\{z\} : \bigcup_{i\in I_1} W_i])\right) \geq p - 1$$

is valid for $2\text{NCON}(G; r)$ (but not for $2\text{ECON}(G; r)$).

The next theorem gives some necessary and some sufficient conditions for a node partition inequality to define a facet of $2\text{NCON}(G; r)$. Reasonable necessary and sufficient conditions are not known.

(3.8) THEOREM. *Consider a node partition inequality* (3.7) *induced by* z, $W_1, \dots, W_p$ *for* $2\text{NCON}(G; r)$.

(a) *The node partition defines a facet of* $2\text{NCON}(G; r)$ *only if the following conditions are satisfied*
- $G[W_i]$ *is connected for* $i = 1, \dots, p$.
- $\lambda(G[W_i \cup \{z\}], V_1) \geq 2$ *for all* $i \in I_2$.
- $\lambda(G[W_i], V_1) \geq 2$ *for all* $i \in I_1$.
- $\lambda(G[W_i], V_2) \geq 2$ *for all* $i \in I_2$.

(b) *The node partition inequality defines a facet of* $2\text{NCON}(G; r)$ *if the following conditions are satisfied.*
- *For* $i \in I_2$ *and all* $e \in E(W_i \cup \{z\})$, $\kappa(G[W_i \cup \{z\}] - e, V_2) \geq 2$.
- $G[W_i]$ *is 2-node connected for* $i \in I_1$.
- (3.7) *defines a facet of* $2\text{NCON}(\widehat{G}; r)$ *for the graph* $\widehat{G}$ *obtained by contracting* $W_1, \dots, W_p$.

The next class of inequalities is motivated by the 2-matching inequalities for the 2-matching (resp. the travelling salesman) problem, see [E], [GP]. Consider a subset $H \subseteq V$ called the **handle** and a subset $T \subseteq \delta(H)$ with $|T|$ odd, $|T| \geq 3$. For each $e \in T$, let T_e denote the set of the two endnodes of e. The sets T_e, $e \in T$, are called **teeth.** Let $H_1, \dots, H_p$, $p \geq 3$, be a partition of H into nonempty disjoint subsets such that $r(H_i) \geq 1$ for $i = 1, \dots, p$, $\text{con}(H_i) = 2$ if $H_i \cap T_e \neq \emptyset$ for some $e \in T$, and $|H_i \cap T_e| \leq 1$ for $i = 1, \dots, p$ and all teeth T_e. We call

$$(3.9) \quad x(E(H)) - \sum_{i=1}^{p} x(E(H_i)) + x(\delta(H)) - x(T) \geq p - \left\lfloor \frac{|T|}{2} \right\rfloor$$

the **lifted 2-cover inequality** (induced by $H, H_1, \dots, H_p, T$); here $\lfloor t \rfloor$ denotes the largest integer not larger than t. (Note that if $r_v = 2$ for all $v \in V$ and $|H_i| = 1$ for $i = 1, \dots, p$, then (3.9) can be brought into the form of a 2-matching inequality for the TSP by subtracting the degree equations $x(\delta(v)) = 2$ for all $v \in V$ and multiplying the resulting inequality by $-\frac{1}{2}$.)

(3.10) THEOREM.

(a) *Every lifted 2-cover inequality* (3.9) *is valid for* $2\,\mathrm{ECON}(G;r)$ *(and hence for* $2\,\mathrm{NCON}(G;r)$*).*

(b) *A lifted 2-cover inequality* (3.9) *defines a facet of* $2\,\mathrm{ECON}(G;r)$ *only if there are node sets* $H, H_1, \dots, H_p$*, and an edge set* T *inducing it such that the following conditions are satisfied.*

- *Let* $\widehat{G}$ *be the graph obtained from* G *by contracting each set* H_i *to a node* h_i *of type* $r(H_i)$*. Then* $\widehat{G}[H]$ *is connected and* $\lambda(\widehat{G}[H], V_2) \geq 2$.
- $G[H_i]$ *is connected for* $i = 1, \dots, p$.
- $\lambda(G[H_i], V_1) \geq 2$ *for* $i = 1, \dots, p$.

In a way analogous to the generalization of 2-matching constraints to comb constraints for the TSP (see **[GP]**) we will now extend 2-cover constraints to certain comb constraints for $2\,\mathrm{NCON}(G;r)$.

Let $H, T_1, \dots, T_t$ be subsets of V (H is called the **handle,** the sets $T_1, \dots, T_t$ are the **teeth**) that satisfy the following conditions. The number t of teeth is at least 3 and odd. Two teeth may have at most one node in common. Each tooth T_i intersects the handle H in exactly one node; we denote this node by t_i for $i = 1, \dots, t$. In each tooth T_i we choose a node $z_i \in T_i \setminus H$ and call z_i a **special node.** The choice is restricted if $T_i \cap T_j \neq \emptyset$, in this case $T_i \cap T_j = \{z_i\} = \{z_j\}$ (i.e., the nodes z_i are not necessarily distinct). Moreover, we assume that $r_{t_i} = 2$ for $i = 1, \dots, t$ and $r_v \geq 1$ for all $v \in H \cup (\bigcup_{i=1}^{t}(T_i \setminus \{z_i\}))$. Under these assumptions one can show that the following **comb inequality**

(3.11)

$$x(E(H)) + x(\delta(H)) + \sum_{i=1}^{t} x(E[T_i]) + \sum_{i=1}^{t} x([T_i \setminus (H \cup \{z_i\}) : V \setminus T_i])$$
$$- \sum_{i=1}^{t} x([\{t_i\} : T_i]) - \sum_{i=1}^{t} x([\{z_i\} : T_i \cap V_2]) \geq |H| + \sum_{i=1}^{t}(|T_i| - 2) - \left\lfloor \frac{t}{2} \right\rfloor$$

is valid for $2\,\mathrm{NCON}(G;r)$. A sufficient (but rather special) condition for (3.11) to define a facet can be found in **[GMS1]**. This paper also describes a lifting theorem with which a node w can be expanded to a node set W such that a valid inequality for $2\,\mathrm{NCON}(G;r)$ is lifted to a valid inequality for $2\,\mathrm{NCON}(G';r')$, where G' is obtained from G by expanding w to W. Using this lifting theorem, the condition $|T_i \cap H| = 1$ ($i = 1, \dots, t$) can be removed; thus the more general combs as studied in **[GP]** arise.

The investigation of comb inequalities is not yet complete. A lot of technicalities creep in. We also have a version of the comb inequalities that gives inequalities valid for $2\,\mathrm{ECON}(G;r)$ (note that (3.11) is not valid for $2\,\mathrm{ECON}(G;r)$ in general), but these inequalities are complicated and not yet well understood.

3.2. The cutting plane approach. The main objective of the polyhedral research described in Section 3.1 was to provide large classes of valid and facet-defining inequalities for $2\,\mathrm{NCON}(G; r)$ and $2\,\mathrm{ECON}(G; r)$ that can be used in a cutting plane algorithm for the solution of the 2 NCON and the 2 ECON problems.

We briefly describe here the computational work we have done so far. More details can be found in **[GMS2]**.

We assume from now on in this section that a graph $G = (V, E)$, node types $r_v \in \{0, 1, 2\}$ for all $v \in V$ and costs $c_e \in \mathbf{R}$ for all $e \in E$ are given. (We assume as in Section 3.1 that there are at least two nodes of type 2, for otherwise we would just deal with a Steiner tree problem for which other special purpose algorithmic machinery has been developed.) These data are the input to our cutting plane algorithm. We input a further parameter that specifies whether the 2 NCON or the 2 ECON problem has to be solved. (We treat the two cases simultaneously below.)

Decomposition. Before starting the cutting plane algorithm we try to reduce the size of the problem by decomposing it. In fact, the practical problems we solved have rather sparse graphs of possible direct links, and the survivability requirements often force certain edges to be present in every feasible solution. Such edges can be fixed and removed from the problem by appropriately changing certain node types. This removal may break the original problem into several smaller ones that can be solved independently.

There are further ways of decomposing a problem into independent subproblems like decomposing on articulation nodes, cut sets of size two, and articulation sets of size two. In each of these cases one can perform the decomposition or determine that no such decomposition is possible using polynomial time methods like depth-first search or connectivity algorithms. All this is quite easy, though a precise description of the necessary transformations need some space. Details can be found in **[GMS2]**.

The main purpose of this decomposition step is to speed up the computation by getting rid of some trivial special cases that the cutting plane algorithm does not need to check any more and by reducing the sizes of the problems to be solved. At the end of this preprocessing phase we have decomposed the original problem into a list of subproblems for which we call the cutting plane algorithm to be described below. The optimal solution of the original problem can then be composed from the optimal solutions of the subproblems in a straightforward manner.

An outline of the cutting plane procedure. The input of this procedure is a graph $G = (V, E)$ with costs $c_e \in \mathbf{R}$ for all $e \in E$ and node types $r_v \in \{0, 1, 2\}$ for all $v \in V$. We want to solve

$$\min_{x \in 2\,\mathrm{NCON}(G; r)} c^T x \quad \text{or} \quad \min_{x \in 2\,\mathrm{ECON}(G; r)} c^T x .$$

We do this by solving a sequence of linear programming relaxations that are based on the polyhedral results of Section 3.1. The initial linear program (in both cases) is

$$\text{(3.12)} \qquad \begin{aligned} &\min c^T x \\ &x(\delta(v)) \geq r_v \quad \text{for all } v \in V, \\ &0 \leq x_e \leq 1 \quad \text{for all } e \in E. \end{aligned}$$

Suppose now that we have solved the current LP-relaxation and obtained a basic optimal solution y. If y is in $2\text{NCON}(G; r)$ (resp. $2\text{ECON}(G; r)$) then we are done since y is the incidence vector of the edge set of a network satisfying the survivability conditions and having minimum cost. If y is not, then we try to generate inequalities (from the classes described in Section 3.1) violated by y. If we can produce such inequalities, we augment the current LP with these inequalities, solve the new LP, and repeat this step.

It may happen that y is not in $2\text{NCON}(G; r)$ (resp. $2\text{ECON}(G; r)$) and we are unable to find a valid inequality violated by y. In this case we have found a (usually very good) lower bound to the 2NCON (resp. 2ECON) problem. If we want to produce an optimum solution to these problems we have to resort to branch and bound (or a similar enumeration technique). We will see, though, that this occurred in only two practical problems we encountered.

The main ingredients of this cutting plane procedure are the so-called separation routines that check whether y satisfies all inequalities of a certain class or not and, if not, generates a violated inequality. These routines are discussed below.

Separation routines. At this point we assume that $y \in \mathbf{Q}^E$ is feasible for the current LP (in fact optimum); in particular, y satisfies the constraints of (3.12). Now we want to check whether y satisfies the cut inequalities (2.1) (i), node cut inequalities (2.1) (ii), partition inequalities (3.4), node partition inequalities (3.7), lifted 2-cover inequalities (3.9), or comb inequalities (3.11), respectively. Observe that each of these classes contains a number of inequalities that is exponential in $|V|$, and thus it is impractical to consider all these inequalities explicitly. Hence we need a practical procedure for checking feasibility of y.

For a class of inequalities C, the **separation problem** is, given a vector y, to determine whether y satisfies all inequalities in C and, if not, to find an inequality in C violated by y. An algorithm that solves such a separation problem will be called an (**exact**) **separation procedure**. Frequently, it is difficult to obtain separation procedures that are practically (or even theoretically) efficient. In such cases, fast heuristics are used that may generate inequalities violated by y but are not guaranteed to find one even if one exists.

For the class of cut inequalities (2.1) (i), an efficient exact separation procedure exists. It works as follows. We consider the components y_e of y as capacities of the edges $e \in E$ and compute—using the Gomory-Hu algorithm—the **Gomory-Hu** tree. This tree has the following property. If u, v is a pair of distinct nodes then the edge on the unique $[u, v]$-path in the tree with minimum weight determines a $[u, v]$-cut in G (with capacities y) of minimum capacity. Using this property of the Gomory-Hu tree, it is straightforward to check whether all cut constraints are satisfied, and if not, to find a violated cut constraint. The Gomory-Hu tree can be computed by solving $n-1$ max-flow problems.

The separation problem for the node cut inequalities (2.1) (ii) can be solved in a similar manner by first deleting a node, applying the Gomory-Hu procedure, and repeating this for all nodes $z \in V$. In our implementation we do not perform this for all nodes $z \in V$; rather we heuristically choose a few "good" candidate nodes to try.

We do not know polynomial time separation procedures for the other classes of inequalities mentioned above. In fact, finding a violated partition inequality (3.4), a node partition inequality (3.7), or a lifted 2-cover inequality (3.9) are $\mathcal{NP}$-hard [**GMS2**]. Therefore, we have designed separation heuristics for these inequalities.

The separation heuristic for the partition inequalities (3.4) we have implemented proceeds as follows. We first perform an (exact) edge contraction procedure which has the following property. If, for the original graph, there is a partition inequality that is violated by y, there is also one in the contracted graph, and vice versa. This contraction procedure reduces the problem size but also seems to help our heuristic find violated partition inequalities. After performing all possible contractions we compute the Gomory-Hu tree of the resulting graph. Removing some edges, say, from the tree, we obtain p components. These components induce a partition of the node set, and we can check whether this yields a partition inequality violated by y. Of course, it is impractical to test all possible partitions arising this way. We therefore use a heuristic that is based on the edge weights of the Gomory-Hu tree and the distribution of the node types to generate some "promising" partitions. The practical experience with this procedure is quite good, as can be seen from the computational results below.

For the separation problem for lifted 2-cover constraints (3.9) we consider, as before, $G = (V, E)$ as a graph with edge capacities y_e, $e \in E$. In a preprocessing step we perform an exact contraction procedure according to criteria similar to those used for the separation of partition inequalities. Then we apply the separation algorithm for 2-matching constraints due to Padberg and Rao, see [**PR**], to the resulting graph. The Padberg-Rao procedure modifies the given graph by replacing certain edge capacities y_e by $1 - y_e$, labeling some nodes, and computing the Gomory-Hu tree of this modified labeled graph. If the edge of minimum weight of this Gomory-Hu tree that separates the tree into two components, each having an odd number of labeled nodes, has weight less than one, a 2-matching inequality violated by y exists. This does not necessarily imply the existence of a lifted 2-cover inequality (3.9) violated by y, but one can use the Gomory-Hu tree, heuristically to select lifted 2-cover inequalities that are likely to be violated by y. Details of these separation heuristics can be found in [**GMS2**].

We have no reasonable separation heuristic for comb inequalities yet. We did find, though, comb inequalities manually. In fact, in one case we were able to add one comb inequality to a current LP manually, for which all other separation routines had failed to produce a violated constraint, such that the new LP gave the desired integral optimum solution. This investigation of comb inequalities is still in its infancy; good and fast separation heuristics need to be invented.

3.3. Computational experience with the cutting plane algorithm. We will now briefly describe our experience with the cutting plane algorithm outlined in Section 3.2. We do this by discussing computational results for several real-world network design problems as shown in Table 1. All of these problems were obtained from the Bell regional telephone companies and represent real data for Intra-LATA fiber optic telephone networks.

The first column of Table 1 gives the name of the original problem. These 7 problems have sizes ranging from 36 nodes to 116 nodes and from 65 edges to 173 edges. These data were analyzed by our preprocessing algorithm. A substantial number of nodes and edges could be deleted or contracted. In each case the decomposition algorithms produced a single much smaller graph with the property that an optimum solution for this graph can be extended to an optimum solution of the original graph. Table 1 shows the data of the reduced graph. In column 2 the numbers of nodes and edges and forced edges is listed, where the forced edges are those that could be proved (by our preprocessing procedure) to be in every optimum solution. Columns 3 and 4 (with heading PART and 2COV) show the number of partition inequalities (3.4) and lifted 2-cover inequalities (3.9) that were generated during the execution of our cutting plane algorithm.

The fifth column shows the number of manually added inequalities. For LATADL one of the two inequalities added manually was a comb inequality, the other one a lifted 2-cover inequality. The sixth column shows the objective function value (lower bound) generated by the linear programming problem after all cuts were generated by the automatic procedure. The next two columns show the optimal objective function value and the relative gap between the automatic solution and the optimal solution (in percent of COPT).

All of the problems are treated as edge-connected problems. For all problems except LATA5L the optimal solution is also feasible for the node-connected case.

The running times in min:sec on a SUN 3/50 are reported in column 9. Our cutting plane procedure is based on a preliminary version of a branch-

TABLE 1

Problem	nodes/edges/ forced edges	PART	2COV	MAN	automatic solution	COPT	% GAP	TIME
LATA1	28/49/2	45	1	0	4296	4296	0	0 : 15
LATA5L	29/77/1	55	6	0	4574	4574	0	0 : 20
LATA5S	23/50/0	51	0	0	4739	4739	0	0 : 12
LATADMA	21/46/4	30	20	0	1489	1489	0	0 : 13
LATADL	39/86/6	117	68	2	7398.5	7400	0.02	1 : 21
LATADS	39/86/3	251	10	32	$7287.\overline{3}$	7320	0.45	1 : 47
LATADSF	39/86/25	41	0	0	7647	7647	0	0 : 28

and-cut framework designed by M. Jünger and uses the LP-solver CPLEX of R. Bixby, a very fast and robust linear programming code written in C. CPLEX is based on the simplex method. All separation routines outlined in Section 3.2 were also coded in C.

In all cases, the lower bound generated by the automatic solution is very close to the optimal solution and the addition of a few inequalities manually results in an optimal solution.

Branch-and-cut was required only for problem LATADL; and there only two branch-and-cut steps were needed. The partition and lifted 2-cover inequalities were enough to optimally solve five of the seven problems. In only one case was the addition of a single comb inequality necessary in order to obtain the optimal solution. These computational results show a great deal of promise for solving these network design problems in very short time. Of course, in the final version of our procedure the manual parts will have to be replaced by automatic separation routines.

The attentive reader may have noticed that we did not mention cut inequalities or node-cut inequalities above. The reason is that we are treating these together with the partition inequalities (using the Gomory-Hu tree) and did not record which of the inequalities found was a cut and which was a partition inequality. In fact, we often changed a violated cut inequality into a violated partition inequality since, for some heuristic reasons, it seemed to be more appropriate.

4. Higher connectivity

The polyhedral investigation of ECON and NCON problems with node types greater than 2 is still in a preliminary stage. We do have a few polyhedral results and some computational experience and will describe this below. This will be documented later in [GMS4], [GMS5]. But there is much more to be done.

4.1. Polyhedral results. In this section we consider—unless stated otherwise—graphs $G = (V, E)$ with node types $r_v \in \mathbf{Z}_+$ for all $v \in V$ such that at least two nodes are of maximum type k at least 3. We first study the cut inequalities (2.1) (i). The exact separation routine for cut inequalities based on the Gomory-Hu procedure described in Section 3.2 clearly carries over to the general case. So the separation problem for cut inequalities can be solved in polynomial time.

Reasonable necessary and sufficient criteria for a cut inequality to define a facet of $k\,\mathrm{ECON}(G;r)$ or $k\,\mathrm{NCON}(G;r)$ are not known. The only special case that has been investigated is the uniform case $r_v = k$ for all $v \in V$. A strengthened version of a result in [GM] due to M. Stoer reads as follows.

(4.1) THEOREM. *Let $G=(V,E)$ be a $(k+1)$-edge connected graph and let $W \neq V$ be a nonempty node set. Define for each $W_i \subseteq W$ with $\emptyset \neq W_i \neq W$ the* **deficit** *of W_i as*

$$\mathrm{def}_G(W_i) := \max\{0, k - |\delta_{G[W]}(W_i)\}.$$

Define similarly for $U_i \subseteq V \setminus W$ with $\emptyset \neq U_i \neq V \setminus W$

$$\mathrm{def}_G(U_i) := \max\{0, k - |\delta_{G[V \setminus W]}(U_i)|\}.$$

The cut inequality $x(\delta(W)) \geq k$ defines a facet of the polytope of k-edge connected graphs if and only if the following conditions are satisfied.

(a)

$$\sum_{i=1}^{p} \mathrm{def}_{G-e}(W_i) + \sum_{i=1}^{q} \mathrm{def}_{G-e}(U_i) - \left|\left[\bigcup_{i=1}^{p} W_i : \bigcup_{i=1}^{q} U_i\right]\right| \leq k$$

for all edges $e \in E(W) \cup E(V \setminus W)$,
for all pairwise disjoint node sets $W_1, W_2, \dots, W_p$ of W, and
for all pairwise disjoint node sets $U_1, U_2, \dots, U_q$ of $V \setminus W$ that satisfy the following conditions:
(i) *if $p = 1$, then $W_1 \neq W$,*
(ii) *if $q = 1$, then $U_1 \neq V \setminus W$.*

(b) *$G[W]$ and $G[V \setminus W]$ are connected.*

D. Bienstock (private communication) devised a polynomial time algorithm that determines whether a given cut inequality induces a facet of $k\,\mathrm{ECON}(G;r)$ or not in the uniform case.

For the NCON problem the following sufficient condition follows from slightly more general results of [GM].

(4.2) THEOREM. *Let $k \geq 2$ and let $G=(V,E)$ be $(k+1)$-node connected and $r_v = k$ for all $v \in V$. Let $W \subseteq V$ such that $G[W]$ and $G[V \setminus W]$ are k-node connected. Then $x(\delta(W)) \geq k$ defines a facet of* $\mathrm{NCON}(G;r)$.

Further, more technical sufficient or necessary conditions of this type can be found in [GM].

The cut inequalities (2.1) (i) can be generalized to partition inequalities for $k\,\mathrm{ECON}(G;r)$. Let $W_1, \dots, W_p$ be a partition of V into p node sets W_i with $r(W_i) \geq 1$. Let $I_1 := \{i \in \{1, \dots, p\} \mid \mathrm{con}(W_i) = 1\}$ and

$I_2 := \{i \in \{1, \ldots, p\} | \operatorname{con}(W_i) \geq 2\}$. We assume I_2 to be nonempty. Then the **partition inequality** induced by $W_1, \ldots, W_p$ is defined as

$$(4.3) \qquad \frac{1}{2}\sum_{i=1}^{p} x(\delta(W_i)) \geq \left\lceil \frac{1}{2}\sum_{l \in I_2} \operatorname{con}(W_l) \right\rceil + |I_1|.$$

Partition inequalities are valid for $\operatorname{ECON}(G; r)$. They define facets of $\operatorname{ECON}(G; r)$ only if $\sum_{i \in I_2} \operatorname{con}(W_i)$ is odd or if I_1 is nonempty. Note that for $r \in \{0, 1, 2\}^V$, this is the partition inequality (3.4) for $2\operatorname{ECON}(G; r)$ with right-hand side p.

The node cut inequalities (2.1) (ii) can be generalized to the larger class of partition inequalities on a subgraph in a straightforward manner as follows.

Let $Z \subseteq V$ be some node set $|Z| \geq 1$. If we delete Z from G then the resulting graph must contain an $[s, t]$-path for every pair of nodes s, t of type larger than $|Z|$. In other words, if $W_1, \ldots, W_p$ is a partition of $V \setminus Z$ into node sets with $r(W_i) \geq |Z| + 1$ then the graph $G' := (G - Z)/W_1/\cdots/W_p$ must be connected. This implies that the partition inequality for the connected subgraph polytope of G' (see [GM] (5.5) and (5.7)) lifted to G is valid for $\operatorname{NCON}(G; r)$. This observation gives the following class of inequalities valid for $\operatorname{NCON}(G; r)$.

$$(4.4) \qquad \frac{1}{2}\sum_{i=1}^{p} x(\delta_{G-Z}(W_i)) \geq p - 1 \quad \text{for every node set } Z \subseteq V,\ |Z| \geq 1 \text{ and every partition } W_1, \ldots, W_p \text{ of } V \setminus Z \text{ such that } r(W_i) \geq |Z| + 1,\ i = 1, \ldots, p.$$

That class can be generalized to further, more general, partition inequalities.

We now describe a relaxation for the ECON (and thus NCON) problem that is based on matching theory. This model applies to the general case $r_v \in \mathbf{Z}_+$ for all $v \in V$ and produces lower bounds for the integer program (2.1) (i), (iii), (iv) that are computable in polynomial time. Moreover, this model gives rise to further inequalities valid for $\operatorname{ECON}(G; r)$ (and thus $\operatorname{NCON}(G; r)$) that can be used in a cutting plane approach.

The survivability requirements imply that, if $v \in V$ is a node of type r_v, then v has degree at least r_v for any feasible solution of the ECON problem. Thus, if we can find an edge set such that each node has degree at least r_v (we call such a set an r**-cover**) and that has minimum cost we obtain a lower bound for the optimum value of the ECON problem. Clearly, such an edge set can be found by solving the integer linear program

(4.5) $\min C^T x$

(i)	$x(\delta(v)) \geq r_v$	for all $v \in V$,
(ii)	$0 \leq x_e \leq 1$	for all $e \in E$, and
(iii)	x_e integer	for all $e \in E$,

which is obtained from (2.1) (i), (iii), (iv) by considering in (2.1) (i) only sets of cardinality one. (4.5) can also be viewed as an integer programming version of the LP (3.12) which is the initial LP relaxation of our cutting plane algorithm. This integer program can be turned into a linear program, i.e., the integrality constraints (iii) are replaced by a system of linear inequalities, using Edmonds' polyhedral results on b-matching. See [E]. Edmonds proved that, for any vector $b \in \mathbf{Z}_+^v$, the vertices of the polyhedron defined by

(4.6)

(i) $$y(E(H)) + y(\overline{T}) \le \left\lfloor \frac{\sum_{v \in H} b_v + |\overline{T}|}{2} \right\rfloor \quad \text{for all } W \subseteq V \text{ and all } \overline{T} \subseteq \delta(H),$$

and

(ii) $$0 \le y_e \le 1 \quad \text{for all } e \in E$$

are precisely the incidence vectors of all (**1**-capacitated) ***b*-matchings** of G, i.e., of edge sets M such that no node $v \in V$ is contained in more than b_v edges of M. For the case $b_v := \deg(v) - r_v$, where $\deg(v)$ denotes the degree of v in G, the b-matchings M are nothing but the complements $M = E \setminus F$ of r-covers F of G. Using the transformation $x := \mathbf{1} - y$ and $T := \delta(H) \setminus \overline{T}$ we obtain the system

(4.7)

(i) $$x(E(H)) + x(\delta(H) \setminus T) \ge \left\lceil \frac{\sum_{v \in H} r_v - |T|}{2} \right\rceil \quad \text{for all } H \subseteq W \text{ and all } T \subseteq \delta(H),$$

and

(ii) $$0 \le x_e \le 1 \quad \text{for all } e \in E.$$

(4.7) gives a complete description of the convex hull of the incidence vectors of all r-covers of G. Since every solution of the ECON problem for G and r is an r-cover, all inequalities (4.7) (i) are valid for ECON(G; r).

It is trivial to see that only those inequalities (4.7) (i) where $r(H) - |T|$ is odd are needed. From the results of [CP], one can derive which of these inequalities define facets of the r-cover polytope.

4.2. Computational results. As mentioned before, our computational experience with network design problems with higher-connectivity requirements is very limited. We describe here briefly our current state of knowledge.

We have started implementing a cutting plane algorithm in the same way as outlined in Section 3.2 for the low-connectivity case. The initial LP is (3.12). The separation routine for cut inequalities also works in this case (with a few modifications). A primitive heuristic for some of the partition

TABLE 2

Problem	nodes	edges	K	IT	PART	C	COPT	CHEUR	% GAP	TIME
graph 1.40	40	780	5	0	0	190.70	190.70	200.92	5.35	0 : 32
graph 2.40	40	780	5	0	0	186.97	186.97	192.65	3.03	0 : 29
graph 3.40	40	780	5	0	0	182.25	182.25	194.52	6.73	0 : 33
graph 4.40	40	780	5	0	0	186.53	?	194.84	4.45	0 : 28
graph 5.40	40	780	5	0	0	152.42	?	157.60	5.15	0 : 33
euclid 40.sparse	40	206	5	0	0	760.12	760.12	829.82	9.17	0 : 18
euclid 40.dense	40	780	5	2	4	338.54	?	367.25	8.48	0 : 44
ship	494	1096	3	73	1189	1510.0686	?	2193.08	45.23	24 : 12

inequalities (4.3) is running. But this has to be tested in detail against various possible modifications. Separation heuristics for further classes of valid inequalities and the exact separation procedure for the complemented b-matching constraints still remain to be implemented.

Nevertheless we have run some experiments with the current version of the code. Seven of our test problems were random problems to test some heuristics [KM] for finding minimum-cost k-edge or node connected graphs. The "graph i.40" problems are 5 ECON problems on a complete graph with 40 nodes and uniformly distributed costs between 0 and 20.

For "euclid i" the costs are euclidean (for randomly generated nodes). The problem "euclid 40.sparse" has some edges randomly deleted from the underlying graph while ensuring that the remaining graph is still 5-connected. All of these problems are 5 ECON problems with uniform node types $r \in \{5\}^V$. The optimal solutions to these 5 ECON problems (if they are known) are also feasible for the 5 NCON problem.

The "ship" problem is a realistic model of the possible direct link connections of the various communication systems on a ship. The possible links form a 3-dimensional lattice with a very regular cost structure; only 3 nodes of the 494 nodes have node type 3 and 30 nodes have node type 1. We treated this problem as a 3 ECON problem.

Table 2 contains some information about the performance of our code when it was applied to those 8 problems. The first three columns contain the problem name, the total number of nodes and the number of edges. K denotes the connectivity type. IT denotes the number of LP-iterations and PART the number of partition inequalities (4.3) used to produce the lower bound C. Note that in the "graph i.40" problems and "euclid 40.sparse" our separation routines could not find any violated partition inequalities after solution of the initial LP (3.12). In fact, as can be seen from the column COPT, in four cases the solution of the initial LP was already optimal. A "?" in the column COPT says that the optimum solution is not known at present. CHEUR is the value of the best feasible solution found by the heuristics of [KM]. $GAP := 100 \times (CHEUR - C)/C$ is the gap between the heuristic value (upper bound) and the lower bound generated by our code.

The total time on a SUN 3/50 in min:sec of our cutting plane procedure is reported in column TIME.

The code produced a somewhat unexpected outcome. It did very well on the random problems where all nodes have high connectivity requirements. But it performed poorly on the real world problem that has many nodes of type 0 and relatively few nodes of higher connectivity. We believe that further polyhedral investigations and the design of more structure dependent heuristic separation procedures will cure this poor behavior.

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